Attitude Kinematics of the Foucault Pendulum

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Abstract

Attitude kinematics is shown to be the fundamental cause of the apparent precession of the plane of motion of the Foucault Pendulum. Ishlinskii's Theorem in attitude kinematics provides a derivation of the angle by which a Foucault pendulum precesses after one complete rotation of the Earth about its axis. A more recent kinematic theorem is used to provide a general derivation of the pendulum precession angle for any Earth rotation angle about its axis. The derivations of these precession equations enhance understanding of the phenomenon by focusing on the essential cause of the precession, while avoiding complicated dynamical analyses and associated mathematical machinery.

Keywords: Foucault Pendulum, Attitude kinematics, Ishlinskii's Theorem, Solid angle, Precession angle

1. Introduction

The Foucault Pendulum [1] is a long pendulum usually with a significant mass attached to maintain its periodic motion in the presence of frictional forces. The pendulum is essentially a spherical pendulum mounted to a frictionless bearing to allow rotation about its axis. The motion of the pendulum is initialized such that the pendulum mass moves in a plane and does not move in circular or elliptical motion. The Foucault Pendulum was first demonstrated in Paris in 1851 and showed that the Earth rotates [1] without relying on a celestial coordinate frame. Since the plane of the pendulum's swing appears to precess with respect to the Earth's local reference frame, it demonstrated that the Earth rotates on its axis. The rate of precession of the pendulum's plane of motion is clockwise in the Northern Hemisphere and given by the Earth's angular rate times the sine of the geodetic latitude. The expression is similar for pendulum precession in the Southern Hemisphere, but the direction of precession is counterclockwise instead of clockwise. The precession of the plane of motion is easy to understand when the Foucault Pendulum is located near the north pole where the Earth's angular rate vector is vertical. In this case, the pendulum is not affected by the rotation of the Earth and the Earth appears to rotate below the pendulum in the counterclockwise direction, as observed from the inertial frame. From an observer at the north pole, the plane of the pendulum's motion appears to precess in the clockwise direction at the Earth's angular rate magnitude. When the pendulum is placed at an arbitrary latitude, the rate of precession of its plane of motion requires more explanation.

Dynamical analyses [2-4] of the motion of the Foucault Pendulum have been performed to help understand its behavior for all values of latitude. The fictitious Coriolis Force appears in the equations of motion when a non-inertial reference frame is used and can provide an intuitive understanding of the precessional motion of the pendulum [2]. The Earth's component of angular rate along the pendulum's axis can also help in understanding its precessional motion [5]. Other researchers found that the precessional motion of the pendulum is the result of the geometrical considerations without the use of dynamical analysis [6-7]. In 1952, Ishlinskii published a theorem in the area of attitude kinematics [8], which provides the fundamental cause of the apparent precession motion of the Foucault Pendulum.

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In this work, it is shown that a kinematical theorem published by Ishlinskii in 1952 [8] can be used to explain the observed precession of the Foucault Pendulum. Ishlinskii's Theorem states that if an axis fixed in the body, describes a closed conical surface in space, the angle of rotation of the body around this axis is equal to the integral of the projection of the angular velocity of the body onto this axis plus the solid angle of the cone enclosed. If the angular rate is zero, the rotation about the axis is equal to the solid angle of the enclosed cone. It is shown that for a complete rotation of the Earth about its axis, the pendulum attach point has both the solid angle rotational contribution and the integral of the angular rate contribution mentioned in Ishlinskii's Theorem. The pendulum itself has only the solid angle rotational contribution. The difference between the attach point and pendulum contributions accounts for the apparent precessional motion of the pendulum's plane of motion. A recently developed kinematic theorem [9] was used to compute the pendulum's precession angle associated with an arbitrary Earth rotation angle, which could be less than the 360 degrees required by Ishlinskii's Theorem.

Section 2 contains an analysis of the precession of the pendulum's plane of motion using Ishlinskii's Theorem, which applies to a complete rotation of the Earth about its axis. Section 3 contains a more complete application of attitude kinematics to the precessional motion of the pendulum's plane of motion. This analysis provides the pendulum's precession angle associated with an arbitrary rotation angle of the Earth about its axis. Section 4 contains a numerical example to clarify the slewing transformation of the pendulum. Section 5 contains the conclusion.

2. Ishlinskii's Theorem Application

Ishlinskii's Theorem [8] can be applied to the case of the Foucault Pendulum by considering an axis extending from the center of the Earth, which is assumed spherical, to the pendulum attach point at a latitude, Θ, in the Northern Hemisphere, as shown in Fig. 1. As the Earth completes one rotation, the axis sweeps out a conical region enclosing a solid angle given by eq. (1).

$$\Omega = 2\pi[1 - \sin(\Theta)] \tag{1}$$

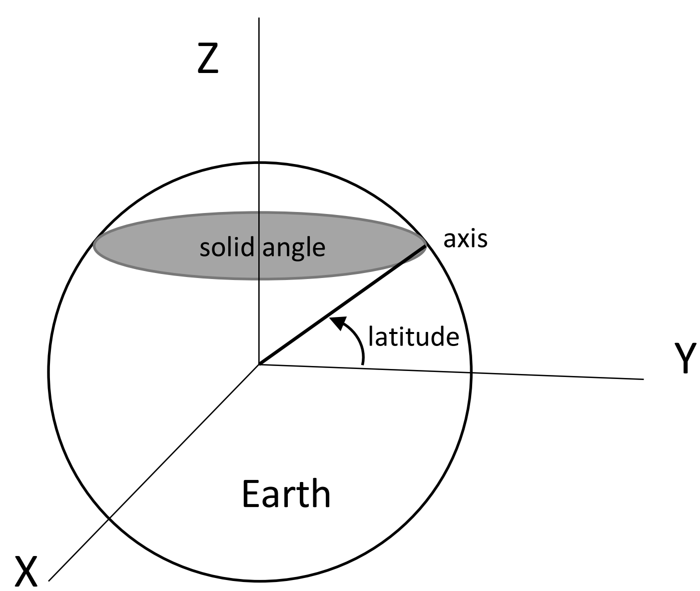

Fig. 1. Solid angle region enclosed by the axis as the Earth completes one revolution about its axis.

Ishlinskii's Theorem states that after the axis completes its closed trajectory it will have rotated by Ω, as given in eq. (1). This rotation is not due to an angular rate along the axis and therefore is not measurable by a rate gyro that has its input axis aligned with the slewing axis. The rotation is the result of fundamental attitude kinematics captured in Ishlinskii's Theorem.

The other component of rotation about the axis is due to the integral of the projection of the Earth's angular rate onto the axis, which is in the vertical direction at the location of the pendulum. This rotation component is given by eq. (2), where ω is the Earth's angular rate magnitude and T is the period of the Earth's rotation. This rotation contribution is measurable by a rate gyro.

$$\phi = \int_0^T \omega \sin(\theta)\, dt = \omega T\, sin(\theta) = 2\pi \sin(\theta) \tag{2}$$

The amount that the axis rotates at the pendulum's attach point, β, after the Earth completes one revolution can be found by applying Ishlinskii's Theorem to the axis and using eqs. (1) and (2), as shown in eq. (3).

$$\beta = \Omega + \phi = 2\pi[1 - \sin(\theta)] + 2\pi \sin(\theta) = 2\pi \tag{3}$$

Eq. (3) shows that the pendulum attach point rotates by 2π along the axis, while the Earth rotates 2π about its spin axis. It should be noted that a rotation of 2π about an axis is equivalent to a rotation of 2π about any other axis or a rotation of zero degrees, which is equivalent to no rotation at all.

According to Ishlinskii's Theorem, the amount that the pendulum and its plane of oscillation rotates about its axis after one complete rotation of the Earth is the solid angle of the cone swept out by the axis, which is given by Ω in eq. (1). Therefore, the pendulum, its attach point and its plane of motion all rotate about the axis by angle, Ω, because they all slew about the Earth's spin axis. The plane of motion of the pendulum does not have the additional rotation given by eq. (2), since the swinging motion of the pendulum establishes an oscillating angular momentum vector, which is normal to the vertical axis and would require an applied torque to slew in synchronous motion with vertical component of the Earth's angular rate. Since the pendulum is decoupled from its attach point in terms of axial rotation, the torque required to slew the angular momentum vector is not available. Therefore, the plane of pendulum motion has zero angular rate in the vertical direction, which creates its apparent precession with respect to the attach point in the local reference frame. The pendulum and its plane of motion, as observed from its attach point appear to precess by an amount, ψ, according to eq. (4).

$$\psi = \Omega - \beta = 2\pi[1 - \sin(\theta)] - 2\pi = -2\pi \sin(\theta) \tag{4}$$

Eq. (4) shows that the plane of the pendulum's motion precesses by the expected amount, after the Earth completes one revolution.

In summary, the pendulum's attach point rotates due to the two components in Ishlinskii's Theorem; the solid angle contribution and the contribution of the integral of the angular rate along the axis. The pendulum and its plane of rotation have only the solid angle rotational contribution, since it is decoupled from the axial rotation of the attach point and cannot rotate according to the vertical component of the Earth's rotation. The difference in the rotation angle of the pendulum's attach point and the rotation angle of the pendulum's plane of motion about the axis accounts for the observed precession of the pendulum's plane of motion.

3. Attitude Kinematics Application

The analysis in Section 2 showed the application of Ishlinskii's Theorem to the precession of a Foucault Pendulum after the Earth rotates 360 degrees about its axis. The amount that the pendulum plane precesses when the Earth rotates by an angle λ can be found by performing a more in depth analysis involving the attitude transformation of the slewing axis. It was shown in an earlier work that any attitude transformation can be represented by the slewing transformation of a body fixed axis followed by a rotational transformation of the body about the axis [9]. Both the slewing transformation and the rotational transformation are driven by the angular rate of the body, which in this case is the rotation of the Earth along its axis. The component of the Earth's angular rate normal to the axis produces the slewing motion of the axis along an infinite number of infinitesimal great circle arcs. The component of the Earth's angular rate parallel to the axis is integrated to produce the rotation angle and the associated rotational transformation, which can be applied after the slewing transformation, as was shown in an earlier work [9]. It was also shown that, if the slewing motion of the axis is fixed in the inertial frame, the slewing and rotational transformations commute [9]. Let the axis slew from **A** to **B** along trajectory S, while the body rotates about the axis, as shown in Fig. 2. The accumulated rotation angle, $\boldsymbol{\Delta}$, about the axis during the slewing motion can be applied about the axis at orientation **B.** Thus, the total transformation is the product of the slewing, $\mathbf{U_S}$, and rotational, $\mathbf{R}(\mathbf{B}, \Delta)$, transformations given in eq. (5). Since the slewing and rotational transformations commute, the rotational transformation can be performed at orientation **A** or orientation **B**, as shown in eq. (5).

$$\mathbf{U} = \mathbf{U_S}\,\mathbf{R}(\boldsymbol{B}, \Delta) = \mathbf{R}(\mathbf{A}, \Delta)\,\mathbf{U_S} \tag{5}$$

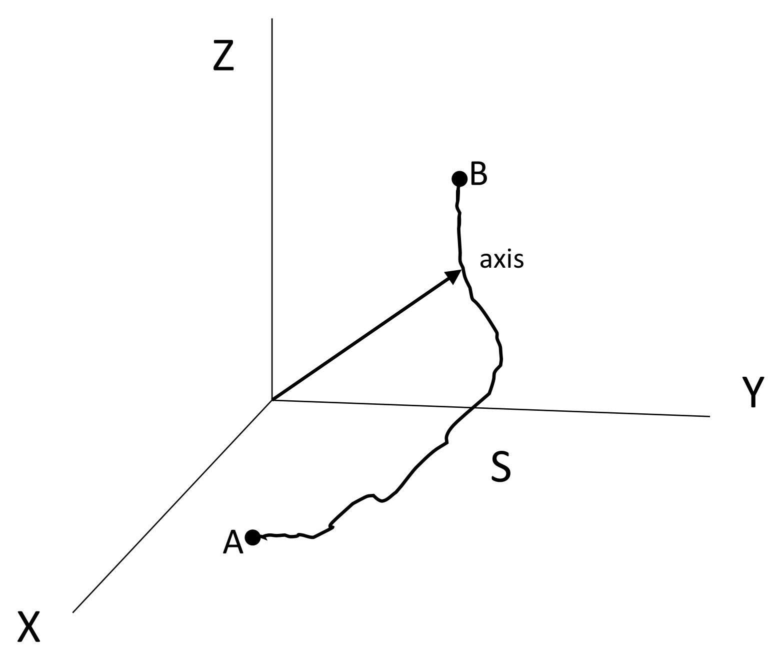

Fig. 2. An axis slews along trajectory S connecting orientations **A** and **B** followed by a rotation of Δ about the axis at orientation **B**.

The result of eq. (5) is applied to the slewing motion of the axis extending from the center of the Earth to the pendulum attach point, as the Earth rotates by angle λ. The initial location of the pendulum attach point is at point **A** and the final location is at point **B**, in Fig. 3. In this case, the total transformation, **U**, is given by a rotation of λ about the z-axis of the Earth, as shown in eq. (6), where **R** is the rotational transformation, z is the axis about which the rotation is applied and λ is the magnitude of the rotation angle.

$$\mathbf{U} = \mathbf{R}(\mathbf{z}, \lambda) \tag{6}$$

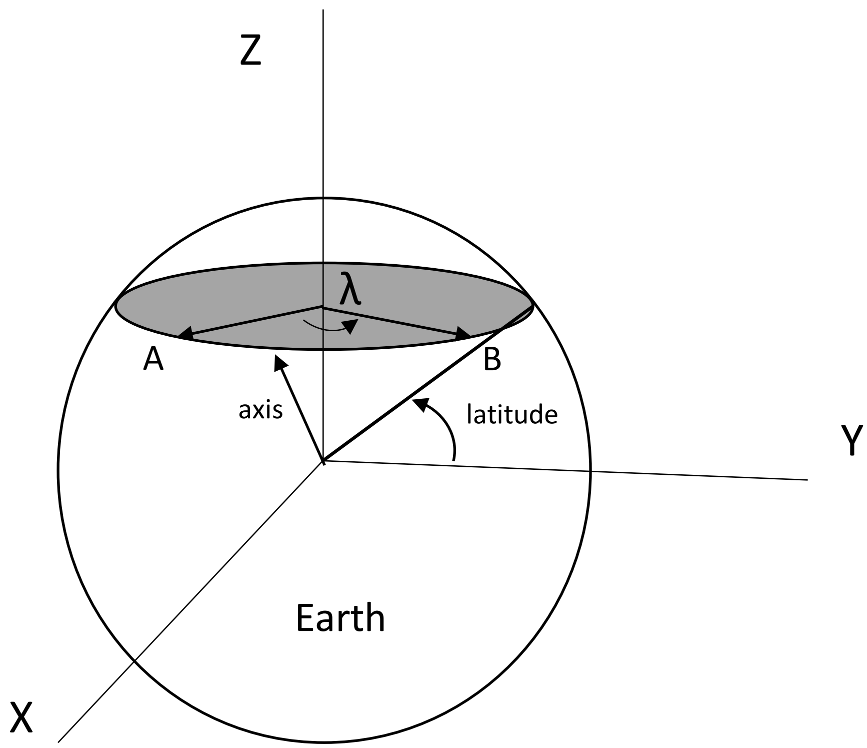

Fig. 3. The pendulum axis moves from orientation **A** to orientation **B** as the Earth rotates by λ.

Assuming that the pendulum is located at latitude Θ, the component of the Earth's angular rate along the axis is given by ω sin(Θ). After the Earth rotates by λ, the axis will have rotated by λ sin(Θ). Therefore, the rotational transformation about the axis at orientation **B** is given by eq. (7).

$$\mathbf{R}(\mathbf{B}, \Delta) = \mathbf{R}[\mathbf{B}, \lambda \sin{(\theta)}] \tag{7}$$

Using eqs. (6) and (7) in eq. (5), one obtains eq. (8).

$$\mathbf{U} = \mathbf{R}(\mathbf{z}, \lambda) = \mathbf{U_S}\, \mathbf{R}[\mathbf{B}, \lambda \sin{(\theta)}] \tag{8}$$

The slewing transformation can be obtained from eq. (8) by multiplying the inverse of the rotation transformation by both sides of eq. (8), as shown in eq. (9).

$$\mathbf{U_S} = \mathbf{R}(\mathbf{z}, \lambda)\ \mathbf{R}^{-1}[\mathbf{B}, \lambda \sin(\theta)] = \mathbf{R}(\mathbf{z}, \lambda)\ \mathbf{R}[\mathbf{B}, -\lambda \sin(\theta)] \quad (9)$$

The inverse of the rotational transformation about the axis at orientation **B** is represented by the negative of the rotation angle as shown in eq. (9). If we used the rotational transformation about **A** in eq. (5), the slewing transformation would be given by eq. (10), which is equivalent to eq. (9).

$$\mathbf{U_S} = \mathbf{R}[\mathbf{A}, -\lambda \sin(\theta)]\ \mathbf{R}(\mathbf{z}, \lambda) \quad (10)$$

The transformation of the pendulum attach point is given by **U** or **R(z, λ)**. The transformation of the pendulum's plane of motion is the slewing transformation, $\mathbf{U_S}$. The precession of the pendulum's plane of motion with respect to the attach point is given by the transformation, $\mathbf{U_P}$, in eq. (11). Therefore, $\mathbf{U_P}$, is given by eq. (12), where the inverse of eq. (8) has been used.

$$\mathbf{U}\ \mathbf{U_P} = \mathbf{U_S} \quad (11)$$

$$\mathbf{U_P} = \mathbf{U^{-1}}\ \mathbf{U_S} = \mathbf{R^{-1}}[\mathbf{B}, \lambda \sin(\boldsymbol{\theta})]\ \mathbf{U_S^{-1}}\ \mathbf{U_S} = \mathbf{R}[\mathbf{B}, -\lambda \sin(\theta)] \quad (12)$$

Eq. (12) shows that the plane of the pendulum motion has rotated about the axis at orientation **B** by an angle of -λ sin(Θ), which agrees with observations of the pendulum motion. Note that the orientation of the axis at **B** is vertical at the location of the pendulum so the precession is in the clockwise direction. As the Earth rotates by angle λ about the z-axis, the plane of the pendulum motion rotates in the clockwise direction about the vertical axis by λ sin(Θ). If λ = 2π, eq. (12) agrees with eq. (4), which indicates that the plane of pendulum motion rotates by 2π sin(Θ) in the clockwise direction.

4. Numerical Example

A computer simulation was developed to compute the slewing transformation of the Foucault Pendulum as given by eq. (9) and the associated Euler Rotation Vector components, which are plotted in Fig. 4. The angle between the Euler Vector, ***E*,** and the axis that extends from the center of the Earth to the location of the pendulum was also computed, as illustrated in Fig. 5. The pendulum was placed at a geodetic latitude of 30 degrees so that its precession angle is 180 degrees after one full rotation of the Earth, which is in agreement with eqs. (4) and (12). The angle between ***E*** and the axis is initially 90 degrees and decreases to 0 degrees, at Earth rotation angle of 360 degrees, as indicated in Fig. 5. At this time, ***E*** is aligned with the axis and has magnitude 180 degrees, which means that the plane of the pendulum motion has rotated by 180 degrees due to its slewing motion cause by the Earth rotation. Since the local frame has rotated by 360 degrees, the apparent precession of the plane of motion with respect to the local frame is 180 - 360 = -180 degrees, as expected. After the Earth rotates 720 degrees, the plane of pendulum motion has rotated by 360 degrees and the apparent precession angle is 360 - 720 = -360 degrees. The cycle repeats from 720 degrees to 1440 degrees, as shown in Figs. 4 and 5. Although the Euler Vector components change sign when the magnitude of the ***E*** reaches 180 degrees at Earth rotation angles of 360 and 1080 degrees, the associated slewing transformation is continuous for all values of Earth rotation. When the components of ***E*** change sign, the angle between the axis and ***E*** increases to 180 degrees, as indicated in Fig. 5.

The apparent precession of the plane of motion of the pendulum with respect to its attach point can be observed in the Euler Vector, ***ER***, obtained from eq. (12), whose components are plotted in Fig. 6.

The magnitude of ***ER*** increases linearly and has a value of 180 degrees at 360 degrees of Earth rotation, as shown in Fig. 6. All of the components of ***ER*** are zero at Earth rotation angle of 720 and the cycle repeats from 720 to 1140 degrees, as given in Fig 6.

The relative precession of the pendulum's plane of motion is most clearly shown in Fig. 7, where the component of the Euler Rotation Vector parallel to the axis decreases from 0 to -180 degrees, as the Earth rotates 360 degrees. The angle plotted in Fig. 7 is equivalent to $-\lambda \sin(\Theta)$, which is the angle in eq. (12). The precession angle about the axis is zero at Earth rotation angles of 720 and 1440 degrees, as expected. For other values of latitude, plots similar to Figs. 4-7 can be created.

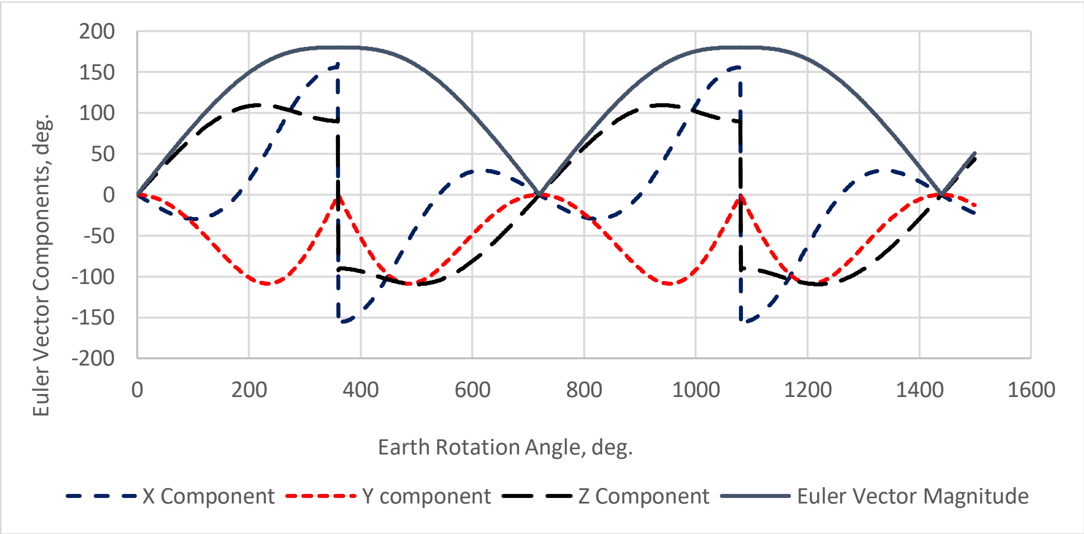

Fig. 4. Euler Rotation Vector components for the slewing transformation of a Foucault Pendulum located at geodetic latitude of 30 degrees north.

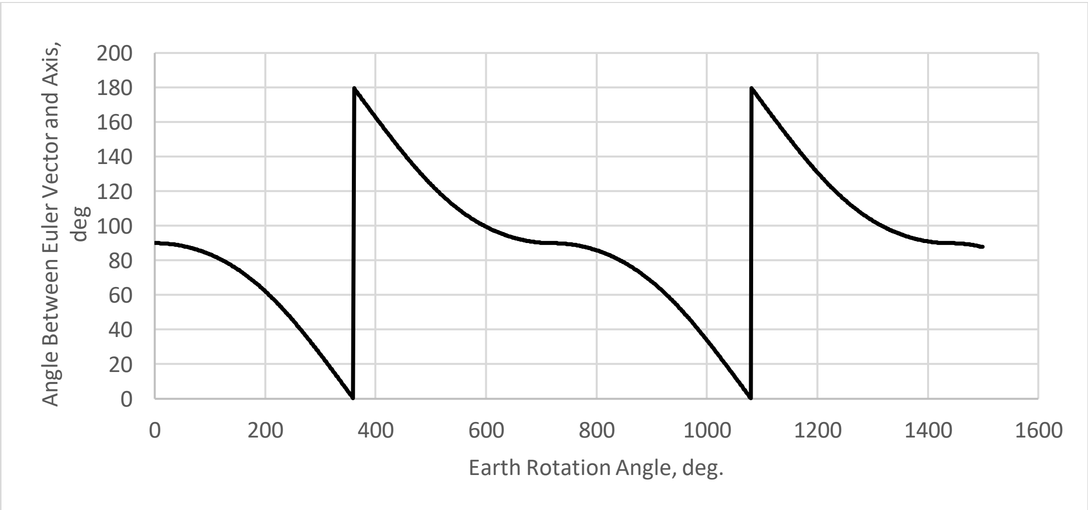

Fig. 5. Angle between axis and Euler Rotation Vector for a slewing pendulum located at geodetic latitude 30 degrees north.

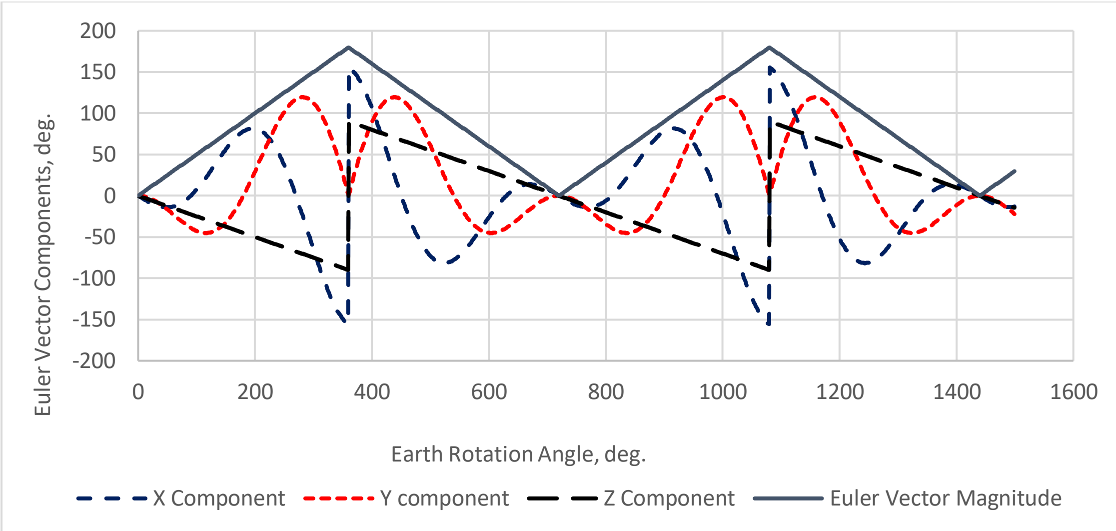

Fig. 6. Euler Rotation Vector components for the apparent precessional transformation of a Foucault Pendulum with respect to its attach point located at geodetic latitude of 30 degrees north.

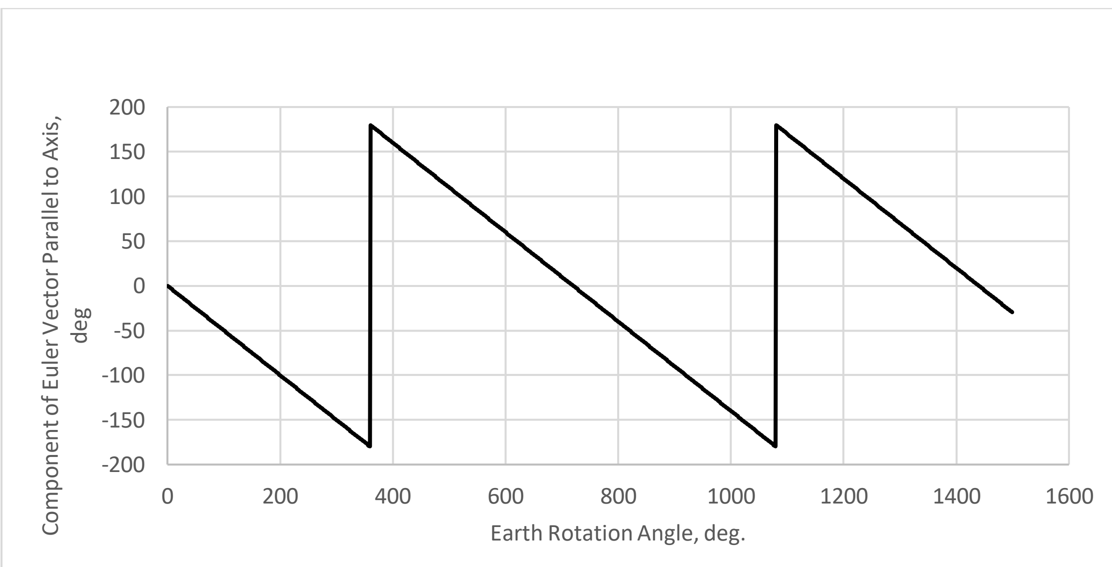

Fig. 7. Component of Euler Vector parallel to the axis indicating precession of a Foucault pendulum located at geodetic latitude 30 degrees north.

5. Conclusion

Attitude kinematics was shown to be the fundamental cause of the precession of the Foucault Pendulum. After one complete rotation of the Earth about its axis, the local reference frame at any latitude location rotates about its vertical axis by an amount specified by Ishlinskii's Theorem. The plane of the pendulum motion, which is rotationally decoupled from the local frame, rotates about the vertical axis by an angle also specified by Ishlinskii's Theorem, but which is different than that of the local frame. This rotation is due to attitude kinematics alone, since there is no torque applied to the pendulum to cause the rotation of its plane of motion. It was shown that the difference in these two rotation angles causes the apparent precession of the plane of motion of the Foucault Pendulum. In a similar fashion, a more recent theorem in attitude kinematics was used to derive the general equation for the precession angle of the Foucault Pendulum for any Earth rotation angle, not just the complete rotation of 360 degrees specified in Ishlinskii's Theorem. A numerical example using only kinematical equations was provided to clarify the slewing transformation of the pendulum's plane of motion by computing the associated Euler Rotation Vector. This work clearly establishes that attitude kinematics is the root cause of the precession of the Foucault Pendulum.

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