Planck's spectrum as function of wavelength is untenable

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Abstract-Scrutinizing Planck's spectra as function of frequency and as function of wavelength learns that the last mentioned one leads to baffling results.

1 Introduction

Planck's book about this subject, originally written in 1913, has been translated to English as shown in [1]. He presents two types of spectra for the so-called black body radiation, one as function of frequency, the other one as function of wavelength. The second one turns out to be untenable.

2 The black body radiation spectrum as function of frequency and wavelength

Planck presented the following two spectra, supplemented with his commentary in Italics:

$$K_\nu = h\nu^3 c^{-2}/(\exp(h\nu/kT) - 1) \qquad W/m^2/Hz$$

"This is the specific intensity of a monochromatic plane polarized ray of the frequency ν which is emitted from a black body at the temperature T into vacuum in a direction perpendicular to the surface."

$$E_\lambda = (hc^2\lambda^{-5})/(\exp(hc/k\lambda T) - 1) \qquad W/m^2/m$$

"This is the specific intensity of a monochromatic ray not to the frequency ν but, as is usually done in experimental physics, to the wavelength λ…"

The spectrum E_λ is incorrect for the following 3 reasons:

1 the maximum of E_λ is not at the same frequency as of K_ν
2 K_ν and E_λ show an incomprehensible relationship
3 the dimension of E_λ is meaningless/unphysical

ad 1 The maximum of K_ν is found for $dK_\nu/d\nu = 3\nu^2 \cdot (e^{a\nu} - 1)^{-1} - \nu^3 \cdot (e^{a\nu} - 1)^{-2} \cdot e^{a\nu} \cdot a = 3 - a\nu/(1 - e^{-a\nu}) = 0$

Approximating $1 - e^{-a\nu}$	by	$a\nu - a^2\nu^2/2$	leads to $\nu = (4/3)\cdot a^{-1} = (4/3)kT/h$	Hz
Approximating $1 - e^{-a\nu}$	by	$a\nu - a^2\nu^2/2 + a^3\nu^3/6$	leads to $\nu = 4\cdot a^{-1} = 4kT/h$	Hz
Approximating $(e^{a\nu} - 1)^{-1}$	by	$e^{-a\nu}$ directly in K_ν	leads to $\nu = 3\cdot a^{-1} = 3kT/h$	Hz

The numerical calculation of K_ν shows that the latter approximation is accurately close to reality.
This approximation applied to E_λ and replacing $1/\lambda$ by y, leads to $E_y = hc^2 y^5 \cdot e^{-by}$, with $b = hc/kT$.
$dE_y/dy = 5y^4 \cdot e^{-by} + y^5 \cdot (-b) \cdot e^{-by} = 5 - y \cdot b = 0$, so the maximum of E_λ is found at $\nu = 5kT/h$.

ad 2 The cause of the deviation from $\nu = 3kT/h$ in K_ν is *only* the power 5 of $1/\lambda$ in E_λ.
Writing blindly hc^2/λ^3 instead of hc^2/λ^5 would lead to the dimension W/m instead of $W/m^2/m$ of E_λ.
The solution to this problem should be found in the introduction of a *constant* with dimension m^{-2}, instead of the introduction, as Planck did, of λ^{-2}. However such a constant does not exist. In order to show the mutual incomprehensible relationship between K_ν and E_λ their maximum values are compared.
Applying $\nu = 3kT/h$ in K_ν results in $K_{\nu max} = 9.5\cdot 10^{-20}\cdot T^3$ $W/m^2/Hz$
Applying $\nu = 5kT/h$ in E_λ results in $E_{\lambda max} = 2.0\cdot 10^{-6}\cdot T^5$ $W/m^2/m$
These results show the already mentioned weird relationship, as well as the fact that E_λ has to be rejected.

Ad3 The correct expression for E_λ is found when ν^3 in K_ν is replaced by c^3/λ^3 and E_λ is written as E_ν:

$$E_\nu = (hc/\lambda^3)/(\exp(hc/k\lambda T) - 1) \qquad W/m^2/Hz$$

The index ν is chosen to emphasize that the integration of this spectrum has to be done w.r.t. the frequency. In a numerical situation, where λ is taken as the primary variable, Δλ (being $\lambda_n - \lambda_{n-1}$) has to be replaced by Δν as $c/\lambda_{n-1} - c/\lambda_n$).

3 The most likely cause of the incorrect spectrum E_λ

This cause can be found by using the simplified shape, in order to show what happens with the integration of the spectrum as proposed by Planck. Replacing λ^{-1} by y results in the following equations:

$$\int E_\lambda\, d\lambda = hc^2 \int \lambda^{-5} e^{-hc/k\lambda T}\, d\lambda \quad \text{becomes} \quad \int E_y\, dy = hc^2 \int f(y)\cdot y^5 e^{-by}\, d(y^{-1}) \quad \text{with } b = hc/kT$$

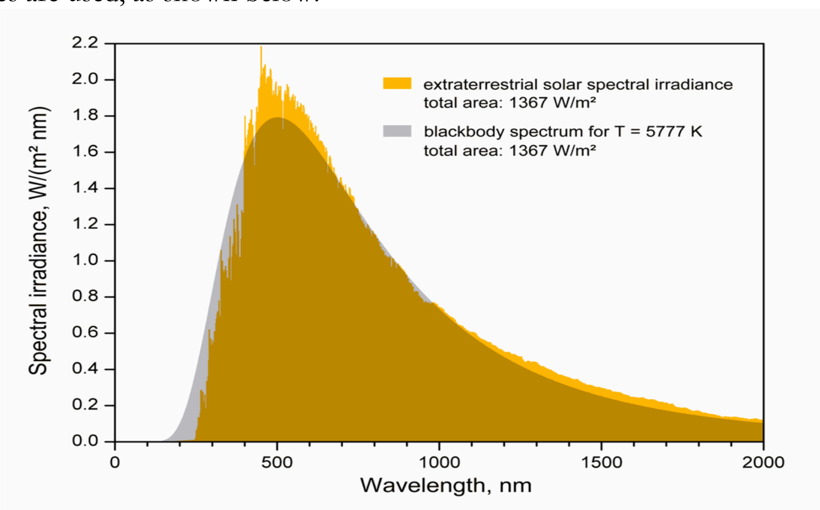
$$dy^{-1}/dy = -y^{-2}, \text{ so } dy^{-1} = -y^{-2}dy, \text{ so } f(y) = -y^{-2}, \text{ resulting in: } \int E_y\, dy = -hc^2 \int y^3 e^{-by}\, dy$$

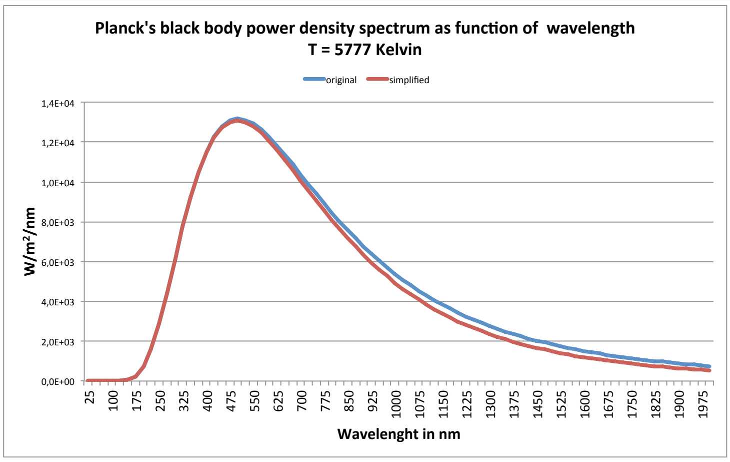
Three times in a row integrating by parts delivers $\int_0^\infty E_y\, dy = hc^2\cdot 6\cdot b^{-4} = 6\cdot h^{-3}c^{-2}k^4\cdot T^4$, being equal to the integration of the simplified spectrum of K_ν, implicitly as function of the frequency.

That implies that, notwithstanding the fact that E_λ as power density *spectrum* is fundamentally wrong, its power density is correct, when integrated w.r.t. the wavelength.

As shown in section 1 the maximum value of E_λ as well its position, expressed in either frequency or wavelength, is wrong. For example at T = 5777 K in the simplified spectra:
$E_{\lambda max} = 2.0\cdot 10^{-6}\cdot T^5 = 1.3\cdot 10^4$ W/m²/nm, at $\lambda \sim c/(5kT/h) \sim 500$ nm, at $\nu \sim 5kT/h \sim 600$ THz
$K_{\nu max} = 9.5\cdot 10^{-20}\cdot T^3 = 1.8\cdot 10^{-8}$ W/m²/Hz, at $\lambda \sim c/(3kT/h) \sim 900$ nm, at $\nu \sim 3kT/h \sim 360$ THz

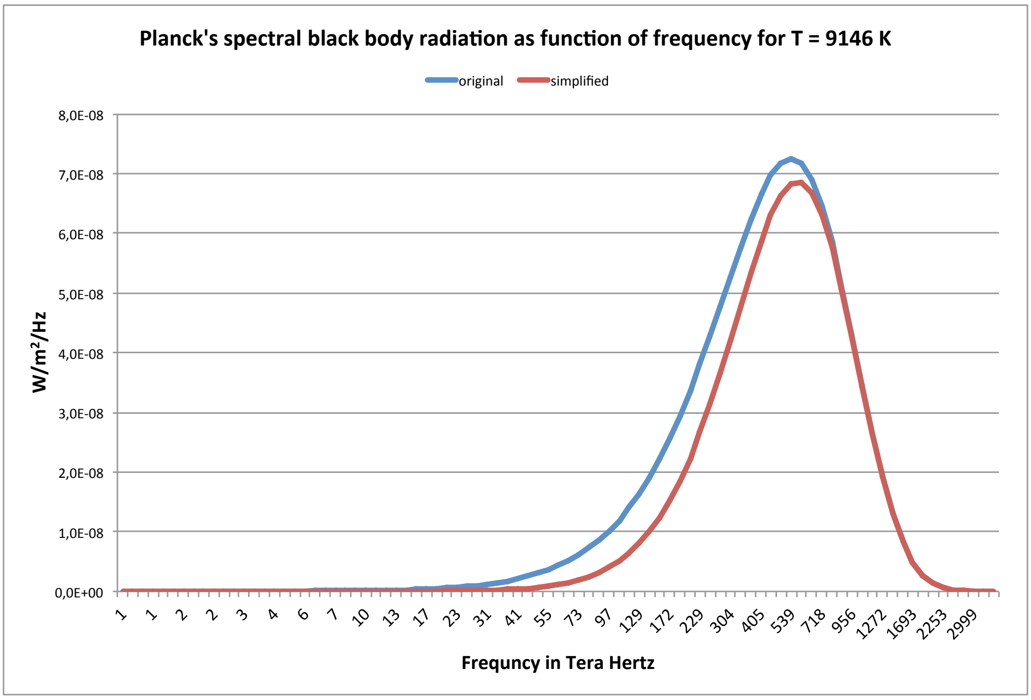
The ratio $1.8\cdot 10^{-8}$ Wm^{-2}Hz^{-1}/$1.3\cdot 10^4$ Wm^{-2}nm^{-1} = $1.4\cdot 10^{-12}$ nm/Hz doesn't show a meaningful outcome, due to the dimension m/Hz, and (as a result) as well as due to the numerical outcome.
But still the curves are used, as shown below.



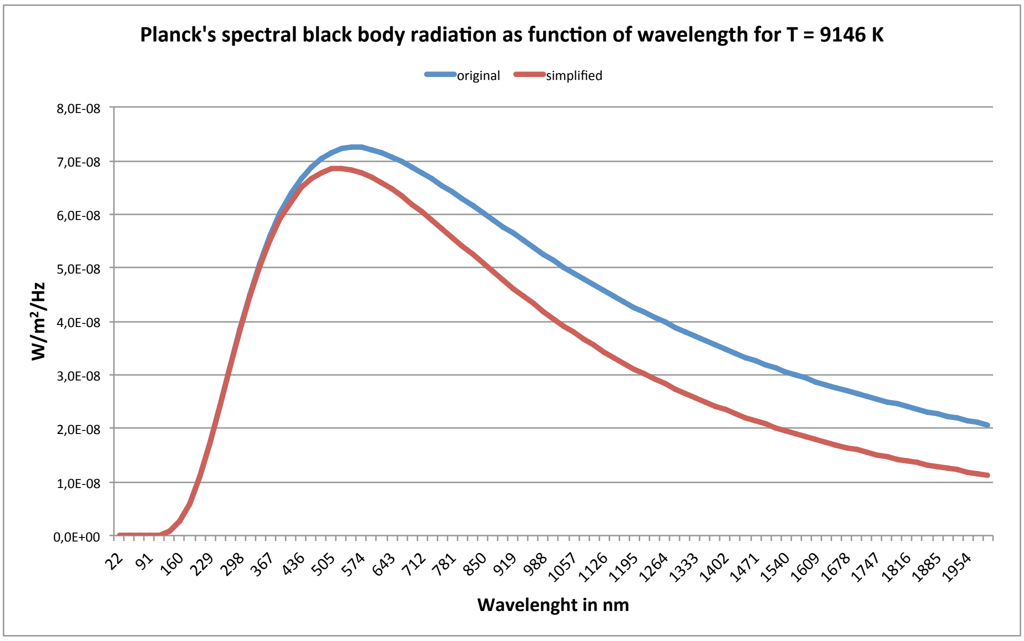


Mind the presented maximum value in the upper graph: 1.8, instead of $1.3\cdot 10^4$ W/m²/nm!
Is it coincidental that the number 1.8 shows up too in $K_{\nu max}$ as $1.8\cdot 10^{-8}$ Wm^{-2}/Hz?
See reference [2] for the interpretation of the extraterrestrial solar spectral radiation.

The surprising accuracy of the position of the maximum value in the simplified spectrum has been the motivation to draw the graphs of K_ν and E_ν for both Planck's original and the simplified spectra.



Graph of K_ν



Graph of E_ν

Conclusion

The originally by Planck proposed spectrum as function of wavelength has to be rejected and replaced by his spectrum as function of frequency in which the variable ν simply is replaced by c/λ.

References

[1] Planck M. The theory of heat radiation. P. Blakiston's Son & Co., Philadelphia, PA, 1914, free available at: http://www.gutenberg.org/zipcat2.php/40030/40030-pdf.pdf

[2] Stefan-Boltzmann constant incorrect by a factor 2π https://vixra.org/abs/1909.0647