Octonionic Strings, Branes and Three Fermion Generations

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Abstract

Actions for strings and p-branes moving in octonionic-spacetime backgrounds and endowed with octonionic-valued metrics are constructed. An extensive study of the bosonic octonionic string moving in flat backgrounds, and its quantization, is presented. A thorough discussion follows pertaining whether or not the analysis leading to the $D = 26$ critical dimension of the ordinary bosonic string is valid in the octonionic case. A remarkable numerical coincidence is found (without invoking supersymmetry) in that the total number of (real) degrees of freedom of 3 fermion generations (involving massless Weyl fermions in $4D$) is $16 \times 4 \times 3 = 192$, and which matches the number of $8 \times 24 = 192$ real dimensions (degrees of freedom) corresponding to the 24 transverse octonionic dimensions associated with the octonionic-worldsheet of a bosonic octonionic-string moving in $D = 26$ octonionic dimensions.

Keywords: Division Algebras, Octonions, Branes, Strings, Standard Model.

1 Introduction : Octonions

Exceptional, Jordan, Division, Clifford and Noncommutative algebras are deeply related and essential tools in many aspects in Physics, see for instance [1], [2], [3], [5]. [7], [6], [8], [16], [13]. It is the belief of many authors that the octonions will ultimately be seen as the key to a unified field theory in physics [4], [21], [6], [23], [20].

A complexification of ordinary gravity (not to be confused with Hermitian-Kahler geometry) has been known for a long time. Complex gravity requires that $g_{\mu\nu} = g_{(\mu\nu)} + i g_{[\mu\nu]}$ so that now one has $g_{\nu\mu} = (g_{\mu\nu})^*$, which implies that the diagonal components of the metric $g_{z_1 z_1} = g_{z_2 z_2} = g_{\bar z_1 \bar z_1} = g_{\bar z_2 \bar z_2}$ must be real. A treatment of a non-Riemannan geometry based on a complex tangent space and involving a symmetric $g_{(\mu\nu)}$ plus antisymmetric $g_{[\mu\nu]}$ metric component was first proposed by Einstein-Strauss [9] (and later on by [11]) in their unified theory of Electromagentism with gravity by identifying the EM field strength $F_{\mu\nu}$ with the antisymmetric metric $g_{[\mu\nu]}$ component.

Borchsenius [10] formulated the quaternionic extension of Einstein-Strauss unified theory of gravitation with EM by incorporating appropriately the $SU(2)$ Yang-Mills field strength into the degrees of a freedom of a quaternionc-valued metric. Oliveira and Marques [12] later on provided the Octonionic Gravitational extension of Borchsenius theory involving two interacting $SU(2)$ Yang-Mills fields and where the exceptional group G_2 was realized naturally as the automorphism group of the octonions.

The Noncommutative and Nonassociative algebra of octonions discovered by Cayley and Graves is determined from the relations

$$e_o^2 \ = \ e_o, \ \ e_o \ e_m \ = \ e_m \ e_o \ = \ e_m, \ \ e_m \ e_n \ = \ - \ \delta_{mn} \ e_o \ + \ f_{mnp} \ e_p, \tag{1.1}$$

with $m, n, p = 1, 2, 3, \cdots, 7$ and where the fully antisymmetric structure constants f_{mnp} are taken to be 1 for the combinations $(123), (516), (624), (435), (471), (673), (672)$ corresponding to 7 quaternionic subalgebras of the octonions and which are associated to the 7 lines of the projective Moufang plane.

The octonionic conjugation operation, $e_m \rightarrow -e_m$, allows to define the quadratic form of a real octonion $\mathbf{X} = x_o e_o + \sum_{m=1}^{m=7} x_m e_m$ as

$$\bar{\mathbf{X}} \ \mathbf{X} \ = \ (x_o)^2 \ + \ \sum_{m=1}^{m=7} \ (x_m)^2 \tag{1.2}$$

Whereas, the quadratic form of a complex octonion is defined by

$$\mathbf{Re}(\bar{\mathbf{X}} \ \mathbf{X}) \ = \ (x_o + iy_o)^2 \ + \ \sum_{m=1}^{m=7} \ (x_m + iy_m)^2 \ \in \mathbf{C} \tag{1.3}$$

Note that the real part of a complex octonion $\mathbf{X}$ is $X_o = x_o + iy_o$ and must not be confused with the real parts of the complex entries defining the complex octonion.

Every nonzero real octonion has a unique inverse, namely

$$\mathbf{X}^{-1} \ = \ \frac{\bar{\mathbf{X}}}{\bar{\mathbf{X}} \ \mathbf{X}} \tag{1.4}$$

The non-vanishing associator is defined by

$$\{\mathbf{X}, \mathbf{Y}, \mathbf{Z}\} = (\mathbf{XY})\mathbf{Z} - \mathbf{X}(\mathbf{YZ}) \tag{1.5}$$

In particular, the associator of the imaginary units is

$$\{e_l, e_m, e_n\} \ = \ 2 \ d_{lmnp} \ e_p, \quad d_{lmnp} \ = \ \epsilon_{lmnprst} \ f_{rst} \tag{1.6}$$

The Hermitian product is defined in terms of the ordinary complex conjugate $*$ and the quadratic form (1.3) as

$$< \mathbf{X}, \mathbf{Y} > \ \equiv \ (\mathbf{X}^*, \ \mathbf{Y}) \ = \ (a_o - ib_o)\,(c_o + id_o) \ + \ \sum_{m=1}^{m=7} \ (a_m - ib_m)\,(c_m + id_m) \tag{1.7}$$

$$< \mathbf{X}, \mathbf{X} > \ \equiv \ (\mathbf{X}^*, \ \mathbf{X}) \ = \ (a_o - ib_o)\,(a_o + ib_o) + \sum_{m=1}^{m=7} \ (a_m - ib_m)\,(a_m + ib_m) \ =$$

$$a_o^2 \ + \ b_o^2 \ + \ \sum_{m=1}^{m=7} \ (a_m^2 \ + \ b_m^2) \tag{1.8}$$

The purpose of this work is to advance further the Octonionic Geometry (Gravity) of [12] by enlarging the ordinary spacetime coordinates to octonionic-valued coordinates furnishing a natural realization of a Noncommutative and Nonassociative spacetime. We construct next the actions for strings and p-branes in octonionic-spacetime backgrounds endowed with octonionic-valued metrics. An extensive study of the bosonic octonionic string moving in flat backgrounds, and its quantization, is presented.

A thorough discussion follows pertaining whether or not the analysis leading to the $D = 26$ critical dimension of the ordinary bosonic string is valid in the octonionic case. A remarkable numerical coincidence is found (without invoking supersymmetry) in that the total number of (real) degrees of freedom of 3 fermion generations (involving massless Weyl fermions in $4D$) is $16 \times 4 \times 3 = 192$, and which matches the number of $8 \times 24 = 192$ real dimensions (degrees of freedom) corresponding to the 24 transverse octonionic dimensions associated with the octonionic-worldsheet of a bosonic octonionic-string moving in $D = 26$ octonionic dimensions.

2 Octonionic p-branes

2.1 Branes in Octonionic Spacetime Backgrounds

Next we shall construct actions for p-branes moving in octonionic spacetime backgrounds $\mathbf{Z}^\mu(\sigma^a) = Z_o^\mu(\sigma^a)e_o + Z_i^\mu(\sigma^a)e_i; a = 0, 1, 2, \cdots, p$, and endowed with octonionic-valued metrics $\mathbf{g}_{\mu\nu}$. Given an spacetime interval defined as

$$(ds)^2 \ = \ \mathbf{Re} \ (\ d\mathbf{Z}^\mu \ \mathbf{g}_{\mu\nu} \ d\mathbf{Z}^\nu \ ) \tag{2.1}$$

the *real* part of the pullback of the spacetime metric onto the $p+1$-dim world-volume yields the embedding metric $h_{ab} = \mathbf{Re}(\partial_a \mathbf{Z}^\mu \mathbf{g}_{\mu\nu} \partial_b \mathbf{Z}^\nu)$. The *real* part of a triple octonionic product (2.1) is unambiguously defined despite the nonassociativity. It is the key relation

$$\mathbf{Re}(\ (\mathbf{xy})\ \mathbf{z}\ )\ = \mathbf{Re}(\ \mathbf{x}\ (\mathbf{yz})\ )\ =\ \mathbf{Re}(\ \mathbf{xyz}\ )\ =$$

$$x_o\ y_o\ z_o\ -\ x_o\ y_m\ z_m\ -\ x_m\ y_o\ z_m\ -\ x_m\ y_m\ z_o\ -\ x_l\ y_m\ z_n\ f_{lmn} \tag{2.2}$$

where the Einstein summation convention of repeated indices is implied, and which allows us to uniquely, and unambiguously, evaluate the real part of the triple product $\mathbf{Re}\ (\partial_a \mathbf{Z}^\mu\ \mathbf{g}_{\mu\nu}\ \partial_b \mathbf{Z}^\nu\ )$ despite the *nonassociativity* of the octonions. The real parts of a *quartic*, and higher products, are *not*. In the most general case, the octonionic metric $\mathbf{g}_{\mu\nu}$ does *not* need to be Hermitian; i.e. it does not need to have the form $\mathbf{g}_{\mu\nu} = g^o_{(\mu\nu)} e_o + g^i_{[\mu\nu]} e_i$. The reason being that by taking the real part of the triple products in eq-(2.1) one ensures that $(ds)^2$ is real-valued.

If the octonionic-valued metric $\mathbf{g}_{\mu\nu}$ is chosen to be Hermitian $(\mathbf{g}_{\mu\nu})^\dagger = \mathbf{g}_{\mu\nu}$, and $\bar{\mathbf{g}}_{\mu\nu} = \mathbf{g}_{\bar{\mu}\bar{\nu}}$, after a careful inspection, one arrives at the following relations

$$\mathbf{g}_{\mu\nu}\ =\ g^o_{(\mu\nu)} e_o\ +\ g^i_{[\mu\nu]} e_i$$
$$g^o_{\mu\nu} = g^o_{\nu\mu} = g^o_{\bar{\mu}\bar{\nu}} = g^o_{\bar{\nu}\bar{\mu}}$$
$$g^i_{\mu\nu} = -g^i_{\nu\mu} = -g^i_{\bar{\mu}\bar{\nu}} = g^i_{\bar{\nu}\bar{\mu}} \tag{2.3}$$

Due to these relations among the components of $\mathbf{g}_{\mu\nu}$ and $\mathbf{g}_{\bar{\mu}\bar{\nu}}$, it is not necessary to include the terms $d\mathbf{Z}^{\bar{\mu}}\ \mathbf{g}_{\bar{\mu}\bar{\nu}}\ d\mathbf{Z}^{\bar{\nu}}$ in eq-(2.1).

By the same token, one may also include an interval of the form

$$(ds)^2\ =\ \mathbf{Re}\ (\ d\mathbf{Z}^\mu\ \mathbf{g}_{\mu\bar{\nu}}\ d\mathbf{Z}^{\bar{\nu}}\ ) \tag{2.4}$$

If the octonionic-valued metric is chosen to be Hermitian : $\mathbf{g}_{\mu\bar{\nu}} = g^o_{(\mu\bar{\nu})} e_o + g^i_{[\mu\bar{\nu}]} e_i$, and $\bar{\mathbf{g}}_{\mu\bar{\nu}} = \mathbf{g}_{\bar{\mu}\nu}$, after a careful inspection it leads to the following Hermiticity conditions

$$g^o_{\mu\bar{\nu}}\ =\ g^o_{\bar{\nu}\mu}\ =\ g^o_{\bar{\mu}\nu}\ =\ g^o_{\nu\bar{\mu}}$$
$$g^i_{\mu\bar{\nu}}\ =\ -\ g^i_{\bar{\nu}\mu}\ =\ -\ g^i_{\bar{\mu}\nu}\ =\ g^i_{\nu\bar{\mu}}$$

Once again, due to these relations among the components of $\mathbf{g}_{\mu\bar{\nu}}$ and $\mathbf{g}_{\bar{\mu}\nu}$, it is not necessary to include the terms $d\mathbf{Z}^{\bar{\mu}}\ \mathbf{g}_{\bar{\mu}\nu}\ d\mathbf{Z}^\nu$ in eq-(2.4). In both cases the real components of the metric is symmetric in its indices, while the imaginary components are antisymmetric.

To sum up, when $\mathbf{g}_{\mu\nu}$ and the spacetime coordinates $\mathbf{Z}^\mu = Z^\mu_o e_o + Z^\mu_i e_i$ are both octonionic-valued , one can construct a more general p-brane action of the form

$$S_{DNG}\ =\ -\ T_p \int d^{p+1}\sigma\ \sqrt{|det\ h_{ab}|}\ =$$

$$-\ T_p \int d^{p+1}\sigma\ \sqrt{|det\ \mathbf{Re}\ (\ \partial_a \mathbf{Z}^\mu\ \mathbf{g}_{\mu\nu}\ \partial_b \mathbf{Z}^\nu\ )|} \tag{2.5}$$

where T_p is the p-brane tension of physical dimension $(mass)^{p+1}$, and the span of the p-brane indices are $a, b = 0, 1, \cdots, p$.

Thus, the real part of the pullback of the octonionic target space Hermitian metric $\mathbf{g}_{\mu\nu}$ is explicitly given by

$$h_{ab} = \partial_a Z_o^\mu \; g_{\mu\nu}^o \; \partial_b Z_o^\nu - \partial_a Z_o^\mu \; g_{\mu\nu}^i \; \partial_b Z_i^\nu - \partial_a Z_i^\mu \; g_{\mu\nu}^o \; \partial_b Z_i^\nu$$
$$- \partial_a Z_i^\mu \; g_{\mu\nu}^i \; \partial_b Z_o^\nu - f_{ijk} \; \partial_a Z_i^\mu \; g_{\mu\nu}^j \; \partial_b Z_k^\nu \quad (2.6)$$

with $i, j = 1, 2, \cdots, 7$, and repeated indices are summed over. The determinant of the above expression for h_{ab} is very complicated since h_{ab} is comprised of the sum of many different terms. Inserting this complicated expression for the $det(h_{ab})$ into eq-(2.4) furnishes the DNG action for a p-brane moving in an octonionic spacetime background and endowed with an octonionic-valued Hermitian metric. A similar action can be constructed based on the metric $\mathbf{g}_{\mu\bar{\nu}}$

$$S'_{DNG} = - T_p \int d^{p+1}\sigma \sqrt{|det \; \mathbf{Re} \; (\; \partial_a \mathbf{Z}^\mu \; \mathbf{g}_{\mu\bar{\nu}} \; \partial_b \mathbf{Z}^{\bar{\nu}} \;)|}, \quad a, b = 0, 1, \cdots, p \quad (2.7)$$

And in the most general case, one can combine both metrics $\mathbf{g}_{\mu\nu}, \mathbf{g}_{\mu\bar{\nu}}$ into the more general action

$$S''_{DNG} = - T_p \int d^{p+1}\sigma \sqrt{|det \; \mathbf{Re} \; (\; \partial_a \mathbf{Z}^\mu \; \mathbf{g}_{\mu\nu} \; \partial_b \mathbf{Z}^\nu + \partial_a \mathbf{Z}^\mu \; \mathbf{g}_{\mu\bar{\nu}} \; \partial_b \mathbf{Z}^{\bar{\nu}} \;)|} \quad (2.8)$$

When the metric is real ($\mathbf{g}_{\mu\nu} \rightarrow g_{\mu\nu}$), and the spacetime coordinates are real ($\mathbf{Z}^\mu \rightarrow X^\mu$) one recovers for the determinant of h_{ab} the usual expression given by the sums of the squares of Nambu-Poisson-brackets

$$h_{ab} = \partial_a X^\mu \; g_{\mu\nu} \; \partial_b X^\nu \;\; \Rightarrow$$

$$det(h_{ab}) = \{X^{\mu_1}, X^{\mu_2}, \cdots, X^{\mu_{p+1}}\} \, \{X^{\nu_1}, X^{\nu_2}, \cdots, X^{\nu_{p+1}}\} \, g_{\mu_1\nu_1} \, g_{\mu_2\nu_2} \, \cdots \, g_{\mu_{p+1}\nu_{p+1}} \quad (2.9a)$$

where the Nambu-Poisson brackets are defined as

$$\{X^{\mu_1}, X^{\mu_2}, \cdots, X^{\mu_{p+1}}\} \equiv \epsilon^{a_1 a_2 \cdots a_{p+1}} \; \partial_{a_1} X^{\mu_1} \; \partial_{a_2} X^{\mu_2} \; \cdots \; \partial_{a_{p+1}} X^{\mu_{p+1}} \quad (2.9b)$$

In general, in a curved background one has $g_{\mu\nu} = g_{\mu\nu}(X^\rho)$. Because the embedding spacetime coordinates $X^\rho(\sigma^1, \sigma^2, \cdots, \sigma^{p+1})$ are functions of the p-brane's $p+1$-dimensional world-volume coordinates, one cannot pull the metric factors inside the Nambu-Poisson brackets in eq-(2.9). Only when the background metric is independent of the X^ρ coordinates that one can pull the metric factors inside the brackets leading to

$$det(h_{ab}) = \{X^{\mu_1}, X^{\mu_2}, \cdots X^{\mu_{p+1}}\} \; \{X_{\mu_1}, X_{\mu_2}, \cdots, X_{\mu_{p+1}}\} \quad (2.10)$$

and the DNG action becomes

$$S_{DNG} \;=\; -\; T_p \int d^{p+1}\sigma \;\sqrt{|det\;(\;\partial_a X^\mu\;\partial_b X^\nu\;\eta_{\mu\nu}\;)|}\;=$$

$$-\;T_p \int d^{p+1}\sigma\;\sqrt{(\;\{X^{\mu_1}, X^{\mu_2}, \cdots, X^{\mu_{p+1}}\}_{NPB}\;)^2} \qquad (2.11)$$

A Polyakov-Howe-Tucker octonionic p-brane action S_p based on the metric $\mathbf{g}_{\mu\nu}$ is of the form

$$S_p \;=\; -\;\frac{T_p}{2}\int d^{p+1}\sigma\;\sqrt{|det(h_{ab})|}\;h^{ab}\;\mathbf{Re}\;(\;\partial_a\mathbf{Z}^\mu\;\mathbf{g}_{\mu\nu}\;\partial_b\mathbf{Z}^\nu\;)\;+$$

$$T_p\;\frac{(p-1)}{2}\int d^{p+1}\sigma\;\sqrt{|det(h_{ab})|} \qquad (2.12)$$

where $a, b = 0, 1, \cdots, p$ and h_{ab} is an auxiliary real-valued world-volume metric. Eliminating h_{ab} via its equations of motion and inserting it back into the action (2.12) yields the DNG action (2.4). A similar action can be constructed based on the metric $\mathbf{g}_{\mu\bar{\nu}}$

$$S'_p \;=\; -\;\frac{T_p}{2}\int d^{p+1}\sigma\;\sqrt{|det(h_{ab})|}\;h^{ab}\;\mathbf{Re}\;(\;\partial_a\mathbf{Z}^\mu\;\mathbf{g}_{\mu\bar{\nu}}\;\partial_b\mathbf{Z}^{\bar{\nu}}\;)\;+$$

$$T_p\;\frac{(p-1)}{2}\int d^{p+1}\sigma\;\sqrt{|det(h_{ab})|} \qquad (2.13)$$

And a more general action combining both metrics $\mathbf{g}_{\mu\nu}; \mathbf{g}_{\mu\bar{\nu}}$ is of the form

$$S''_p \;=\; -\;\frac{T_p}{2}\int d^{p+1}\sigma\;\sqrt{|det(h_{ab})|}\;h^{ab}\;\mathbf{Re}\;(\;\partial_a\mathbf{Z}^\mu\;\mathbf{g}_{\mu\nu}\;\partial_b\mathbf{Z}^\nu + \partial_a\mathbf{Z}^\mu\;\mathbf{g}_{\mu\bar{\nu}}\;\partial_b\mathbf{Z}^{\bar{\nu}}\;)\;+$$

$$T_p\;\frac{(p-1)}{2}\int d^{p+1}\sigma\;\sqrt{|det(h_{ab})|} \qquad (2.14)$$

All the actions described in this subsection are invariant under diffeomorphisms $\sigma^a \rightarrow \sigma'^a(\sigma^b)$ of the $p+1$-dim world volume swept by the p-branes.

2.2 The Bosonic Octonionic String in Flat Backgrounds

For simplicity, let us study the action (2.12) for a bosonic string ($p = 1$) moving in a flat target octonionic spacetime bacground whose octonionic-valued coordinates are $\mathbf{Z}^\mu, \mu = 1, 2, \cdots, D$. When the spacetime background metric is real-valued and flat one has for metric $\mathbf{g}_{\mu\nu} = g^o_{\mu\nu} e_o = \eta_{\mu\nu} e_o$. We shall study two cases based on two choices for the metric $\eta_{\mu\nu}$. One choice is $\eta_{\mu\nu} = diag(-1, +1, +1, \cdots, +1)$ corresponding to the usual Lorentzian spacetime metric, and the second choice for $\eta_{\mu\nu}$ can have an arbitrary signature.

Given $\mathbf{g}_{\mu\nu} = g^o_{\mu\nu} e_o = \eta_{\mu\nu} e_o$, the interval $(ds)^2$ (2.1) becomes, after expanding into the real and imaginary components of $\mathbf{Z}^\mu$, the following

$$(ds)^2 \ = \ \eta_{\mu\nu} \ (\ dZ^\mu_o \ dZ^\nu_o \ - \ dZ^\mu_i \ dZ^\nu_i \ ), \quad i = 1,2,\cdots,7; \quad \mu,\nu = 1,2,\cdots,D$$

and such that the corresponding equations of motion for $\mathbf{Z}^\mu(\tau,\sigma)$, obtained from the action (2.20) in the conformal gauge $h_{ab} = e^\phi \eta_{ab}(a,b = 0,1)$, are given by

$$\frac{\partial^2 \mathbf{Z}^\mu}{\partial \tau^2} \ - \ \frac{\partial^2 \mathbf{Z}^\mu}{\partial \sigma^2} \ = \ 0 \tag{2.15}$$

Following a similar procedure, as in the standard string theory text books [24], for a closed string, the left/right moving sector solutions to the equations of motion (2.15) admit the following mode expansions

$$\mathbf{Z}^\mu_{left}(\tau,\sigma) \ = \ \frac{1}{2}\mathbf{z}^\mu \ + \ \frac{1}{2} \ l^2_s \ \mathbf{p}^\mu(\tau - \sigma) \ + i \ \frac{l_s}{2} \sum_{n=-\infty, n\neq 0}^{n=\infty} \frac{1}{n} \ \mathbf{a}^\mu_n \ e^{-2ni(\tau-\sigma)} \tag{2.16}$$

$$\mathbf{Z}^\mu_{right}(\tau,\sigma) \ = \ \frac{1}{2}\mathbf{z}^\mu \ + \ \frac{1}{2} \ l^2_s \ \mathbf{p}^\mu(\tau + \sigma) \ + i \ \frac{l_s}{2} \sum_{n=-\infty, n\neq 0}^{n=\infty} \frac{1}{n} \ \tilde{\mathbf{a}}^\mu_n \ e^{-2ni(\tau+\sigma)} \tag{2.17}$$

where l_s is the string length, and the first two terms are the *zero* modes (center of mass coordinates) of the string. The general solution $\mathbf{Z}^\mu = \mathbf{Z}^\mu_{left} + \mathbf{Z}^\mu_{right}$ obeys the boundary conditions $\mathbf{Z}^\mu(\tau, \sigma + \pi) = \mathbf{Z}^\mu(\tau,\sigma)$.

To ensure that the coordinates $\mathbf{Z}^\mu$ are real octonionic-valued, the *complex*-octonionic-valued *coefficients*, which will turn into the creation, annihilation operators $\hat{\mathbf{a}}^\mu_{-n}, \hat{\mathbf{a}}^\mu_n$, respectively, upon quantization, must obey the following conditions

$$\mathbf{a}^\mu_{-n} \ = \ (\mathbf{a}^\mu_n)^*, \ \ \tilde{\mathbf{a}}^\mu_{-n} \ = \ (\tilde{\mathbf{a}}^\mu_n)^*, \ \ n > 0 \tag{2.18}$$

involving the ordinary complex conjugation $i \rightarrow -i$, and which must *not* be confused with the *octonionic* conjugation $e_i \rightarrow -e_i$. A *complex*-octonion is described by $\mathbf{z} = z_o e_o + z_i e_i; i = 1,2,\cdots,7$ and where $z_o = x_o + iy_o, z_i = x_i + iy_i$ are complex-valued entries such that $\mathbf{z} + \mathbf{z}^* = 2x_o e_o + 2x_i e_i$ is a real octonion. The center of mass coordinate and momentum $\mathbf{z}^\mu, \mathbf{p}^\mu$ are real octonionic-valued so that the zero modes obey the relation $\mathbf{a}^\mu_0 = \tilde{\mathbf{a}}^\mu_0 = \frac{1}{2} l^2_s \mathbf{p}^\mu$.

The complex-conjugate conditions (2.18) imposed on the coefficients turns, upon quantization, into the following conditions on the operators

$$\hat{\mathbf{a}}^\mu_{-n} \ = \ (\hat{\mathbf{a}}^\mu_n)^\dagger, \ \ \tilde{\mathbf{a}}^\mu_{-n} \ = \ (\tilde{\mathbf{a}}^\mu_n)^\dagger, \ \ n > 0 \tag{2.19}$$

For an open string obeying the Neumann boundary conditions $\partial_\sigma \mathbf{Z}^\mu = 0$ at $\sigma = 0, \pi$, the mode expansion is

$$\mathbf{Z}^\mu(\tau,\sigma) \ = \ \mathbf{z}^\mu \ + \ l_s^2 \ \mathbf{p}^\mu \tau \ + i \ l_s \sum_{n=-\infty, n\neq 0}^{n=\infty} \frac{1}{n} \ \mathbf{a}_n^\mu \ e^{-ni\tau} \ cos(n\sigma) \tag{2.20}$$

where the zero mode in this case obeys $\mathbf{a}_0^\mu = l_s \mathbf{p}^\mu$. For Dirichlet boundary conditions, $\mathbf{Z}^\mu = 0$ at the end-points $\sigma = 0, \pi$, one may set $\mathbf{z}^\mu = \mathbf{p}^\mu = 0$ and replace the cosine for a sine in eq-(2.20).

There are *no* matrix *representations* of the octonions due to the non-associativity, however Dixon has shown how many Lie algebras can be obtained from the left/right action of the octonion algebra on itself [6]. By an algebra acting on itself, one does not mean that quantum mechanical operators and states are expressed in the same algebra. An example of an algebra action on itself is the Clifford algebra. The left/right action of the octonion algebra on itself are isomorphic to the matrix algebra $R(8)$ of 8×8 real matrices.

For instance, from the structure constants of the octonion algebra one can associate to the $left$ action of e_i on e_o and e_j

$$e_{Li} \ [e_o] \ = \ e_i \ e_o \ = \ e_i, \ \ e_{Li} \ [e_j] \ = \ e_i \ e_j \ = \ f_{ijk} \ e_k \tag{2.21}$$

the following 8×8 *antihermitian* matrix $\mathbf{M}_i^L : e_{Li} \leftrightarrow \mathbf{M}_i^L$, and whose entries are given by

$$(M_i^L)_{jk} = f_{ijk}, \ i,j,k = 1,2,\cdots,7; \ (M_i^L)_{00} = 0, \ (M_i^L)_{0k} = \delta_{ik}, \ (M_i^L)_{k0} \ = \ -\delta_{ik} \tag{2.22}$$

The matrix representation of the left action of e_o is given by the unit 8×8 matrix. It is important to emphasize that because the associative matrix product $\times$ is *not* the same as the non-associative octonionic product $\cdot$, one has that $M^L(e_i) \times M^L(e_j) \neq M^L(e_i \cdot e_j) = f_{ijk} M^L(e_k)$.

Equipped with the above 8×8 matrix realization of the $left$ action of the algebra of octonions on itself, the complex-octonionic-valued coefficients $\mathbf{a}_n^\mu \equiv a_{n,o}^\mu e_o + a_{n,i}^\mu e_i$, and $\tilde{\mathbf{a}}_n^\nu \equiv \tilde{a}_{n,o}^\mu e_o + \tilde{a}_{n,i}^\mu e_i$ associated with the octonionic *closed* string can now be represented explicitly in terms of 8×8 matrices. Upon quantization, the complex-octonionic valued coefficients turn into the raising/lowering operators in the first quantization (creation/annihilation operators) denoted by hats as shown next.

Firstly, after a careful analysis based on the Heisenberg algebra $[\mathbf{P}^\mu(\tau,\sigma), \mathbf{Z}^\nu(\tau,\sigma')] = -i\eta^{\mu\nu}\delta(\sigma - \sigma')e_o$, $(\hbar = c = 1)$ where $\mathbf{P}^\mu = T\dot{\mathbf{Z}}^\mu$ is the octonionic canonical momentum conjugate to $\mathbf{Z}^\mu$, and T is the string tension $T = \frac{1}{\pi l_s^2}$, one learns that the canonical commutation relations among the real and 7 imaginary *components* of the complex-octonionic oscillators are given by

$$[\hat{a}_{m,o}^\mu, \ \hat{a}_{n,o}^\nu] \ = \ \frac{m}{8} \ \delta_{m+n,0} \ \eta^{\mu\nu}, \ \ [\hat{\tilde{a}}_{m,o}^\mu, \ \hat{\tilde{a}}_{n,o}^\nu] \ = \ \frac{m}{8} \ \delta_{m+n,0} \ \eta^{\mu\nu} \tag{2.23}$$

$$[\hat{a}_{m,i}^\mu, \ \hat{a}_{n,j}^\nu] \ = \ - \ \frac{m}{8} \ \delta_{m+n,0} \ \eta^{\mu\nu} \ \delta_{ij}, \ \ [\hat{\tilde{a}}_{m,i}^\mu, \ \hat{\tilde{a}}_{n,j}^\nu] \ = \ - \ \frac{m}{8} \ \delta_{m+n,0} \ \eta^{\mu\nu} \ \delta_{ij} \tag{2.24}$$

$$[\hat{a}^\mu_{m,o},\ \hat{a}^\nu_{n,j}] \ = \ 0, \quad [\hat{\tilde{a}}^\mu_{m,o},\ \hat{\tilde{a}}^\nu_{n,j}] \ = \ 0 \qquad (2.25)$$

Roughly speaking, one may interpret the oscillator-operator-components $\hat{a}^\mu_{m,o}, \hat{a}^\mu_{m,i}, \cdots$ as a "coloring" process of the ordinary oscillators $\hat{a}^\mu_m, \hat{\tilde{a}}^\mu_m$ by attaching $1+7=8$ internal $SU(3)$ color indices to each oscillator.

From the above commutators of the real and 7 imaginary *components* of the oscillators (2.23-2.25), after a very laborious algebra, one obtains the canonical commutation relations of the complex-octonionic oscillators

$$[\hat{\mathbf{a}}^\mu_m,\ \hat{\mathbf{a}}^\nu_n] \ = \ m\ \delta_{m+n,0}\ \eta^{\mu\nu}\ \mathbf{1}_{8\times 8}, \quad [\hat{\tilde{\mathbf{a}}}^\mu_m,\ \hat{\tilde{\mathbf{a}}}^\nu_n] \ = \ m\ \delta_{m+n,0}\ \eta^{\mu\nu}\ \mathbf{1}_{8\times 8}$$

$$[\hat{\mathbf{a}}^\mu_m,\ \hat{\tilde{\mathbf{a}}}^\nu_n] \ = \ 0 \qquad (2.26a)$$

where $\mathbf{1}_{8\times 8}$ is the unit 8×8 matrix and which corresponds to the identity element e_o of the octonion algebra. The normalization factors of $\frac{1}{8}$ in (2.23, 2.24) result from the 8-dimensional octonion algebra and are introduced so that factors of 8 do not appear in eqs-(2.26a). The 8×8 matrix realization of the *right* action of the algebra of octonions on itself leads to the same commutation relations.

To sum up,

$$[\mathbf{P}^\mu(\tau,\sigma), \mathbf{Z}^\nu(\tau,\sigma')] \ = \ -i\eta^{\mu\nu}\ \delta(\sigma-\sigma')\ e_o, \quad \hbar = c = 1 \ \Leftrightarrow$$

$$[\hat{\mathbf{a}}^\mu_m,\ \hat{\mathbf{a}}^\nu_n] \ = \ m\ \delta_{m+n,0}\ \eta^{\mu\nu}\ e_o, \quad [\hat{\tilde{\mathbf{a}}}^\mu_m,\ \hat{\tilde{\mathbf{a}}}^\nu_n] \ = \ m\ \delta_{m+n,0}\ \eta^{\mu\nu}\ e_o \qquad (2.26b)$$

after one recurs to the Fourier decomposition of

$$\delta(\sigma-\sigma') \ = \ \frac{1}{\pi} \sum_{n=-\infty}^{\infty} e^{-2in(\sigma-\sigma')} \qquad (2.26c)$$

In the case of the open string, the generalized Virasoro operators are given by the following sum of the normal ordered products of the 8×8 matrix operators

$$\mathbf{L}_m \ = \ \sum_{n=-\infty}^{n=\infty} \ :\ \hat{\mathbf{a}}^\mu_{m-n}\ \hat{\mathbf{a}}^\nu_n\ \eta_{\mu\nu}\ : \qquad (2.27)$$

As usual, the normal ordering prescription assigns the lowering operators to the right of the raising operators. Due to a normal ordering ambiguity in the definition of $\mathbf{L}_0$ one must include a constant c in the definition of $\mathbf{L}_0$ as follows

$$\mathbf{L}_0 \ = \ \frac{1}{2}\ \mathbf{a}^\mu_0\ \mathbf{a}^\nu_0 \ + \ \sum_{n>0}^{n=\infty} \ \mathbf{a}^\mu_{-n}\ \mathbf{a}^\nu_n\ \eta_{\mu\nu} \ + \ c \qquad (2.28)$$

The closed string involves the addition of the tilde oscillators so the Virasoro operators are $\mathbf{L}_m, \tilde{\mathbf{L}}_m$.

In ordinary QM, the commutator

$$[\mathbf{AB},\mathbf{CD}] \ = \ \mathbf{ABCD}-\mathbf{CDAB} \ = \ \mathbf{A}[\mathbf{B},\mathbf{C}]\mathbf{D}+\mathbf{C}[\mathbf{A},\mathbf{D}]\mathbf{B}+[\mathbf{A},\mathbf{C}]\mathbf{BD}+\mathbf{CA}[\mathbf{B},\mathbf{D}] \tag{2.29}$$

is well defined. However, when the operators are octonionic-valued, the quartic products in the middle terms are not well defined due to the non-associativity. To resolve this ambiguity we choose a nesting operation of the form $\mathbf{A}(\mathbf{B}(\mathbf{CD}))-\mathbf{C}(\mathbf{D}(\mathbf{AB}))$ in order to evaluate the quartic products. In doing so, given the definitions (2.27,2.28) of the Virasoro operators as bilinears in the oscillators, and the commutators (2.23-2.26), the generalized Virasoro algebra becomes

$$[\mathbf{L}_m,\mathbf{L}_n] \ = \ (m-n)\ \mathbf{L}_{m+n} \ + \ \frac{c}{12}\ m(m^2-1)\ \delta_{m+n,0}\ e_o, \quad e_o \leftrightarrow \mathbf{1}_{8\times 8} \tag{2.30}$$

where the central charge is $c=D$. For the specific details of the very laborious calculation of the Virasoro algebra commutators in the ordinary string see [25]. A different nesting operation like $((\mathbf{AB})\mathbf{C})\mathbf{D}-((\mathbf{CD})\mathbf{A})\mathbf{B}$ requires the 8×8 matrix representation of the *right* action of the the octonion algebra on itself. It leads to the same Virasoro commutators.

The action of the raising/lowering operators on the ground state requires also a *nesting* procedure due to the nonassociativity of the octonions. For example, in the open string, one may have the state

$$|\Psi\rangle \ = \ \mathbf{a}^{\mu_4\dagger}_{n_4}\ (\ \mathbf{a}^{\mu_3\dagger}_{n_3}\ (\ \mathbf{a}^{\mu_2\dagger}_{n_2}\ (\ \mathbf{a}^{\mu_1\dagger}_{n_1}\ |0\rangle\ )\ )\ ), \quad \{n_1,n_2,n_3,n_4\}>0 \tag{2.31}$$

And so forth by including the action of more raising operators acting on the left The lowering (annihilation) operators acting on the ground state yield zero.

Before proceeding with the quantization process, some important remarks about Octonionic Quantum Mechanics are in order. The physical interpretation of the Octonionic Quantum Mechanics has posed many problems [21]. An important problem has to do with the possible product states, which is crucial for the algebraic explanation of the unobservability of colored states. Such octonionic Hilbert space can be divided into an observable subspace corresponding to the usual complex Hilbert space of quantum mechanics and an unobservable subspace corresponding to the nonassociative components of the underlying octonionic algebra [21].

The authors [22] have argued that the use of complex geometry allows to obtain a consistent formulation of Octonionic Quantum Mechanics. The use of complex scalar products (or complex geometry as called by Rembielinski) permits to define a consistent tensor product. In the octonionic formulation of QM they solve the Hermiticity problem and define an appropriate momentum operator within Octonionic QM. The nonextendability of the completeness relation and the norm conservation was also discussed in detail.

The usual axioms of one-particle Quantum Mechanics can be implemented with projection operators belonging to the exceptional Jordan algebra $J_3[O]$ over the real octonions [21], [23]. It turns out that this is not possible with the usual

Hilbert space formulation of Quantum Mechanics, because the octonion algebra is non associative. So one has to go to a more abstract level, starting with the concept of proposition (yes-no experiment). Propositions correspond in the usual case to projection operators, and the structure of the propositional system of Quantum Mechanics is equivalent to the structure of an ortho-complemented projective geometry. At this point one finds the link with octonions as found by [21].

Moufang has constructed a projective plane coordinated by octonions, and which turns out to be non-Desarguian. The authors [21] studied the Quantum Mechanical properties of this non-Desarguian geometry by using Jordan's formulation of it in terms of the exceptional Jordan algebra, and explained how successive compatible experiments can yield a result which is independent of the order in which they are performed. Hence, on the one-particle level all the axioms of Quantum Mechanics were fullfilled [21].

The study of the octonionic left/right eigenvalue equation and the construction of octonionic Hilbert spaces is very subtle [22]. The Exceptional Jordan-Matrix eigenvalue problem and the characteristic equation associated with the Jordan-von Neumann-Wigner formulation of Quantum Mechanics in terms of anti-commutators was studied in detail by [23]. Due to the nonassociativity, the octonionic Hermitian matrices *no* longer correspond with Hermitian operators in Octonionic Quantum Mechanics, so it is not surprising to find *non* real left/right eigenvalues for these matrices. A plausible way to recapture the relation between Hermitian matrices and Hermitian operators by using a complex projection procedure was proposed by [22].

After this discussion of the subtle issues behind Octonionic QM, let us recall that the no-ghost theorem of the ordinary open bosonic string in flat backgrounds leads to the *critical* dimension $D = 26$ and to the Regge intercept $a = 1$ (resulting from the normal ordering procedure in the definition of level number operator) [24]. Ghost states are negative-norm states that are due to the contribution of the timelike component of the flat background Lorentzian metric $\eta_{tt} < 0$, and which cause problems with causality and unitarity.

The physical states were characterized by the condition $L_m|\phi\rangle = 0$ for $m > 0$, and the mass-shell condition $(L_0 - a)|\phi\rangle = 0$. The spurious states $|\psi\rangle$ are orthogonal to the physical states $\langle\psi|\phi\rangle = 0$. There are also zero-norm states in $D = 26$ that are both spurious and orthogonal to themselves and which *decouple* from all physical processes [24]. Similar findings occur for the closed string by including the tilde oscillators $(\tilde{L}_0 - a)|\phi\rangle = 0$; $\tilde{L}_m|\phi\rangle = 0$ for $m > 0$.

One can borrow these results. mutatis mutandis, for the octonionic bosonic string and impose the conditions

$$\begin{aligned} \mathbf{L}_m|\Phi\rangle &= 0, \quad m > 0, \quad (\mathbf{L}_0 - \mathbf{a})|\Phi\rangle = 0 \\ \tilde{\mathbf{L}}_m|\Phi\rangle &= 0, \quad m > 0, \quad (\tilde{\mathbf{L}}_0 - \mathbf{a})|\Phi\rangle = 0 \end{aligned} \tag{2.32}$$

where $\mathbf{L}_m$ are the octonionic Virasoro operators that are bilinear in the octonionic raising/lowering operators $\sum_{n=-\infty}^{\infty} : \hat{\mathbf{a}}^\mu_{m-n}\hat{\mathbf{a}}^\nu_n : \eta_{\mu\nu}$, and can be represented in terms of 8×8 matrices. And $\mathbf{a} = a e_o$ is the octonionic analog of the Regge intercept and can be represented by $a\mathbf{1}_{8\times 8}$. The state $|\Phi\rangle$ can be decomposed in an octonionic basis as $|\Phi_o\rangle e_o + |\Phi_i\rangle e_i$. The spurious states $|\Psi\rangle$ are orthogonal to the physical states $\langle\Psi|\Phi\rangle = 0$. The problem which arises now is, firstly, how to define the inner products and the proper octonionic Hilbert space, before one can define the notion of orthogonality. In ordinary complex-QM the wave-functions are complex-valued, and the inner product of two states is given, for example, in one-dimension by

$$\langle\Psi|\Phi\rangle \ = \ \int_{x=-\infty}^{x=\infty} dx \ \Psi^*(x) \ \Phi(x) \qquad (2.33a)$$

In octonionic QM matters are more complicated . One has complex-octonionic-valued functions of octonionic-valued variables $\mathbf{Z}$ given by $\Psi(\mathbf{Z}) = \Psi_o(\mathbf{Z})e_o + \Psi_i(\mathbf{Z})e_i$, where the components Ψ_o, Ψ_i are themselves complex-valued. There are two integrals of the form

$$\langle\Psi|\Phi\rangle_{(1)} \ = \ \frac{1}{2}\int \ d\mathbf{Z} \ (\Psi^*(\mathbf{Z}) \ \Phi(\mathbf{Z})) \ + \ o.c \ \neq \ \frac{1}{2}\int \ (d\mathbf{Z} \ \Psi^*(\mathbf{Z})) \ \Phi(\mathbf{Z}) \ + \ o.c \qquad (2.33b)$$

$$\langle\Psi|\Phi\rangle_{(2)} \ = \ \frac{1}{2}\int \ d\mathbf{Z} \ (\bar{\Psi}(\mathbf{Z}) \ \Phi(\mathbf{Z})) + o.c \ \neq \ \frac{1}{2}\int \ (d\mathbf{Z} \ \bar{\Psi}(\mathbf{Z})) \ \Phi(\mathbf{Z}) + o.c \qquad (2.33c)$$

where $\bar{\Psi}$ involves the octonionic conjugation (o.c) $e_i \rightarrow -e_i$, while Ψ^* involves the ordinary complex conjugation $i \rightarrow -i$. The integrals depend on the arrangements of the products due to the nonassociativity. One must add as well the octonionic conjugates (o.c) to the above integrals because the real part of a *complex*-octonion is given by $\frac{1}{2}(\mathbf{X} + \bar{\mathbf{X}}) \equiv (x_o + iy_o)$ is a *complex* number. In this way the values of the integrals are complex-valued when $\Psi \neq \Phi$.

There is yet a *third* integral (leading to a third inner product) involving $\Psi^\dagger \equiv \Psi_o^* e_o - \Psi_i^* e_i$ ($\neq \Psi^*$) that is based on the transpose of the real antisymmetric 8×8 matrix representation of the imaginary octonion units e_i, and which is tantamount of replacing $e_i \rightarrow -e_i$, followed by the ordinary complex conjugation of the 8 components of Ψ : Ψ_o^*, Ψ_i^*

$$\langle\Psi|\Phi\rangle_{(3)} \ = \ \frac{1}{2}\int \ d\mathbf{Z} \ (\Psi^\dagger(\mathbf{Z}) \ \Phi(\mathbf{Z})) \ + \ o.c \ \neq \ \frac{1}{2}\int \ (d\mathbf{Z} \ \Psi^\dagger(\mathbf{Z})) \ \Phi(\mathbf{Z}) \ + \ o.c \qquad (2.33d)$$

These integrals simplify considerably when $\mathbf{Z} \rightarrow x$. Another subtlelty is how to generalize the notion of the Lorentz group $SO(1, D-1)$ (and Poincare group) to a D-dimensional octonionic space comprised of D octonionic coordinates $\mathbf{Z}^\mu, \mu = 1, 2, \cdots, D$, and how to describe the notion of particles, one-particle states, $\cdots$.

Due to the many subtleties and difficulties of Octonionic QM (Nonassociative QM will be discussed in the next section) we find it very difficult to verify if the no-ghost theorem for the ordinary bosonic string can be extended to the bosonic octonionic string case, and whether or not there will be a judicious value for the critical octonionic-spacetime dimension D, and the analog of the Regge intercept a. Besides, there is *no* reason why $D = 26$ and $a = 1$ should also turn out to be the critical values in the octonionic bosonic string case.

Before proceeding, there is a very interesting numerical *coincidence* (without invoking supersymmetry) related to $D = 26$ worth exploring and mentioning. Imagine having the embedding of an octonionic-worldsheet onto an octonionic-spacetime background. This requires a rigorous study of Octonionic Analysis because now the worlsheet coordinates τ, σ are also octonionic-valued. The massless modes in standard QFT correspond to oscillations in the transverse dimensions $26 - 2 = 24$. An octonionic 26-dim spacetime has 8×26 real dimensions, so the 24 transverse octonionic dimensions to the octonionic worldsheet amounts to $8 \times 24 = 192$ real dimensions (degrees of freedom). Each fermion generation has 16 fermions (leptons, quarks of three colors and their anti-particles). A massless Weyl (chiral) fermion in $4D$ has 2 complex components (4 real components). Hence the total number of (real) degrees of freedom of 3 fermion generations (involving massless Weyl fermions in $4D$) is $16 \times 4 \times 3 = 192$, and which matches the number of $8 \times 24 = 192$ real dimensions (degrees of freedom) corresponding to the 24 transverse octonionic dimensions associated with the octonionic-worldsheet of a bosonic octonionic-string moving in $D = 26$ octonionic dimensions.

Gresnigt [26] has shown that the non-trivial braid groups that can be represented using the four normed division algebras are B_2 and B_3^c, exactly those required to represent a single generation of fermions in terms of simple three strand ribbon braids. These braided fermion states can be identified with the basis states of the minimal left ideals of the complex Clifford algebra $Cl(6)$, generated from the nested left actions of the complex octonions on itself. Thus, the ribbon spectrum can be related to octonion algebras.

In [14] we have shown that the algebra $\mathbf{J}_3[\mathbf{C} \otimes \mathbf{O}] \otimes Cl(4, \mathbf{C})$, given by the tensor product of the complex exceptional Jordan $\mathbf{J}_3[\mathbf{C} \otimes \mathbf{O}]$ and the complex Clifford algebra $Cl(4, \mathbf{C})$, can describe all of the spinorial degrees of freedom of three generations of fermions in four-spacetime dimensions. More recently, Singh [15] has combined the seminal work of Adler's Trace dynamics with division algebras as a path towards quantum gravity and unification. Based on these findings, it is warranted to explore further the above numerical coincidence in connection to the algebra of octonions, bosonic octonionic strings, braids, complex exceptional Jordan and Clifford algebras.

We shall see next that depending on the choices of the signature of the diagonal metric $g^o_{\mu\nu} = \eta_{\mu\nu}$, this could lead to different critical dimensions for the bosonic octonionic string. So far we have studied the Bosonic Octonionic String moving in flat backgrounds. The motion in curved backgrounds is far more complicated. To get a picture of what an octonionic spacetime background endowed with an octonionic metric looks like, let us concentrate in the very special case of diagonal metrics. Namely $\mathbf{g}_{\mu\nu} = 0$ when $\mu \neq \nu$, and such that the nonzero diagonal components are all real-valued

$$\mathbf{g}_{\mu\mu} = g^o_{(\mu\mu)} e_o \ + \ g^i_{[\mu\mu]} e_i \ = \ g^o_{(\mu\mu)} e_o, \ \ \mu = 1, 2, \cdots, D \tag{2.34}$$

there is no sum over μ in eq-(2.42). Hence, the interval $(ds)^2$ in eq-(2.1) becomes

$$(ds)^2 \ = \ (\ dZ_o^1 \ g_{11}^o \ dZ_o^1 - dZ_i^1 \ g_{11}^o \ dZ_i^1 \ ) + (\ dZ_o^2 \ g_{22}^o \ dZ_o^2 - dZ_i^2 \ g_{22}^o \ dZ_i^2 \ ) + \ \cdots \ +$$

$$(\ dZ_o^D \ g_{DD}^o \ dZ_o^D \ - \ dZ_i^D \ g_{DD}^o \ dZ_i^D \ ) \tag{2.35}$$

One may then identify the coordinates

$$Z_o^1 \leftrightarrow t^{(1)}, \quad Z_o^2 \leftrightarrow t^{(2)}, \quad \cdots, \quad Z_o^D \leftrightarrow t^{(D)} \tag{2.36}$$

with D temporal directions $t^{(1)}, t^{(2)}, \cdots, t^{(D)}$. And the coordinates

$$Z_i^1 \leftrightarrow x_i^{(1)}, \quad Z_i^2 \leftrightarrow x_i^{(2)}, \quad \cdots, \quad Z_i^D \leftrightarrow x_i^{(D)}; \quad i \ = \ 1, 2, \cdots, 7 \tag{2.37}$$

can be identified with $7 \times D$ spatial coordinates. By setting

$$g_{11}^o < 0, \quad g_{22}^o < 0, \quad g_{33}^o < 0, \quad g_{DD}^o < 0 \tag{2.38}$$

the interval $(ds)^2$ (2-35) can be written as the direct sum of D eight-dimensional spacetimes intervals $(ds)_8^2$, each one of signature $(-, +, +, +, \cdots, +)$,

$$(ds)_8^2 \ = \ dt^{(1)} \ g_{11}^o \ dt^{(1)} \ - g_{11}^o \ dx_i^{(1)} \ \delta_{ij} \ dx_j^{(1)}, \quad g_{11}^o < 0 \tag{2.39a}$$

$$(ds)_8^2 \ = \ dt^{(2)} \ g_{22}^o \ dt^{(2)} \ - g_{22}^o \ dx_i^{(2)} \ \delta_{ij} \ dx_j^{(2)}, \quad g_{22}^o < 0, \quad \cdots\cdots \tag{2.39b}$$

and

$$(ds)_8^2 \ = \ dt^{(D)} \ g_{DD}^o \ dt^{(D)} \ - g_{DD}^o \ dx_i^{(D)} \ \delta_{ij} \ dx_j^{(D)}, \quad g_{DD}^o < 0 \tag{2.39c}$$

This direct sum of D eight-dimensional spacetimes intervals has the appearance of an 8-fold periodicity : $\mathbf{O}^D \leftrightarrow M^8 \oplus M^8 \oplus M^8 \cdots \oplus M^8$. On the other hand, there is also the correspondence $\mathbf{O}^2 \leftrightarrow M^{(14,2)} \leftrightarrow SO(14,2)$, conformal group in $D = 14$. $\mathbf{O}^3 \leftrightarrow M^{(21,3)} \leftrightarrow SO(21,3)$. $\mathbf{O}^4 \leftrightarrow M^{(28,4)} \leftrightarrow SO(28,4)$, quasi-conformal group in $D = 28$.

The most renowned case, when *all* the coordinates of $\mathbf{Z}^\mu$ are *spatial* (after a Wick rotation of the temporal variables), is the Wilson's construction of the 24-dim Leech lattice based on $\mathbf{O}^3$ [17]. Dixon [6] has also offered a different construction of the 24-dim Leech lattice based on the *ternary* products of $\mathbf{O}$. The automorphism group of the vertex operator algebra associated with the states of the 26-dim bosonic string compactification on the 24-dim Leech lattice is the Monster group [18]. Infinite extensions of the Exceptional algebras based on the notion of an 8-fold Exceptional Periodicity can be found in [27].

Lets us study the case of a metric with the same signature as the Lorentzian one

$$g_{11}^o < 0, \quad g_{22}^o > 0, \quad g_{33}^o > 0, \quad g_{DD}^o > 0 \tag{2.40}$$

and which $differs$ from the signature choice displayed in eq-(2.46). In this new case, the assignment of the temporal/spatial directions will now $differ$ from the one displayed by eqs-(2.36, 2.37). And the direct sum of the D eight-dimensional spacetime intervals leads to an effective $8D$ diagonal metric with $7D - 6$ minus signs (temporal directions) and $D+6$ positive signs (spatial directions). Whereas previously, eqs-(2.36, 2.37, 2.38) furnished D temporal and $7D$ spatial directions. The *split* signature case occurs when $7D - 6 = D + 6 \Rightarrow D = 2$ which is very special. In this case the $2D$-octonionic-dimensional spacetime is 16-real dimensional whose metric has a *split* signature of $(8, 8)$.

To conclude, assuming that one could extend the analysis of the no-ghost theorem of the ordinary bosonic string to the octonionic case, we found that depending on the choices of the signature of the diagonal metric $g^o_{\mu\nu} = \eta_{\mu\nu}$, one can have many different choices for the number of temporal/spatial directions, and leading to direct sums of many different D eight-dimensional "spacetime" intervals, and consequently, it will considerably affect the overall analysis of the physical, ghosts and zero-norm states of the bosonic octonionic string. And, which in turn, would lead to different values for the critical dimensions and the Regge intercept. One of the most significant findings, in our opinion, has been the numerical coincidence (without invoking supersymmetry) in the number of degrees of freedom of three fermion generations and the number of transverse dimensions of an octonionic string moving in $D = 26$ octonionic dimensions. This warrants further investigation.

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