Any finite connected poset is isomorphic to $\mathrm{Aut}(X)\backslash X$ for some finite poset X

Pierre-Yves Gaillard

We show that, given any finite connected poset X, there is a finite poset Y such that the quotient poset $\mathrm{Aut}(Y)\backslash Y$ is isomorphic to X.

Let X be a finite poset, G its automorphism group and X_* the set of all G-orbits in X. For $A, B \in X_*$ write $A \leq B$ if the inequality $a \leq b$ holds in X for some $a \in A, b \in B$. Equipped with this relation X_* is again a finite poset.

If X is a poset we write X^- for the subset of minimal elements, and X^+ for the subset of maximal elements. An element of $X^- \cap X^+$ is called an **isolated point**.

Theorem. *If X is a nonempty finite poset without isolated points, then there is a finite poset Y without isolated points such that $Y_* \simeq X$.*

To prove the theorem we will define a poset Y and show that it has the required properties.

Denote by $|S|$ the cardinality of any set S and put $[n] := \{1, 2, \ldots, n\}$ for any nonnegative integer n. Let X be a nonempty finite poset without isolated points. Set

$$n := |X|, \quad m := |X^-|, \quad X^- = \{x_1, \ldots, x_m\}.$$

Let $x_{m+1}, x_{m+2}, \ldots, x_n$ be an enumeration of the elements of $X \setminus X^-$ such that $x_i < x_j$ implies $i < j$.

We will first define integers $c_1 < \cdots < c_n < r$ and then define Y using these integers.

For $i \in [m]$ we set $c_i := i$.

• Definition of $c_{m+1} < c_{m+2} < \cdots < c_n$: Let Π be the set of all non-constant polynomial $P(t) \in \mathbb{Q}[t]$ with positive leading coefficient; here t is an indeterminate.

Set for $m < i \le n$

$$P_i(t) := \sum_{x_j < x_i} \binom{t}{j},$$

where the sum runs over the $j \in [m]$ such that $x_j < x_i$. Since $P_i \in \Pi$ for all i, there are integers $c_{m+1}, \dots, c_n \in m\mathbb{Z}$ such that $m < c_{m+1} < \cdots < c_n$ and $P_i(c_i) \ne P_j(c_j)$ whenever $m < i < j \le n$.

• Definition of r: For $i \in [m]$ denote by Π_i the set of all the polynomials in Π whose degree is congruent to $-i$ modulo m, and set

$$Q_i(t) := \sum_{x_j > x_i} \binom{t-i}{c_j - i},$$

where the sum runs over the $j \in [n]$ such that $x_j > x_i$. Since each Q_i is in Π_i, the Q_i are pairwise distinct. Hence there is an integer $r > c_n$ such that the $Q_i(r)$ are pairwise distinct.

• Definition of Y: For $i \in [n]$ set $Y_i := \{S \subset [r] \mid |S| = c_i\}$, and let Y be the union of the Y_i. For $S \in Y_i$, $T \in Y_j$ set

$$S < T \iff x_i < x_j \text{ and } S \subset T.$$

Then $\le$ is a partial order on Y. Note that for $S \in Y_i$ we have: S is minimal in Y if and only if x_i is minimal in X, and S is maximal in Y if and only if x_i is maximal in X. In particular Y is a nonempty finite poset without isolated points.

• Proof of the isomorphism $Y_* \simeq X$: Write $S \sim T$ for $S, T \in Y$ to indicate that there is an automorphism of Y which maps S to T.

There is a strictly increasing surjection $f : Y \to X$ mapping $S \in Y_i$ to x_i. We claim that f induces an isomorphism $Y_* \to X$. To prove this it suffices to show that for $S \in Y_i$ and $T \in Y_j$ we have $S \sim T$ if and only if $i = j$. If $i = j$, then, since S and T are two cardinality c_i subsets of $[r]$, there is a permutation σ of $[r]$ which moves S to T, and σ induces an automorphism of Y which maps S to T. It only remains to prove that $S \sim T$ implies $i = j$.

Recall that we have $S \in Y_i$, $T \in Y_j$, $S \sim T$ and we claim $i = j$.

Case 1: S is minimal. Then T is also minimal and we get $i, j \in [m]$. The number of elements of Y which are greater than S (respectively greater than T) is $Q_i(r)$ (respectively $Q_j(r)$). But the assumption $S \sim T$ implies $Q_i(r) = Q_j(r)$, and thus $i = j$.

Case 2: S is not minimal. Then T is also not minimal and we get $m < i, j \le n$. The number of minimal elements of Y which are less than S (respectively less than T) is $P_i(c_i)$ (respectively $P_j(c_j)$). But the assumption $S \sim T$ implies $P_i(c_i) = P_j(c_j)$, and thus $i = j$. This completes the proof of the theorem.