The non-oscillatory behavior of the pendulum equation

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Abstract

We study in this work the pendulum equation. We show for the first time the existence of general non-periodic solution for this equation. This comes down to say that the pendulum equation can exhibit non-oscillatory behavior.

Keywords: pendulum equation, periodic solution, non-oscillatory behavior, Lienard type equation.

Introduction

The study of the pendulum began in the time of Huyghens. After the establishment of its differential equation [1-3]

$$\ddot{x} + \beta \sin x = 0 \tag{1}$$

where the dot over a symbol means the derivative with respect to time, and β is an arbitrary parameter. The determination of general solution of (1) has been for a long time a challenging problem in mathematics and physics. The general periodic solution of the pendulum equation (1) with an analytical method which does not imply the physical law of energy conservation has been only derived in a recent work performed by Akande and coworkers [2] within the framework of the generalized Sundman transformation theory. The pendulum equation (1) is well known in the literature to have periodic solutions. However, non-periodic solutions have been exhibited for many nonlinear differential equations of Lienard type recently in the literature. In this perspective no one can say whether the equation (1) has non-periodic solution for arbitrary value of $\beta > 0$. The objective in this paper is to show that the pendulum equation can exhibit non-periodic solution. To do so we establish the pendulum equation (1) from the general class of Lienard type equation introduced recently in [4-6] and solve it to calculate the general non-periodic solution (section 2). Finally we perform a conclusion for the work.

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2- The theory

2.1 The statement of the equation

Consider the general class of dissipative Lienard type equations [4-6]

$$\ddot{x}+\frac{g'(x)}{g(x)}\dot{x}^{2}+a\,\ell\frac{f(x)}{g(x)}x^{\ell-1}\,\dot{x}-a^{2}\,x^{2\ell}\,\frac{f'(x)f(x)}{g^{2}(x)}+ab\frac{f'(x)}{g^{2}(x)}x^{\ell}=0 \qquad (2)$$

where a, b and ℓ are arbitrary constants and $f(x)$ and $g(x)$ are arbitrary functions of x,. Substituting $g(x)=1$, and $\ell=0$, yields the non-dissipative Lienard type equation

$$\ddot{x}-a^{2}f'(x)f(x)+ab\,f'(x)=0 \qquad (3)$$

Taking into account $b=0$, leads to obtain the equation

$$\ddot{x}-a^{2}f'(x)f(x)=0 \qquad (4)$$

Now choosing $f(x)=\cos qx$, when q is an arbitrary parameter, the equation (4) reduces to

$$\ddot{x}+\frac{a^{2}}{4}\sin x=0 \qquad (5)$$

under the condition that $q=\frac{1}{2}$, which is the equation (1) when $\beta=\frac{a^{2}}{4}$.

2.2 The general solution of (1)

According to [4-6], the corresponding first-order differential equation associated to the equation (2) can read

$$g(x)\dot{x}+af(x)x^{\ell}=b \qquad (6)$$

which becomes under the conditions that $g(x)=1$, $\ell=0$ and $b=0$, after separation of variables

$$\int\frac{dx}{f(x)}=-a\left(t+K\right) \qquad (7)$$

where K is an integration constant. Knowing $f(x)=\cos\left(\frac{x}{2}\right)$, the equation (7) gives

$$\int \frac{dx}{\cos\left(\frac{x}{2}\right)} = -a(t+K) \tag{8}$$

The integration of the indefinite integral in (8) gives

$$\ell n\left|\tan\left(\frac{\pi}{4}+\frac{1}{4}x\right)\right| = -\frac{a}{2}(t+K) \tag{9}$$

which ensures the solution of the equation (5) to have the form

$$x(t) = 4\tan^{-1}\left[e^{-\frac{a}{2}(t+K)}\right] - \pi \tag{10}$$

Substituting $\beta = \frac{a^2}{4}$, yields the general solution of the equation (1) to take the definitive form

$$x(t) = 4\tan^{-1}\left(e^{\pm\sqrt{\beta}(t+K)}\right) - \pi \tag{11}$$

Now we can conclude the work.

Conclusion

We have investigated in this paper the well-studied pendulum equation. We have shown for the first time the existence of non-periodic solution for this equation. This result is obtained using an analytical method without implying the physical law of energy conservation.

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