On the modified Emden-type equation with quadratic damping

J. Akande[1], M. Nonti[1], E. A. Doutètien[1], M.D. Monsia[1*]

1-Department of Physics, University of Abomey-Calavi, Abomey-Calavi, 01.B P.526, Cotonou, BENIN

Abstract

We exhibit in this paper a complex-valued solution for a modified Emden-type equation with quadratic damping. We show also the conditions of existence of periodic solution and calculate it for the first time.

Keywords: Modified Emden-type equation, quadratic damping, general solution, exact periodic solution

Introduction

Recently, it is observed that many nonlinear differential equations claimed by certain authors in the literature to be nonlinear conservative oscillators are rather pseudo-oscillators, since these can exhibit non-oscillatory behavior for the same numerical parameters. Gadella and coworkers [1] were the first to raise this problem for the truly nonlinear differential equation

$$\ddot{x} + \frac{1}{x} = 0 \tag{1}$$

The authors were able to show that the general solutions of the equation are non-oscillatory so that they conclude that the equation (1) is in fact a pseudo-oscillator even if it has a constant time-independent Hamiltonian. After this, Doutètien and coworkers [2] confirmed the results obtained in [1] by successfully calculating explicitly the two general non-periodic solutions predicted in [1]. The same pseudo-oscillator feature was observed for many others nonlinear differential equations mentioned to be conservative oscillators. A remarkable case is the so-called Mathews-Lakshmanan equation [3]. This equation has been widely studied in the literature as a unique nonlinear oscillator having harmonic periodic solution but with amplitude-dependent frequency. However, it has been recently shown in a lot of papers that there exists a number of such equations that can exhibit sinusoidal periodic solutions [4, 5]. In [6, 7] the authors have shown even the existence of quadratic Lienard type equation

* Corresponding author : E-mail: monsiadelphin@yahoo.fr

exhibiting the harmonic periodic solution with amplitude-independent frequency for the first time. In other words these equations have the linear harmonic oscillator solution. On the other hand Akande et al. [8] have shown recently the existence of real-valued and complex-valued solutions of the Mathews-Lakshmanan equation. They concluded then the Mathews-Lakshmanan equation is in fact a pseudo-oscillator. Another interesting case is the generalized Emden type equation presented by Chandrasekar and his group [9] as a Lienard type nonlinear oscillator in 2005. Later this equation has been widely studied in the literature as a conservative oscillator [10, 11]. However, in [12] the authors succeeded to calculate an unbounded periodic solution for the so-called Lienard type nonlinear oscillator. Also these authors [12] concluded that the generalized Emden type equation claimed to be a Lienard type nonlinear oscillator is in fact a pseudo-oscillator. In this context it is reasonable to ask whether the modified Emden-type equation with quadratic damping.

$$\ddot{x} + k\,x\,\dot{x}^2 + k_1 x + k_2 x^3 = 0 \tag{2}$$

where k, k_1 and k_2 are constants, and the overdot means derivative with respect to time, introduced by Ghosh et al. [13] is effectively a conservative oscillator as claimed by the authors. The equation belongs to the also of quadratic Lienard type equations intensively investigated in the literature [14]. However, the authors [13] were unable to prove their claim by a presentation of exact periodic solutions. Instead of exact periodic solutions, the authors [13] used qualitative theory of differential equation in the phase space to conclude to the conservative oscillator feature of this equation. They have been only able to calculate approximate analytical solution for the equation. In [15] the equation (2) is also investigated using symmetry methods, but no exact periodic solution has been exhibited by the author. In this work we exhibit for the first time the exact and general periodic solution of this equation when $\frac{k_2}{k_1} < 0$, as well as the complex-valued solutions, to conclude that such a modified Emden-type equation could not be a conservative oscillator but rather a pseudo-oscillator. In this perspective we show that the equation (2) belongs to a general class of quadratic Lienard type equations introduced recently in the literature by Monsia and his group (section 2) and exhibit its general solutions (section 3). We present finally a conclusion for the work.

2- The statement of the equation (2)

To establish the equation (2) let us consider the general class of quadratic Lienard type equations [16-18]

$$\ddot{x}+\frac{1}{2}\frac{g'(x)}{g(x)}\dot{x}^2+\frac{a}{2}\frac{f'(x)}{g(x)}=0 \tag{3}$$

associated to the corresponding first-order differential equation

$$g(x)\dot{x}^2+a\,f(x)=b \tag{4}$$

where the prime denotes differentiation with respect to the argument, a and b are arbitrary parameters, and $f(x)$ and $g(x)\neq 0$, are arbitrary functions of the variable x. Now let $g(x)=e^{k\,x^2}$, and $f(x)=\left(c_1+c_2x^2\right)e^{k\,x^2}$. Then one may obtain $f'(x)=2\left[\left(c_1k+c_2\right)x+c_2k\,x^3\right]e^{k\,x^2}$, so that the equation (3) reduces to the equation

$$\ddot{x}+k\,x\,\dot{x}^2+a\left(c_1k+c_2\right)x+a\,c_2k\,x^3=0 \tag{5}$$

which becomes the equation (2) when $k_1=a\left(c_2+c_1k\right)$, and $k_2=a\,c_2k$. c_1 and c_2 are arbitrary constants. In this regard the equation (4) transforms into

$$e^{k\,x^2}\dot{x}^2+a\left(c_1+c_2\,x^2\right)e^{k\,x^2}=b \tag{6}$$

Using the equation (6) one may get the general solutions of the equation (2).

3- The general solutions of the equation (2)

We calculate the complex-valued solution (3.1) and secondly exhibit the exact and general periodic solution (3.2).

3.1 Complex-valued solutions

From the equation (6) one may, after variables separation, write

$$\frac{dx}{\sqrt{b\,e^{-k\,x^2}-ac_1-ac_2x^2}}=\pm dt \tag{7}$$

Letting $b=0$, reduces the equation (7) to

$$\frac{dx}{\sqrt{-\left(ac_1+ac_2x^2\right)}}=\pm dt \tag{8}$$

which may be written as

$$\frac{dx}{\sqrt{1+\frac{c_2}{c_1}x^2}} = \pm\sqrt{-ac_1}\,dt \tag{9}$$

As k, k_1 and k_2, are positive constants for the equation introduced by Ghosh et al. [13], then $\frac{c_2}{c_1} > 0$ and $a > 0$. In this case, by integration, one may, from (8) obtain

$$sh^{-1}\left(\frac{\sqrt{c_1 c_2}}{c_2}x\right) = \pm\sqrt{-ac_2}\,(t+\gamma) \tag{10}$$

where γ is a constant of integration, such that

$$x(t) = \frac{\sqrt{c_1 c_2}}{c_1} sh\left[\pm\sqrt{-ac_2}\,(t+\gamma)\right] \tag{11}$$

As $a > 0$, then $c_1 > 0$ and $c_2 > 0$, so that the solution (11) becomes

$$x(t) = i\frac{\sqrt{c_1 c_2}}{c_2}\sin\left[\pm\sqrt{ac_2}\,(t+\gamma)\right] \tag{12}$$

which may be rearranged in the form

$$x(t) = i\frac{\sqrt{kk_2(kk_1 - k_2)}}{kk_2}\sin\left[\pm\frac{\sqrt{kk_2}}{k}(t+\gamma)\right] \tag{13}$$

As may be seen, the general solution of the equation (2) is a complex-valued function for k, k_1 and k_2 positive. Now the objective is to show that when

$\frac{c_2}{c_1} < 0$, that is $\frac{k_2}{k_1} < 0$, the equation (2) may exhibit periodic solution.

3.2 Periodic solution

This case corresponds to $\frac{c_2}{c_1} < 0$, that is $\frac{k_2}{k_1} < 0$, when we choose $a > 0$. In this way from the equation (9) one may get

$$\sin^{-1}\left(\sqrt{-\frac{c_2}{c_1}}x\right) = \pm\sqrt{ac_2}\,(t+\gamma_1) \tag{14}$$

so that

$$x(t) = \sqrt{-\frac{c_1}{c_2}} \sin\left[\pm \sqrt{ac_2}\left(t + \gamma_1\right)\right] \tag{15}$$

where $c_2 > 0$, and γ_1 is a constant of integration. The solution (15) may be written in the form

$$x(t) = \frac{\sqrt{kk_2^2 - k^2 k_1 k_2}}{kk_2} \sin\left[\pm \frac{\sqrt{kk_2}}{k}\left(t + \gamma_1\right)\right] \tag{16}$$

The above shows that the general solution of the equation (2) is periodic when $\frac{k_2}{k_1} < 0$, with $k_2 > 0$. It is worth to notice that the equation (2) may be generalized as

$$\ddot{x} + (m+1)k\, x^{2m+1}\dot{x}^2 + (m+1)k_1 x^{2m+1} + (m+1)k_2\, x^{4m+3} = 0 \tag{17}$$

when $f(x) = \left(c_1 + c_2 x^{2m+2}\right)e^{k\, x^{2m+2}}$ and $g(x) = e^{k\, x^{2m+2}}$. But this will be studied in the subsequent work.

Conclusion

We have shown in this work that a modified Emden-type equation introduced in the literature by some authors as a Lienard type nonlinear oscillator may exhibit complex-valued solution so that such an equation could not be a conservative oscillator but rather a pseudo-oscillator. We finally have shown the conditions of existence of periodic solution and calculate it for the first time.

References

[1] M. Gadella and L. P. Lara, on the solutions of a nonlinear pseudo-oscillator equation, Phys. Scr. 89 105205 (2014).

[2] E. A. Doutetien, A. R. Yehossou, P. Mallick, B. Rath, M. D. Monsia, On the General Solutions of a Nonlinear Pseudo-Oscillator Equation and Related Quadratic Lienard Systems, PINSA(2020), Doi:10.16943/ptinsa/2020/154987

[3] P. M. Mathews and M. Lakshmanan, On a unique nonlinear oscillator, Quarterly of Applied Mathematics, 32 (1974) 215-218, Doi: 10.1090/qam/430422.

[4] J. Akande, D. K. K. Adjaï, M. D. Monsia, Theory of exact trigonometric periodic solutions to quadratic Lienard type equations, Journal of Mathematics and Statistics 14 (1) (2018) 232-240, Doi:10.3844jmssp.2018.232.240.

[5]] M. Nonti, A.V. R. Yehossou, J. Akande and M.D. Monsia, On a singular Kamke equation equivalent to linear harmonic oscillator, Math.Phys.,viXra.org/1808.0572v1.pdf (2018).

[6] J. Akande, K. K. D. Adjaï, A.V. R. Yehossou and M. D. Monsia, On the existence of identical periodic solutions between differential equations, Math.Phys.,viXra.org/2012.0156v2.pdf (2020).

[7] Y. J. F. Kpomahou, M. Nonti, K .K. D. Adjaï and M.D. Monsia, On the linear harmonic oscillator solution for a quadratic Lienard type equation Math.Phys.,viXra.org/2101.0010v1.pdf (2021).

[8] J. Akande, K. K. D. Adjaï, L.H. Koudahoun and M.D. Monsia, Existence of non-periodic real-valued and complex-valued solutions of Mathews-Lakshmanan oscillator equation, Math.Phys.,viXra.org/ 2012.0104v1.pdf (2020).

[9] V. K. Chandrasekar, M. Senthilvelan, and M. Lakshmanan, Unusual Liénard-type nonlinear oscillator, Physical Review E 72, (2005) 066203

[10] R. Iacono and F. Russo, Class of solvable nonlinear oscillators with isochronous orbits, Physical Review E, 83(2011), 027601-1-027601-4.

[11] S. N. Pandey, P. S. Bindu, M. Senthilvelan, and M. Lakshmanan, A group theoretical identification of integrable cases of the Liénard-type equation $\ddot{x}+f(x)\dot{x}+g(x)=0$. I. Equations having nonmaximal number of Lie point symmetries, Journal of Mathematical Physics 50, (2009) 082702.

[12] E. A. Doutètien, A. B. Yessoufou, Y. J. F. Kpomahou and M. D. Monsia, Unbounded periodic solution of a modified Emden type equation, Math.Phys.,viXra.org/2101.0002v1.pdf (2021).

[13] S. Ghosh, B. Talukdar, Umapada Das and A. Saha, Modified Emden-type equation with dissipative term quadratic in velocity, Journal of Physics A:

Mathematical and Theoretical, 45(15) (2012), 155207. doi:10.1088/1751-8113/45/15/155207

[14] A. K. Tiwari, S. N. Pandey, M. Senthilvelan and M. Lakshmanan, Classification of Lie point symmetries for quadratic Lienard type equation $\ddot{x}+f(x)\dot{x}^2+g(x)=0$, Journal of Mathematical Physics 54, 053506 (2013).

[15] K. S. Govinder, Analysis of modified Painlevé–Ince equations, Ricerche Di Matematica. (2020) doi:10.1007/s11587-020-00517-5.

[16] M. D Monsia, Analysis of a purely nonlinear generalized isotonic oscillator equation, Math.Phys.,viXra.org/2010.0195v1.pdf (2020).

[17] M. Nonti, K. K. D. Adjaï, J. Akande and M.D. Monsia, On truly nonlinear oscillator equations of Ermakov-Pinney type, Math.Phys.,viXra.org /2012.0117v1.pdf (2020).

[18] M. D. Monsia, On a nonlinear differential equation of Lienard type, Math.Phys.,viXra.org/2011.0150v3.pdf (2020).