On isochronous periodic solution of a generalized Emden type equation

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Abstract

We present in this paper a generalized Emden type equation which is explicitly integrable. We show the existence of isochronous periodic solution of the equation. As a result, such a dissipative equation may be used as a Lienard type nonlinear oscillator.

Keywords: Modified Emden type equation, Lienard type nonlinear oscillator, periodic solution, isochronous properties.

Introduction

The generalized Emden type equation

$$\ddot{x} + \alpha x\dot{x} + \beta x^3 + \lambda x = 0 \qquad (1)$$

has been, during the last decades, the object of intensive study in the literature. Since Chandrasekar et al. [1] have presented this dissipative equation as a Lienard type nonlinear oscillator when $\alpha = k$, $\beta = \frac{k^2}{9}$ and λ positive, several works investigating the equation have been carried out from various and rich mathematical and physical points of view [1-7]. However, recently in [6] Doutètien et al., succeeded to show the existence of unbounded periodic solution for the equation (1) where α and λ are arbitrary parameters, and $\beta = \frac{\alpha^2}{9}$. Thus, this equation could not be an oscillator as, by definition, an oscillator can only have bounded periodic solution. The feature of pseudo-oscillator has been

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recently found in the literature for many nonlinear differential equations studied as nonlinear oscillators. As a famous case, one may notice some truly or purely nonlinear differential equations that exhibit non-periodic or complex-valued solutions [8-12]. Another celebrated case is the Mathews-Lakshmanan equation widely studied as a nonlinear oscillator since its presentation in 1974 [13]. In [14] Akande et al., showed that the Mathews-Lakshmanan equation is not a conservative oscillator but rather a pseudo-oscillator since it can exhibit real non-periodic and complex-valued solutions. As regards equation (1), one can observe that it constitutes a generalization of the modified Emden type equation

$$\ddot{x} + \alpha x\dot{x} + \beta x^3 = 0 \tag{2}$$

by a linear external forcing term λx. However, the equation (2) has been and continues to be a subject of interesting studies in the literature. This, may be, according to [15-17], is due to the fact that the equation (2) arises in many problems in physics and mathematics. In [17] the authors have been able to calculate the general solutions of the equation (2) following three distinct dynamical regimes according to parameters α and β. In [15] the author succeeded to calculate the general solution of this equation as a single exact and explicit formula. The equation (1) is of the form

$$\ddot{x} + h(x)\dot{x} + \beta x^3 + \lambda \vartheta(x) = 0 \tag{3}$$

where $h(x)$ and $\vartheta(x)$ are arbitrary functions of x. It seems that no exact periodic solution has been shown in the literature for equations of form (3) when $\vartheta(x) \neq x$. In consequence it is reasonable to investigate the problem of finding periodic solution of this dissipative nonlinear differential equation. More precisely one can ask whether there exists the functions $h(x)$ and $\vartheta(x) \neq x$, which ensure the existence of periodic solution for the equation (3). In this work we predict the existence of such functions $f(x)$ and $\vartheta(x)$ so that one can obtain isochronous periodic solution of the dissipative equation (3). To do so, we identify the convenient function $f(x)$ and $\vartheta(x)$ and calculate the general solution of the corresponding Lienard type differential equation to show our prediction and end the work by a conclusion.

2. The general periodic solution

We establish the dissipative equation of the form (3) of interest in paragraph (2.1) and solve it in paragraph (2.2) to show its isochronous periodic solution.

2.1 The dissipative Emden type equation

Let us consider the general class of Lienard type equations

$$\ddot{x}+\frac{g'(x)}{g(x)}\dot{x}^2+a\,\ell x^{\ell-1}\frac{f(x)}{g(x)}\,\dot{x}+ab\,x^{\ell}\,\frac{f'(x)}{g^2(x)}-a^2\,x^{2\ell}\,\frac{f'(x)f(x)}{g^2(x)}=0 \qquad (4)$$

where a and b are arbitrary parameters and $f(x)$ and $g(x)$ are arbitrary functions of x, introduced recently by some authors of this work in the literature [6, 18]. The choice of $g(x)=1$, leads to obtain

$$\ddot{x}+a\,\ell\,f(x)x^{\ell-1}\,\dot{x}-a^2x^{2\ell}f'(x)f(x)+ab\,x^{\ell}f'(x)=0 \qquad (5)$$

If $b=0$, then the equation (5) becomes

$$\ddot{x}+a\ell f(x)x^{\ell-1}\dot{x}-a^2f'(x)f(x)x^{2\ell}=0 \qquad (6)$$

Making $\ell=1$, allows one to get

$$\ddot{x}+af(x)\dot{x}-a^2f'(x)f(x)x^2=0 \qquad (7)$$

Substituting the function $f(x)=\sqrt{k_1x^2+k_2x+k_3}$, into (7) yields as equation

$$\ddot{x}+a\sqrt{k_1x^2+k_2x+k_3}\,\dot{x}-a^2k_1x^3-\frac{1}{2}a^2k_2x^2=0 \qquad (8)$$

where k_1,k_2 and k_3 are arbitrary parameters, and $\vartheta(x)=x^2$. The equation (8) is the dissipative Lienard type equation of interest. One can see that when $k_2=k_3=0$, this equation reduces to the well-known modified Emden type equation (2). Thus the equation (8) is a generalization of the equation (2). In this context we can solve explicitly the equation (8) to secure the isochronous properties of the periodic solution.

2.2 The isochronous periodic solution

Using the corresponding first-order differential equation

$$g(x)\dot{x}+af(x)x^{\ell}=b \qquad (9)$$

associated to (4), one can get

$$\int\frac{g(x)dx}{b-af(x)x^{\ell}}=\int dt \qquad (10)$$

Taking into account $g(x)=1, b=0$ and $\ell=1$, the equation (10) reduces to

$$\int \frac{dx}{xf(x)} = -a(t+\gamma) \tag{11}$$

In this context the equation (11) can take the form

$$\int \frac{dx}{x\sqrt{k_1 x^2 + k_2 x + k_3}} = -a(t+\gamma) \tag{12}$$

where γ is a constant of integration. By integration, one can obtain

$$\frac{1}{\sqrt{-k_3}} \sin^{-1}\left(\frac{2k_3 + k_2 x}{x\sqrt{k_2^2 - 4k_1 k_2}} \right) = -a(t+\gamma) \tag{13}$$

From the equation (13) one can secure the general solution as

$$x(t) = \frac{-2k_3}{k_2 + \sqrt{k_2^2 - 4k_1 k_3}\,\sin\left[a\sqrt{-k_3}\,(t+\gamma)\right]} \tag{14}$$

where $k_3 < 0$, and $k_2^2 \text{ - } 4\mathrm{k}_1\mathrm{k}_3 > 0$, such that $k_1 < 0$ and $k_2 \neq 0$. One can make the solution (14) isochronous by taking $k_3 = -1$, as these parameters are arbitrary constants. In this situation the solution (14) becomes

$$x(t) = \frac{2}{k_2 + \sqrt{k_2^2 + 4k_1}\,\sin\left[a(t+\gamma)\right]} \tag{15}$$

It is also possible to make the solution (14) isochronous by choosing $a\sqrt{-k_3} = 1$, that is $-k_3 a^2 = 1$. In this context the solution (14) takes the expression

$$x(t) = \frac{2}{a^2\left[k_2 + \sqrt{k_2^2 + 4\frac{k_1}{a^2}}\,\sin(t+\gamma) \right]} \tag{16}$$

where $k_2^2 + \frac{4\mathrm{k}_1}{a^2} > 0$.

Conclusion

We have presented in this paper a generalized Emden type equation which is explicitly integrable. The general periodic solution is shown to exhibit isochronous properties. In this way the presented equation can be considered as a Lienard type nonlinear oscillator.

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