On differential equations of Lienard type with identical exact solutions

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Abstract

We investigate in this paper the property of Lienard type differential equations to have identical exact solutions. We establish the conditions of existence of identical exact solutions and exhibit some examples to illustrate the theory.

Keywords: Lienard nonlinear differential equations, quadratic Lienard type equations, exact solutions, identical solutions.

Introduction

The Lienard equations of type

$$\ddot{x} + h(x) = 0 \tag{1}$$

where the overdot means a derivative with respect to time, and $h(x)$ is a nonlinear function of x, have been, for a long time intensively investigated in the literature to secure periodic solutions. When $h(x) = \omega_0^2 x$, where ω_0 is a constant, the equation (1) is known as the linear harmonic oscillator equation in physics. The equations of type (1) have been mainly investigated in physics to describe conservative nonlinear oscillators. A celebrated equation of the form (1) is the cubic Duffing equation

$$\ddot{x} + \alpha x + \beta x^3 = 0 \tag{2}$$

where α and β are arbitrary constants. A vast literature exists for this equation [1-4]. The equation (2) has gained a high importance in physics since it has been used to describe many phenomena that could not be explained by the linear harmonic oscillator in dynamics. The solution of the equation (2) is well known in literature as the Jacobi elliptic functions. In [2] the authors used the generalized Sundman transformation theory to find the periodic solutions of the cubic Duffing equation (2). However, it has been shown recently in several papers [5-7] that the

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cubic Duffing equation (2) can have general non-oscillatory solutions. Another famous equation that belongs to the general class of Lienard type equation (1) is the pendulum equation [4, 8-10]

$$\ddot{x} + \gamma \sin x = 0 \tag{3}$$

The equation (3) has been widely investigated in the literature from mathematical and physical point of view. For a long time only the periodic solution has been calculated in the literature for the pendulum equation (3). However, the authors in [9] were recently able to exhibit non-oscillatory solution for this equation. The above shows the great importance of the Lienard equations of type (1) in mathematics and physics. Another general class of differential equations may read

$$\ddot{x} + u(x)\dot{x}^2 + \vartheta(x) = 0 \tag{4}$$

where $u(x)$ and $\vartheta(x)$ are arbitrary functions of x. This general class of equations is known in the literature as the quadratic Lienard type equation [11-14]. Different approaches have been used to investigate in the literature the general class of the quadratic Lienard type equations (4). The equation (4) is investigated in [11] from Lie point symmetry method. In [14] the authors investigated a generalized equation of the form (4) using symmetry method as well as generalized Sundman transformation theory. A celebrated equation that belongs to the class of equations (4) is the Mathews-Lakshmanan equation [15] exhibiting harmonic periodic solution but with amplitude dependent frequency. The equation was presented by the authors [15] as a unique oscillator of quadratic Lienard type exhibiting this behavior. However, in [12,13] the authors recently showed the existence of a class of quadratic Lienard type equations which may exhibit the harmonic periodic solution behavior with amplitude dependent frequency using the generalized Sundman transformation. Several examples of equation of this class have been shown to exhibit this behavior [13, 16]. In other papers [17, 18] it is also shown the existence of such quadratic Lienard type equations exhibiting harmonic periodic solutions. In a recent paper [19] the authors have shown for the first time the existence of a singular quadratic Lienard type equation that exhibits the linear harmonic oscillator equation solution but with an amplitude of oscillations equal to unity. More recently, Monsia and coworkers have shown in [20] that the reduced quadratic Lienard type equation

$$\ddot{x} + \frac{g'(x)}{g(x)}\dot{x}^2 = 0 \tag{5}$$

known also as quadratically damped Lienard equation can exhibit the linear harmonic oscillator solution with arbitrary amplitude of oscillations μ when the nonlinear function $g(x)=(\mu^2-x^2)^{-\frac{1}{2}}$, where $\mu \succ 0$. One can see from the above the existence of differential equations with identical solutions. In [19] this property is shown for the Emarkov-Pinney equation and a singular quadratic Lienard type equation, which exhibit identical periodic solution. It is remarkable to notice that the Emarkov-Pinney equation belongs to the class of Lienard type equations (1). In view of the above, one can ask whether there are conditions under which the Lienard equation (1) and the quadratic Lienard type equation (4) have identical solutions. In this paper we predict the existence of such conditions. In this regard, we establish these conditions (section 2) and give some examples to illustrate this property of differential equations to have identical exact solutions (section 3). Finally a conclusion is carried out for this work.

2. Conditions of existence of identical exact solutions

Let us consider the theory of differential equations introduced by Monsia and coworkers [20-23] to establish the conditions of existence of identical exact solutions between the Lienard equation (1) and the quadratic Lienard type equation (2). Accordingly one may consider the general class of mixed Lienard type equations

$$\ddot{x}+\frac{g'(x)}{g(x)}\dot{x}^2+a\ell\frac{f(x)}{g(x)}x^{\ell-1}\dot{x}-a^2\frac{f'(x)f(x)}{g^2(x)}x^{2\ell}+ab\frac{f'(x)}{g^2(x)}x^{\ell}=0 \tag{6}$$

associated to the first integral

$$g(x)\dot{x}+af(x)x^{\ell}=b \tag{7}$$

where prime means differentiation with respect to the argument, $a,$ b and ℓ are arbitrary parameters, and $f(x)$ and $g(x)\neq 0$ are arbitrary functions of $x.$ As can be seen the mixed Lienard type equation (6) can be reduced to the dissipative Lienard type equation

$$\ddot{x}+a\ell f(x)x^{\ell-1}\dot{x}+abf'(x)x^{\ell}-a^2f'(x)f(x)x^{2\ell}=0 \tag{8}$$

when $g(x)=1.$The dissipative term in the equation (8) can be also canceled taking $\ell=0$ to obtain [23]

$$\ddot{x}-a^2f'(x)f(x)+abf'(x)=0 \tag{9}$$

which becomes [23]

$$\ddot{x} - a^2 f'(x) f(x) = 0 \tag{10}$$

when $b = 0$. The equation (10) is of the form (1) when $h(x) = -a^2 f'(x) f(x)$. In this situation the exact and general solution of the equation (10) is given by the differential relation

$$-\frac{dx}{af(x)} = dt \tag{11}$$

where $f(x) \neq 0$, and $a \neq 0$.

On the other hand, substituting $\ell = 0$, into the equation (6) yields the quadratic Lienard type equations

$$\ddot{x} + \frac{g'(x)}{g(x)} \dot{x}^2 - a^2 \frac{f'(x) f(x)}{g^2(x)} + ab \frac{f'(x)}{g^2(x)} = 0 \tag{12}$$

which reduces to the quadratically damped Lienard type equation [23]

$$\ddot{x} + \frac{g'(x)}{g(x)} \dot{x}^2 = 0 \tag{13}$$

when $a = 0$. In this context the exact and general solution of the equation (13) is given by the differential form

$$\frac{g(x)}{b} dx = dt \tag{14}$$

where $b \neq 0$. The equation (13) can be identified to the Lienard type form (4) when $u(x) = \frac{g'(x)}{g(x)}$, and $\vartheta(x) = 0$, for simplicity reason. In this respect, comparing the differential forms (11) and (14) yields the sufficient conditions for the ordinary Lienard equation (1) and the quadratic Lienard type equation (4) to have identical exact and general solutions, as

$$\frac{g(x)}{b} = -\frac{1}{af(x)} \tag{15}$$

that is

$$f(x)g(x) = -\frac{b}{a} \tag{16}$$

where $a \neq 0,$ and $b \neq 0.$ In this context the equations (1) and (4) take respectively the definitive form

$$\ddot{x} + \frac{g'(x)}{g(x)^3} = 0 \tag{17}$$

and

$$\ddot{x} + \frac{g'(x)}{g(x)}\dot{x}^2 = 0 \tag{18}$$

where $g(x) \neq 0,$ is an arbitrary function of $x.$ The equation (17) represents a general class of Lienard equations while the equation (18) represents a general class of quadratic Lienard type equations. In this sense these equations are quite different but have the same solution given by the quadrature defined by

$$b(t+K) = \int g(x)dx \tag{19}$$

where K is a constant of integration. In this situation the following theorem has been shown.

Theorem

Let $\vartheta(x) = 0$. *If* $h(x) = b^2 \dfrac{g'(x)}{g^3(x)}$, *and* $u(x) = \dfrac{g'(x)}{g(x)}$, *then the equations (1) and (4) become respectively (17) and (18), and have identical exact solutions.*

Now some illustrative examples can be given in the following section.

3- Examples of equations

3.1 $g(x) = \left(\mu^2 - x^2\right)^{-1/2}$

Recently, in [20] the authors investigated the quadratically damped oscillator equation

$$\ddot{x} + \frac{x}{\mu^2 - x^2}\dot{x}^2 = 0 \tag{20}$$

and found that it has the harmonic and isochronous periodic solution

$$x(t) = \mu \sin\left[b\left(t + k_1\right)\right] \tag{21}$$

where $\mu > 0$, and k_1 are arbitrary parameters. The equation (20) is obtained by substituting $g(x) = \left(\mu^2 - x^2\right)^{-1/2}$ into the reduced quadratic Lienard type equation (18). Now, substituting $g(x) = \left(\mu^2 - x^2\right)^{-1/2}$ into (17) yields, as can be expected, the equation of the linear harmonic oscillator

$$\ddot{x} + b^2 x = 0 \tag{22}$$

of the well-known solution

$$x(t) = \mu \sin\left(bt + \gamma\right) \tag{23}$$

which is identical to the solution (21) where the amplitude is taken as μ, and $\gamma = bk_1$. Therefore the equations (20) and (22) have the same harmonic and isochronous periodic solution.

3.2 $g(x) = -\frac{b}{a}\frac{1}{\cos\left(\frac{x}{2}\right)}$

In [9] the pendulum equation

$$\ddot{x} + \frac{a^2}{4}\sin x = 0 \tag{24}$$

has been investigated by Adjaï and coworkers, to show for the first time that the pendulum equation can exhibit non-oscillatory behavior. The solution obtained in [9] is

$$x(t)=4\tan^{-1}\left[e^{-\frac{a}{2}(t+K_2)}\right]-\pi \tag{25}$$

where K_2 is an a constant of integration. The equation (24) can be obtained from (17) by putting $g(x)=-\dfrac{b}{a\cos\left(\frac{x}{2}\right)}$, where $\frac{x}{2}\neq\frac{\pi}{2}+n\pi$ and $a\neq 0$. In this situation the equation (18) takes the form

$$\ddot{x}+\frac{\dot{x}^2}{2}\tan\frac{x}{2}=0 \tag{26}$$

which has the formula (25) as solution.

3.3 $g(x)=e^{\frac{x^2}{2}}$

In this case the quadratically damped Lienard type equation (18) takes the form

$$\ddot{x}+x\dot{x}^2=0 \tag{27}$$

A vast literature exists [24-26] on the equation (27). Using the equation (19) one can write

$$\int e^{\frac{x^2}{2}}dx=b\left(t+K_3\right) \tag{28}$$

where K_3 is an integration constant. The evaluation of the integral in (28) leads to [27]

$$\frac{1}{2}\sqrt{2\pi}\,erfi\left(\frac{\sqrt{2}}{2}x\right)=b\left(t+K_3\right) \tag{29}$$

from which, knowing that [27] $erfi(z)=\dfrac{erf(iz)}{i}$, one can get the solution

$$x(t)=-i\sqrt{2}erf^{-1}\left\{ib\sqrt{\frac{2}{\pi}}\left(t+K_4\right)\right\} \tag{30}$$

As can be seen the solution (30) is in agreement with the result given in [25]. Now by application of $g(x)=e^{\frac{x^2}{2}}$, the nonlinear Lienard type equation (17) reduces to

$$\ddot{x}+b^2xe^{-x^2}=0 \tag{31}$$

which has , as expected, the formula (30) as solution. It is interesting to notice that the equation (27), in fact, the equation (5) is a special case of the quadratic Lienard type equation

$$\ddot{x}+\left(\ell\frac{p'(x)}{p(x)}-\gamma\,\varphi'(x)\right)\dot{x}^2+\frac{\beta^2\,e^{2\gamma\varphi(x)}\int p(x)^{\ell}\,dx}{p(x)^{\ell}}=0 \tag{32}$$

where ℓ and γ are arbitrary parameters, and $p(x)$ and $\varphi(x)$ are arbitrary functions of x introduced recently in the literature by Akande et al. [13]. It suffices to set $\ell=\beta=0$, $\gamma=-1$ and $\varphi(x)=\frac{1}{2}x^2$, to recover the equation (27) so that the solution (30) can be obtained by using the corresponding generalized Sundman transformation [13]. In this way the equation (27) is the nonlocal transformation of the second-order linear differential equation

$$y''(\tau)=0 \tag{33}$$

admitting the general solution

$$y(\tau)=c\tau \tag{34}$$

where c is an arbitrary constant. The solution of the equation (27) becomes

$$x(t)=c\tau \tag{35}$$

where τ is given by

$$t+K_4=\int e^{-\frac{c^2}{2}\tau^2}\,d\tau \tag{36}$$

and K_4 is a constant of integration. In this way one can secure

$$\tau=-i\frac{\sqrt{2}}{c}\,erf^{-1}\left\{ic\sqrt{\frac{2}{\pi}}\left(t+K_4\right)\right\} \tag{37}$$

from which one can get

$$x(t)=-i\sqrt{2}erf^{-1}\left\{ic\sqrt{\frac{2}{\pi}}\left(t+K_4\right)\right\} \tag{38}$$

The solution (30) and (38) are identical for $b=c$.

3.4 $g(x)=e^{\frac{x}{2}}$

In this case the Lienard equation (17) takes the form

$$\ddot{x}+\frac{b^2}{2}e^{-x}=0 \tag{39}$$

The equation (39) is a Bratu type equation [28, 29] and has been widely used in mathematics to test the efficiency and reliability of analytical approximate methods [28]. Using the equation (19), one may obtain

$$\int e^{\frac{x}{2}}\,dx=b\left(t+K_5\right) \tag{40}$$

which gives the solution

$$x(t)=2\ell n\left[\frac{b}{2}\left(t+K_5\right)\right] \tag{41}$$

where K_5 is a constant of integration. In this regard the equation (18) becomes

$$\ddot{x}+\frac{\dot{x}^2}{2}=0 \tag{42}$$

and admits the formula (41) as exact and general solution.

3.5 $g(x)=\dfrac{1}{x}$

Substituting $g(x)=\frac{1}{x}$, into the equation (17) leads to

$$\ddot{x}-b^2x=0 \tag{43}$$

with well-known solution

$$x(t)=e^{b\,(t+K_6)} \tag{44}$$

where K_6 is an arbitrary parameter. In this context the reduced quadratic Lienard type equation

$$\ddot{x}-\frac{\dot{x}^2}{x}=0 \tag{45}$$

has the expression (44) as exact and general solution. According to Mickens [30] the equation (45) is a truly nonlinear conservative oscillator but it cannot have periodic solutions.

Conclusion

We studied in this work the existence of identical solutions between the ordinary and quadratic Lienard type equations. The conditions of existence have been established and examples of equations are given to illustrate the theory. Thus, the global knowledge of one equation can be obtained by knowing the solution of the other.

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