Some new results on Dieudonné-type theorems for k-triangular lattice group-valued set functions

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Abstract

Using the Maeda-Ogasawara-Vulikh representation theorem and sliding hump-type techniques, we prove some Dieudonné-type theorems for k-triangular set functions, taking values in lattice groups.

1 Introduction

The non-additive set functions have been the object of several studies and applications in Mathematics, and are useful, for example, to model various forms of uncertainty. They are also an important tool in decision support systems (for instance, belief, plausibility, possibility, see e.g. [1, 6, 8, 9, 10, 11] and the references therein). In [1] it is dealt with the so-called *M-measures*, namely increasing set functions, continuous from above and from below and compatible with respect to finite suprema and infima. In [9], different kinds of non-additive set functions, among which k-triangular set functions, are investigated. In [5] some kinds of limit theorems are proved for k-triangular set functions, both when the concepts of (s)-boundedness, regularity, continuity with respect to a topology and continuity from above at $\emptyset$ are given with respect to a single regulator or order sequence and when they are formulated like in the classical setting.

In this paper we continue the investigation done in [5] and prove some Dieudonné-type theorems, in which the concepts of (s)-boundedness and regularity are intended like in the

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classical case, extending some results earlier proved in [3, 4, 7]. We use the tool of the Maeda-Ogasawara-Vulikh representation theorem of lattice groups as suitable subgroups of continuous extended real-valued functions. We find a suitable meager set in whose complement it is possible to apply the versions of limit theorems obtained for real-valued k-triangular set functions (see also [5, 9]). Observe that, differently than in the finitely additive case, a bounded k-triangular set function is not necessarily (s)-bounded. So, we consider the tool of the disjoint variation. Moreover, we show that our approach includes also the finitely additive case.

2 Preliminaries

We begin with recalling the following basic concepts on lattice groups (see also [5] and the references therein).

A sequence $(p_n)_n$ of positive elements of R is an (O)-*sequence* iff it is decreasing and $\bigwedge_n p_n = 0$.

A bounded double sequence $(a_{t,l})_{t,l}$ in R is a (D)-*sequence* or a *regulator* iff $(a_{t,l})_l$ is an (O)-sequence for any $t \in \mathbb{N}$.

A lattice group R is *weakly* σ-*distributive* iff $\bigwedge_{\varphi \in \mathbb{N}^{\mathbb{N}}} \Big(\bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \Big) = 0$ for any (D)-sequence $(a_{t,l})_{t,l}$.

A sequence $(x_n)_n$ in R is said to be *order convergent* (or (O)-*convergent*) to x iff there exists an (O)-sequence $(\sigma_p)_p$ in R such that for every $p \in \mathbb{N}$ there is a positive integer n_0 with $|x_n - x| \leq \sigma_p$ for each $n \geq n_0$, and in this case we write $(O) \lim_n x_n = x$.

A sequence $(x_n)_n$ in R is *order Cauchy* (or (O)-*Cauchy*) iff there is an (O)-sequence $(\tau_p)_p$ in R such that for every $p \in \mathbb{N}$ there is a positive integer n_0 with $|x_n - x_q| \leq \sigma_p$ for each n, $q \geq n_0$, and in this case we write $(O) \lim_n x_n = x$.

Observe that, in a Dedekind complete lattice group, every (O)-Cauchy sequence is (O)-convergent (see also [5]).

We now recall the Maeda-Ogasawara-Vulikh theorem, which gives a representation of lattice groups as subsets of continuous extended real-valued functions defined on suitable topological spaces (see also [5, Theorem 1.2.11]). From now on, we denote by the symbols $\bigvee$ and $\bigwedge$ the supremum and infimum in R and by sup and inf the pointwise supremum and infimum, respectively.

Theorem 2.1 *Given a Dedekind complete lattice group R, there exists a compact extremely disconnected topological space Ω, unique up to homeomorphisms, such that R can be embedded isomorphically as a subgroup of $C_\infty(\Omega) = \{f \in \widetilde{\mathbb{R}}^{\Omega} : f$ is continuous, and $\{\omega : |f(\omega)| = +\infty\}$*

is nowhere dense in $\Omega\}$*. Moreover, if we denote by* $\widehat{a}$ *an element of* $C_\infty(\Omega)$ *which corresponds to* $a \in R$ *under the above isomorphism, then for any family* $(a_\lambda)_{\lambda\in\Lambda}$ *of elements of* R *with* $R \ni a_0 = \bigvee_\lambda a_\lambda$ *(where the supremum is taken with respect to* R*), we get* $\widehat{a_0} = \bigvee_\lambda \widehat{a_\lambda}$ *with respect to* $C_\infty(\Omega)$ *and* $\widehat{a_0}(\omega) = \sup_\lambda \widehat{a_\lambda}(\omega)$ *in the complement of a meager subset of* Ω*. The same is true for* $\bigwedge_\lambda a_\lambda$*.*

Now we deal with some basic properties of lattice group-valued set functions (see also [5, 9]). From now on, R is a Dedekind complete lattice group, Ω is as in Theorem 2.1, G is an infinite set, Σ is a σ-algebra of subsets of G, $m : \Sigma \to R$ is a bounded set function, k is a fixed positive integer, $\mathcal{G}$ and $\mathcal{H}$ are two sublattices of Σ such that $\mathcal{G}$ is closed under countable disjoint unions and the complement of any subset of $\mathcal{H}$ belongs to $\mathcal{G}$.

Definitions 2.2 (a) Given a set function $m : \Sigma \to R$ and a lattice $\mathcal{E} \subset \Sigma$, the *semivariation of* m *with respect to* $\mathcal{E}$ is defined by $v_{\mathcal{E}}(m)(A) := \bigvee\{|m(B)| : B \in \mathcal{E}, B \subset A\}$, $A \in \Sigma$. We often denote by $v(m)$ the semivariation of m with respect to Σ.

(b) We say that m is k*-triangular* on Σ iff $m(A) - k\,m(B) \leq m(A \cup B) \leq m(A) + k\,m(B)$ whenever $A, B \in \Sigma$, $A \cap B = \emptyset$, and $0 = m(\emptyset) \leq m(A)$ for each $A \in \Sigma$.

Now we recall the following property of the semivariation.

Proposition 2.3 ([5, Proposition 1.4.10]) *If* $m : \Sigma \to R$ *is* k*-triangular, then so is* $v(m)$*.*

Definitions 2.4 (a) Let $\mathcal{E}$ be a sublattice of Σ. A set function $m : \Sigma \to R$ is said to be (s)*-bounded on* $\mathcal{E}$ iff for every disjoint sequence $(C_h)_h$ in $\mathcal{E}$ we get $(O)\lim_h v(m)(C_h) = 0$. We say that m is (s)*-bounded* iff it is (s)-bounded on Σ.

(b) The set functions $m_j : \Sigma \to R$, $j \in \mathbb{N}$, are *uniformly* (s)*-bounded on* $\mathcal{E}$ iff

$$(O)\lim_h \Big(\bigvee_{j=1}^{\infty} v(m_j)(C_h)\Big) = 0$$

for each disjoint sequence $(C_h)_h$ in $\mathcal{E}$. The m_j's are said to be *uniformly* (s)*-bounded* iff they are uniformly (s)-bounded on $\mathcal{E}$.

(c) A k-triangular set function $m : \Sigma \to R$ is *regular* iff for each $A \in \Sigma$ and $W \in \mathcal{H}$ there exist four sequences $(F_n)_n$, $(F'_n)_n$ in $\mathcal{H}$, $(G_n)_n$, $(G'_n)_n$ in $\mathcal{G}$, with

$$F_n \subset F_{n+1} \subset A \subset G_{n+1} \subset G_n \quad \text{for all } n \in \mathbb{N}, \tag{1}$$

$$W \subset F'_{n+1} \subset G'_n \subset F'_n \quad \text{for any } n \in \mathbb{N}, \tag{2}$$

and $\bigwedge_n \Big[v_\Sigma(m)(G_n \setminus F_n)\Big] = \bigwedge_n \Big[v_\Sigma(m)(G'_n \setminus W)\Big] = 0$.

(d) The k-triangular set functions $m_j : \Sigma \to R$, $j \in \mathbb{N}$, are *uniformly regular* iff for any $A \in \Sigma$ and $W \in \mathcal{H}$ there exist sequences $(F_n)_n$, $(G_n)_n$, $(F'_n)_n$, $(G'_n)_n$ satisfying (1) and (2), and with

$$\bigwedge_n \Big[\bigvee_j v_\Sigma(m_j)(G_n \setminus F_n)\Big] = 0.$$

(e) The set functions $m_j : \Sigma \to R$, $j \in \mathbb{N}$, are *equibounded* iff there is $u \in R$ with $|m_j(A)| \le u$ for all $j \in \mathbb{N}$ and $A \subset \Sigma$.

3 The main results

Before giving our main results on limit theorems and their equivalence, we present some notions and properties on k-triangular lattice group-valued set functions.

Proposition 3.1 ([5, Proposition 1.4.13]) *Let $m_j : \Sigma \to R$ be a sequence of equibounded set functions. Then the m_j's are k-triangular if and only if there is a nowhere dense set $N_* \subset \Omega$ such that the set functions $m_j(\cdot)(\omega)$ are real-valued and k-triangular for every $\omega \in \Omega \setminus N_*$.*

Observe that in general, differently from the finitely additive case, it is not true that any bounded k-triangular set function is (s)-bounded. For example, let $G = [1, 2]$, set

$$m(\emptyset) = 0 \text{ and } m(A) = \sup A \tag{3}$$

for each nonempty subset A of G. It is not difficult to see that m is subadditive, positive and monotone, and hence m is 1-triangular. However, for any disjoint sequence $(A_n)_n$ of nonempty subsets of G we get $m(A_n) \ge 1$ for every $n \in \mathbb{N}$, and so it is not possible to have $\lim_n m(A_n) = 0$. Thus m is not (s)-bounded.

We consider the *disjoint variation* of a lattice group-valued set function (see also [9, 12]).

Definitions 3.2 (a) Let us add to R an extra element $+\infty$, obeying to the usual rules. For any set function $m : \Sigma \to R$, define the *disjoint variation* $\overline{m} : \Sigma \to R \cup \{+\infty\}$ of m by

$$\overline{m}(A) := \bigvee_I \Big(\sum_{i \in I} |m(D_i)|\Big), \tag{4}$$

where the involved supremum is taken with respect to all disjoint finite families $\{D_i : i \in I\}$ with $D_i \subset A$ for each $i \in I$.

(b) A set function m is said to be *of bounded disjoint variation* on Σ (shortly, BDV) iff $\overline{m}(G) \in R$, where $\overline{m}$ is as in (4).

Some examples and properties of BDV set functions can be found, for instance, in [5, 9]. We now recall the following

Proposition 3.3 ([5, Proposition 1.4.18]) *Let $m : \Sigma \to R$ be a BDV set function, and $R \subset \mathcal{C}_\infty(\Omega)$. Then there is a meager set $N^* \subset \Omega$ such that the set function $m_\omega := m(\cdot)(\omega)$ is real-valued, BDV and (s)-bounded on Σ for every $\omega \in \Omega \setminus N^*$. Moreover, m is (s)-bounded on Σ.*

Our setting includes also the finitely additive case. Indeed we have the following

Proposition 3.4 ([5, Proposition 1.4.18]) *Every bounded finitely additive measure $m : \Sigma \to R$ is BDV.*

From now on, we assume that the involved set functions are BDV, without saying it explicitly.

We recall the next Brooks-Jewett-type theorem, which extends [3, Theorem 3.1] to the context of k-triangular set functions.

Theorem 3.5 ([5, Theorem 3.2.1]) *Let $m_j : \Sigma \to R$, $j \in \mathbb{N}$, be a sequence of k-triangular equibounded set functions. Suppose that there is a set function $m_0 : \mathcal{G} \to R$ such that the sequence $(m_j)_j$ (O)-converges to m_0 on $\mathcal{G}$ with respect to a single (O)-sequence.*

Then there is a meager subset $N \subset \Omega$ such that for every $\omega \in \Omega \setminus N$ the real-valued set functions $m_j(\cdot)(\omega)$, $j \in \mathbb{N}$, are uniformly (s)-bounded on $\mathcal{G}$ for ω belonging to the complement of a meager subset of Ω. Moreover, the m_j's are uniformly (s)-bounded on $\mathcal{G}$.

Before proving our versions of the Dieudonné theorem, we recall the following results, whose proof is analogous to those of [2, Proposition 2.6] and [3, Theorem 4.4], and that of [5, Theorem 3.2.6], respectively.

Theorem 3.6 *Let $m_j : \Sigma \to R$, $j \in \mathbb{N}$, be a sequence of equibounded, regular and k-triangular set functions, (O)-convergent to m_0 on $\mathcal{G}$ with respect to a single (O)-sequence, and let $A \in \Sigma$, $W \in \mathcal{H}$, $(F_n)_n$, $(F'_n)_n$ in $\mathcal{H}$, $(G_n)_n$, $(G'_n)_n$ in $\mathcal{G}$ satisfy* (1) *and* (2).

Then, $\bigwedge_n \Big[\bigvee_j v(m_j)(G_n \setminus F_n)\Big] = \bigwedge_n \Big[\bigvee_j v(m_j)(G'_n \setminus W)\Big] = 0.$

Theorem 3.7 *Let $(m_j)_j$ be a sequence of regular and k-triangular set functions, (O)-convergent to m_0 on $\mathcal{G}$ with respect to a single (O)-sequence. Then the following assertions hold.*

3.7.1) *The set functions m_j, $j \in \mathbb{N}$, are uniformly regular on Σ.*

3.7.2) *The sequence $(m_j(A))_j$ is (O)-Cauchy in R for each $A \in \Sigma$.*

3.7.3) *If* $m_0(A) := (O)\lim_j m_j(A)$*,* $A \in \Sigma$*, then* m_0 *is regular on* Σ*.*

Now we prove uniform (s)-boundedness of the m_j's on $\mathcal{H}$, extending [4, Theorem 3.1].

Theorem 3.8 *Let* $m_j : \Sigma \to R$*,* $j \in \mathbb{N}$*, be as in Theorem* 3.7*. Then the set functions* m_j*,* $j \in \mathbb{N}$*, are uniformly* (s)*-bounded on* $\mathcal{H}$*.*

Proof: Fix a disjoint sequence $(H_n)_n$ in $\mathcal{H}$, and put $W_n := \bigcup_{h=1}^{n} H_h$, $n \in \mathbb{N}$.

At the first step, in correspondence with W_1, by monotonicity of $v_{\mathcal{G}}$ and uniform regularity on $\mathcal{G}$ (which follows from Theorem 3.7) we find two sequences $(G^*_{l_1})_{l_1}$ in $\mathcal{G}$, $(F^*_{l_1})_{l_1}$ in $\mathcal{H}$ satisfying (2), that is with $W_1 \subset F^*_{l_1+1} \subset G^*_{l_1} \subset F^*_{l_1}$ for any $l_1 \in \mathbb{N}$, and with the property that

$$\bigwedge_{l_1} \left[\bigvee_j v_{\mathcal{G}}(m_j)(F^*_{l_1} \setminus W_1) \right] = \bigwedge_{l_1} \left[\bigvee_j v_{\mathcal{G}}(m_j)(G^*_{l_1} \setminus W_1) \right] = 0. \tag{5}$$

Let $\mathcal{D}_1 := \{F^*_{l_1}, G^*_{l_1} : l_1 \in \mathbb{N}\}$.

At the second step, we use again uniform regularity, "countably many times", that is taking in (2), instead of W, the sets $W_2^{l_1} := W_2 \cup F^*_{l_1}$, as l_1 varies in $\mathbb{N}$, where the $F^*_{l_1}$'s, $l_1 \in \mathbb{N}$, are the same as in (5).

So, for any $l_1 \in \mathbb{N}$, in correspondence with $W_2^{l_1}$, we determine two sequences $(G^{l_1}_{l_2})_{l_2}$ in $\mathcal{G}$, $(F^{l_1}_{l_2})_{l_2}$ in $\mathcal{H}$ (l_1 is fixed, l_2 variable), with $W_2 \subset W_2^{l_1} \subset F^{l_1}_{l_2+1} \subset G^{l_1}_{l_2} \subset F^{l_1}_{l_2}$ for any $l_2 \in \mathbb{N}$, and

$$\bigwedge_{l_2} \left[\bigvee_j v_{\mathcal{G}}(m_j)(F^{l_1}_{l_2} \setminus W_2^{l_1}) \right] = \bigwedge_{l_2} \left[\bigvee_j v_{\mathcal{G}}(m_j)(G^{l_1}_{l_2} \setminus W_2^{l_1}) \right] = 0.$$

Set $\mathcal{D}_2 := \{F^{l_1}_{l_2}, G^{l_1}_{l_2} : l_1, l_2 \in \mathbb{N}\}$.

By induction, taking into account the $(n-1)$-th step, we start with the sets $W_n^{l_1,l_2,\ldots,l_{n-1}} := W_n \cup F^*_{l_1} \cup F^{l_1}_{l_2} \cup \ldots \cup F^{l_1,l_2,\ldots,l_{n-1}}_{l_n}$, where $l_1, \ldots, l_n \in \mathbb{N}$. For every (fixed) $l_1, \ldots, l_{n-1} \in \mathbb{N}$, and we find two sequences $(G^{l_1,\ldots,l_{n-1}}_{l_n})_{l_n}$ in $\mathcal{G}$, $(F^{l_1,\ldots,l_{n-1}}_{l_n})_{l_n}$ in $\mathcal{H}$, as l_n varies in $\mathbb{N}$, with

$$W_n \subset W_n^{l_1,l_2,\ldots,l_{n-1}} \subset F^{l_1,l_2,\ldots,l_{n-1}}_{l_n+1} \subset G^{l_1,l_2,\ldots,l_{n-1}}_{l_n} \subset F^{l_1,l_2,\ldots,l_{n-1}}_{l_n} \quad \text{for all } l_n \in \mathbb{N},$$

$$\begin{aligned} & \bigwedge_{l_n} \left[\vee_j v_{\mathcal{G}}(m_j)(F^{l_1,\ldots,l_{n-1}}_{l_n} \setminus W_n^{l_1,\ldots,l_{n-1}}) \right] \\ = & \bigwedge_{l_n} \left[\vee_j v_{\mathcal{G}}(m_j)(G^{l_1,\ldots,l_{n-1}}_{l_n} \setminus W_n^{l_1,\ldots,l_{n-1}}) \right] = 0. \end{aligned} \tag{6}$$

For every $n \in \mathbb{N}$, set $\mathcal{D}_n := \{F^{l_1,\ldots,l_{n-1}}_{l_n}, G^{l_1,\ldots,l_{n-1}}_{l_n} : l_1, \ldots, l_n \in \mathbb{N}\}$. Let $\mathcal{D}$ be the algebra generated by the W_n's and the $\mathcal{D}_n$'s: note that $\mathcal{D}$ is countable. So there exists a meager set

$N_* \subset \Omega$ with the property that all the expressions in (5), (6) are equal to 0 even if we replace the lattice infima and suprema with the corresponding pointwise infima and suprema. Let N be a meager subset of Ω such that the real-valued set functions $m_j(\cdot)(\omega)$, $j \in \mathbb{N}$, are uniformly (s)-bounded on $\mathcal{G}$ for ω belonging to the complement of N (such a set exists, thanks to Theorem 3.5), and set $N_{**} := N \cup N_*$. Fix arbitrarily $\varepsilon > 0$ and $\omega \in \Omega \setminus N_{**}$. Now, taking into account Theorem 2.1, (5), (6) and monotonicity of v (since $\mathcal{D}$ is countable and we consider elements of $\mathcal{G}$), we prove by induction that for any $n \in \mathbb{N}$ there are $G_n, T_n \in \mathcal{G} \cap \mathcal{D}_n$, $F_n \in \mathcal{H} \cap \mathcal{D}_n$ with $W_n \subset G_n \subset F_n \subset T_n$, and

$$\begin{aligned} v(m_j(\cdot)(\omega))(F_n \setminus W_n) &\leq \frac{\varepsilon}{2^2} + \ldots + \frac{\varepsilon}{2^{n+1}} \\ &= \frac{\varepsilon}{4} \frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}} = \frac{\varepsilon}{2}\left(1 - \frac{1}{2^n}\right) \end{aligned} \tag{7}$$

for all $j \in \mathbb{N}$. By Theorem 2.1 and (5) there exists a triple (G_1, F_1, T_1) with

$$\begin{aligned} &G_1, T_1 \in \mathcal{G} \cap \mathcal{D}_1, F_1 \in \mathcal{H} \cap \mathcal{D}_1, W_1 \subset G_1 \subset F_1 \subset T_1, \\ &v(m_j(\cdot)(\omega))(F_1 \setminus W_1) \leq v(m_j(\cdot)(\omega))(T_1 \setminus W_1) \leq v(m_j)(T_1 \setminus W_1)(\omega) \leq \frac{\varepsilon}{2^2} \end{aligned}$$

whenever $j \in \mathbb{N}$. Moreover, if we suppose by induction the existence of a triple (G_n, F_n, T_n), with

$$\begin{aligned} &G_n, T_n \in \mathcal{G} \cap \mathcal{D}_n, F_n \in \mathcal{H} \cap \mathcal{D}_n, W_n \subset G_n \subset F_n \subset T_n, \\ &v(m_j(\cdot)(\omega))(F_n \setminus W_n) \leq v(m_j(\cdot)(\omega))(T_n \setminus W_n) \leq v(m_j)(T_n \setminus W_n)(\omega) \leq \frac{\varepsilon}{2}\left(1 - \frac{1}{2^n}\right) \end{aligned}$$

for all $j \in \mathbb{N}$, by Theorem 2.1 and (6) there is a triple $(G_{n+1}, F_{n+1}, T_{n+1})$ with

$$\begin{aligned} &G_{n+1}, T_{n+1} \in \mathcal{G} \cap \mathcal{D}_{n+1}, F_{n+1} \in \mathcal{H} \cap \mathcal{D}_{n+1}, W_{n+1} \cup F_n \subset G_{n+1} \subset F_{n+1} \subset T_{n+1}, \\ &v(m_j(\cdot)(\omega))(F_{n+1} \setminus (W_{n+1} \cup F_n)) \leq v(m_j(\cdot)(\omega))(T_{n+1} \setminus (W_{n+1} \cup F_n)) \\ \leq\ &v(m_j)(T_{n+1} \setminus (W_{n+1} \cup F_n))(\omega) \leq \frac{\varepsilon}{k\, 2^{n+2}} \end{aligned}$$

for any $j \in \mathbb{N}$. Now observe that, by Proposition 3.1, the set functions $m_j(\cdot)(\omega)$ are k-triangular, and hence, by Proposition 2.3, the set functions $v(m_j)(\cdot)(\omega)$ are k-triangular too. From this, taking into account monotonicity of the semivariation, it follows that

$$\begin{aligned} &v(m_j(\cdot)(\omega))(F_{n+1} \setminus W_{n+1}) \leq k\, v(m_j(\cdot)(\omega))(F_{n+1} \setminus (W_{n+1} \cup F_n)) + \\ + \ &v(m_j(\cdot)(\omega))(F_n \setminus W_{n+1}) \leq k \frac{\varepsilon}{k\, 2^{n+2}} + v(m_j(\cdot)(\omega))(F_n \setminus W_n) \\ \leq\ &\frac{\varepsilon}{2}\left(1 - \frac{1}{2^n}\right) + \frac{\varepsilon}{2^{n+2}} = \frac{\varepsilon}{2}\left(1 - \frac{1}{2^{n+1}}\right) \quad \text{for all } j \in \mathbb{N}. \end{aligned}$$

We now turn to the given disjoint sequence $(H_n)_n$. From (7), for every $j, n \in \mathbb{N}$, taking into account k-triangularity of the semivariation, we get:

$$\begin{aligned} 0 &\leq v(m_j(\cdot)(\omega))(H_{n+1}) \leq k\, v(m_j(\cdot)(\omega))(G_{n+1} \setminus F_n) + \\ &+ v(m_j(\cdot)(\omega))(F_n \setminus W_n) \leq k\, v(m_j(\cdot)(\omega))(G_{n+1} \setminus F_n) + \frac{\varepsilon}{2}. \end{aligned}$$

Note that the sets $G_{n+1} \setminus F_n$, $n \in \mathbb{N}$, belong to $\mathcal{G}$ and are pairwise disjoint. Since $\omega \in \Omega \setminus N_{**}$ and $N_{**} \supset N$, then it follows that for n large enough (depending on $\omega \in \Omega \setminus N_{**}$) it is

$$v(m_j(\cdot)(\omega))(G_{n+1} \setminus F_n) \leq \frac{\varepsilon}{2\,k}$$

for all $j \in \mathbb{N}$, and so

$$v(m_j(\cdot)(\omega))(H_n) \leq \varepsilon \tag{8}$$

for each $j \in \mathbb{N}$. Hence

$$\inf_s [\sup_{n \geq s} \{\sup_j (v(m_j(\cdot)(\omega))(H_n))\}] = 0 \tag{9}$$

for every $\omega \in \Omega \setminus N_{**}$. Since countable unions of meager sets are still meager sets, for any $n \in \mathbb{N}$ it is

$$\sup_j (v(m_j(\cdot)(\omega))(H_n)) = \{\bigvee_j (v(m_j)(H_n))\}(\omega) \tag{10}$$

for all $\omega \in \Omega \setminus \widehat{N}$, where $\widehat{N}$ is a suitable meager set. Without loss of generality, we can suppose $\widehat{N} \supset N_{**}$. From (9) and (10) it follows that, up to complements of meager sets,

$$\bigwedge_s [\bigvee_{n \geq s} \{(\bigvee_j (v(m_j)(H_n)))\}](\omega) = 0. \tag{11}$$

By a density argument, from (11) we obtain

$$\bigwedge_s [\bigvee_{n \geq s} (\bigvee_j v(m_j)(H_n))] = 0. \tag{12}$$

By the arbitrariness of the chosen sequence $(H_n)_n$, we get the uniform (s)-boundedness of the m_j's on $\mathcal{H}$. This ends the proof. □

Arguing as in [4, Theorem 3.2], it is possible to see that the m_j's are uniformly (s)-bounded on the whole of Σ.

Theorem 3.9 *Under the same hypotheses and notations as in Theorem 3.8, suppose also that R is weakly σ-distributive. Then the set functions m_j, $j \in \mathbb{N}$, are uniformly (s)-bounded on Σ.*

As a consequence of Theorems 3.5 and 3.9, we get the following Dieudonné-type theorem, which extends [4, Corollary 3.3] to the non-additive setting.

Theorem 3.10 *Let $m_j : \Sigma \to R$, $j \in \mathbb{N}$, be a sequence of equibounded, regular and k-triangular set functions, (O)-convergent to m_0 on $\mathcal{G}$ with respect to a single (O)-sequence, and R be as in Theorem 3.9. Then the m_j's are uniformly regular and uniformly (s)-bounded on Σ. Moreover, if $m_0(A) := (O) \lim_j m_j(A)$, $A \in \Sigma$, then m_0 is well-defined, (s)-bounded and regular on Σ.*

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