Quaternionic Zeta Function

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ABSTRACT. We introduce and suggest to study famous Zeta Function, extending it to a quaternionic variable and other hypercomplex variables.

1. Introduction

A well-known famous Zeta Function was first introduced and studied for a real variable. Then, it was extended to a complex variable.

We will recall its history and details and suggest to extend it to quaternion and other hypercomplex variables.

2. Real variable

Zeta Function that was first introduced and studied by Leonhard Euler in 1734 for solution of Basel problem, first posed by Pietro Mengoli in 1650, which asks for the precise sum of the infinite series:

$$\zeta(2) = (1)^{-2} + (2)^{-2} + (3)^{-2} + \dots \ .$$

Euler found the exact sum to be $\pi^2/6$ and announced his discovery in 17 35 and produced a truly rigorous proof in 1741. Complex analysis was not available at the time. In 1979 Roger Apéry proved irrationality of $\zeta(3)$.

For any positive even integer 2n:

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$\zeta(2n) = (-1)^{n+1} (2\pi)^{2n} B_{2n}(2n!)^{-1}/2,$

where B_{2n} is 2nth Bernoulli number.

For odd positive integers, no such simple expression is known.

For non positive integers one has:

$\zeta(-n) = (-1)^n (n+1)^{-1} B_{n+1}$

for $n \geq 0$ (using the convention that $B_1 = 1/2$).

For the negative even integers ζ is vanished because $B_n = 0$ for all odd $n \neq 1$(trivial zeros of zeta function).

Another specific values of Zeta function:

$\zeta(-1) = -1/12$, $\zeta(0) = -1/2$, $\zeta(1/2) \cong -1.4603545088095868129$,

$\zeta(1) = \infty$, $\zeta(3/2) \cong 2.6123753486854883433 5$,

$\zeta(3) \cong 1.2020569031595942854 0$, $\zeta(4) = \pi^4/90$, $\zeta(\infty) = 1$.

3. **Complex variable**

Later, Chebyshev extended the Euler definition to $Re(s) > 1$, $s = \sigma + \mathbf{i}t$, $\mathbf{i}^2 = -1$, $\sigma = Re(s)$, $t = Im(s)$, $s \in \mathbf{C}$, $\sigma \in \mathbf{R}$, $t \in \mathbf{R}$.

In 1859, Bernhard Riemann extended the Euler definition of Zeta funct-tion to a complex variable $s = \sigma + \mathbf{i}t$, $\mathbf{i}^2 = -1$:

$\zeta(s) = (1)^{-s} + (2)^{-s} + ... + (n)^{-s} + ...$.

The Riemann Zeta function is defined as the analytic continuation of the function defined for $\sigma > 1$ by the sum of the preceding series.

The Riemann Zeta function is a meromorphic function on the whole complex s-plane, which is holomorphic everywhere except for a simple pole at $s = 1$ with residue 1.

The Riemann Zeta function has trivial zeros at -2, -4, ... It is known that any non-trivial zero lies in the open strip $\{ s \in \mathbf{C}, 0 < \mathrm{Re}(s) < 1 \}$, which is called the critical strip. The famous Riemann hypothesis asserts that any non-trivial zero s has $\mathrm{Re}(s) = 1/2$. The set $\{ s \in \mathbf{C}, \mathrm{Re}(s) = 1/2 \}$ is called the critical line. In 1914 Hardy proved that $\zeta(1/2 + \mathbf{i}t)$ has infinitely many real zeros. There are no zeros of the Zeta function on the $\mathrm{Re}(s) = 1$. There are infinitely many zeros on the critical line (see [4, 8, 12, 14, 16]).

The Generalized Riemann hypothesis (GRH) for Drichlet L-functions was probably first formulated by Adolph Piltz in 1884.

The Extended Riemann hypothesis(ERH) asserts that for every number field K with ring of integers O_K and every complex number s with $\zeta_K(s) = 0$, wherein

$\zeta_K(s) = \Sigma(\mathrm{Na})^{-s}$, a is an ideal of O_K other than the zero ideal, Na is the norm of ideal of O_K: if $\{ s \in \mathbf{C}, 0 < \mathrm{Re}(s) < 1 \}$, then it is in fact 1/2.

Zeta function occurs in statistic, quantum field theory and is also useful for analysis of dynamical systems, and, of course, in Number Theory.

ZetaGrid was at one time the largest distributed computing project, designed to explore the non-trivial roots of the Riemann Zeta Function, checking over one billion roots a day. The project ended in 2005.

There are a number of generalizations of Zeta function: Hurwitz Zeta function, Drichlet L-functions, Dedekind Zeta function, Polylogarithm, Lerch Zeta function, Multiple Zeta function, Drichlet Eta function, Arithmetic Zeta function, Prime Zeta function, Xi function (see [2, 5, 8, 12, 14, 16]).

Let us introduce our contribution: the following two Zeta-like functions:

$\varphi(s) := (1)^{-s} + (2)^{-2s} + ... + (n)^{-ns} + ...$,

$\psi(s) := (1)^{-r} + (2)^{-r} + ... + (n)^{-r} + ...$, $r = s^n$.

Note that $\varphi(0) = \psi(0) = \zeta(0) = -1/2$, $\varphi(1) = \psi(1) = \zeta(1) = \infty$.

Similar to Zeta function, the aforementioned generalizations could be introduced for the functions $\varphi(s)$ and $\psi(s)$.

4. Next step: Quaternionic Zeta function

Quaternions are generally represented in the form: $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, where, $a \in \mathbf{R}$, $b \in \mathbf{R}$, $c \in \mathbf{R}$, $d \in \mathbf{R}$, and **i**, **j** and **k** are the fundamental quaternion units and are a number system that extends the complex numbers(see [1,6,11]). Quaternions find uses in both pure and applied mathematics: in three-dimensional computer graphics, computer vision, robotics, control theory, signal processing, attitude control, physics, bioinformatics, molecular dynamics, computer simulations, orbital mechanics, crystallographic texture analysis. In quantum mechanics, the spin of an electron and other matter particles can be described using quaternions. In 1999 is was shown that Einstein equations of general relativity could be formulated using quaternions.

The set of all quaternions **H** is a normed algebra, where the norm is multiplicative: $\| pq \| = \| p \| \| q \|$, $p \in \mathbf{H}$, $q \in \mathbf{H}$, $\| q \|^2 = a^2 + b^2 + c^2 + d^2$.

This norm makes it possible to define the distance $d(p, q) = \|p - q\|$ which makes **H** into a metric space.

As we see, there were no suggestions yet to extend Zeta function definition to a quaternionic or other hypercomplex variables.

Let us introduce Quaternionic Zeta function:

$\zeta(q) := (1)^{-q} + (2)^{-q} + ... + (n)^{-q} + ...$,

$q := a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, where, $a \in \mathbf{R}$, $b \in \mathbf{R}$, $c \in \mathbf{R}$, $d \in \mathbf{R}$, $q \in \mathbf{H}$.

Respectively, $\zeta_K(q) = \Sigma(Na)^{-q}$, $q \in \mathbf{H}$.

Note that $q^p = \exp(\ln(q)\, p)$, $p, q \in \mathbf{H}$.

Correspondingly,

$\varphi(q) := (1)^{-q} + (2)^{-2q} + \ldots + (n)^{-nq} + \ldots$,

$\psi(q) := (1)^{-r} + (2)^{-r} + \ldots + (n)^{-r} + \ldots$, $r = q^n$, $q \in \mathbf{H}$.

It would be important and interesting to explore and calculate, e.g.:

$\zeta(0 + 0\mathbf{i} + 0\mathbf{j} + 1\mathbf{k})$, $\zeta(0 + 0\mathbf{i} + 1\mathbf{j} + 0\mathbf{k})$, $\zeta(0 + 0\mathbf{i} + 1\mathbf{j} + 1\mathbf{k})$,
$\varphi(0 + 0\mathbf{i} + 0\mathbf{j} + 1\mathbf{k})$, $\varphi(0 + 0\mathbf{i} + 1\mathbf{j} + 0\mathbf{k})$, $\varphi(0 + 0\mathbf{i} + 1\mathbf{j} + 1\mathbf{k})$,
$\psi(0 + 0\mathbf{i} + 0\mathbf{j} + 1\mathbf{k})$, $\psi(0 + 0\mathbf{i} + 1\mathbf{j} + 0\mathbf{k})$, $\psi(0 + 0\mathbf{i} + 1\mathbf{j} + 1\mathbf{k})$,

as well as to explore the corresponding critical "zero sets" and Riemann-like hypothesis of the above introduced Quaternionic Zeta Functions.

Similarly we can define quaternionic extensions for Hurwitz Zeta function, Drichlet L-functions, Dedekind Zeta function, Polylogarithm, Lerch Zeta function, Multiple Zeta function, Drichlet Eta function, Arithmetic Zeta function, Prime Zeta function, Riemann Xi function, as well as for Gamma function, Laplace, Mellin and other transforms. The same can be done for octonions and other hypercomplex systems.

Correspondingly, we would like to inspire and motivate researchers to investigate properties of Quaternionic Zeta function and other aforementioned new-defined functions and transforms. In particular, it may help to solve Riemann hypothesis.

5. **Conclusions**

This is a pioneering work extending definition of Euler-Riemann Zeta function, its generalizations as well as some other functions and transforms to quaternionic and other hypercomplex variables.

It would stimulate researchers to develop the corresponding new methods and algorithms.

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