The Continuous Bernoulli approaching distribution when λ → 0 and Continuous Binomial distribution

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Abstract

We provide the mathematical deduction and numerical explanations to verify that as $\lambda \to 0$, the continuous Bernoulli approximates to the exponential distribution in Chapter 1 and as $\lambda \to 0$ and $\lambda \to 1$, the continuous binomial distribution will approximate to Gamma distribution in Chapter 3. Meanwhile, Chapter 2 describes how to compute the continuous Binomial distribution which can be derived by the continuous Bernoulli.

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Chapter 1 The Continuous Bernoulli approaching distribution when λ → 0

Section 1.The Continuous Bernoulli distribution will approach to exponential distribution when λ → 0

1. Continuous Bernoulli distribution

$$f_X(x;\lambda)=C(\lambda)\lambda^x(1-\lambda)^{1-x},0\le x\le 1,0<\lambda<1,$$

$$C(\lambda)=\begin{cases}\dfrac{2tanh^{-1}(1-2\lambda)}{1-2\lambda},\lambda\ne\dfrac{1}{2}\\ 2,\lambda=\dfrac{1}{2}\end{cases}$$

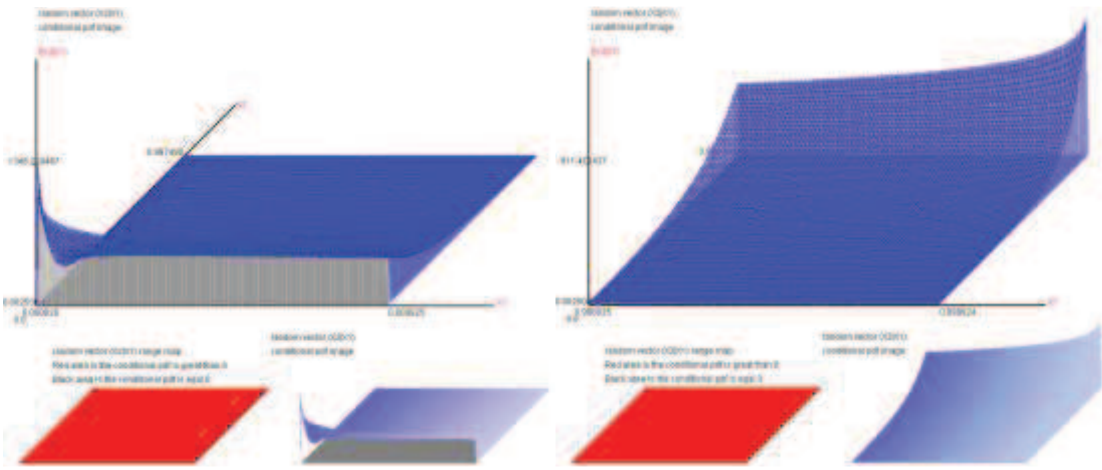
$$tanh^{-1}(x)=\frac{1}{2}log_e\left(\frac{1+x}{1-x}\right)=\frac{1}{2}ln\left(\frac{1+x}{1-x}\right),-1<x<1,$$

$$\mu=E(X)=\begin{cases}\dfrac{\lambda}{2\lambda-1}+\dfrac{1}{2tan^{-1}(1-2\lambda)} & if\ \lambda\ne\dfrac{1}{2}\\ \dfrac{1}{2} & if\ \lambda=\dfrac{1}{2}\end{cases}$$

$$\sigma^2=Var(X)=\begin{cases}\dfrac{(1-\lambda)\lambda}{(1-2\lambda)^2}+\dfrac{1}{\left(2tan^{-1}(1-2\lambda)\right)^2} & if\ \lambda\ne\dfrac{1}{2}\\ \dfrac{1}{12} & if\ \lambda=\dfrac{1}{2}\end{cases}$$

2. $\lambda\to 0$ or $\lambda\to 1$

(1) the diagram is (X1= λ , f(X2|X1)) and X2=X ~ $CB(\lambda)$,



$0.000001\le\lambda\le 0.01$ $\qquad$ $0.99\le\lambda\le 0.999999$

The red area is the range of (X, λ).

Section 2. The approaching distribution

1. The exponential distribution

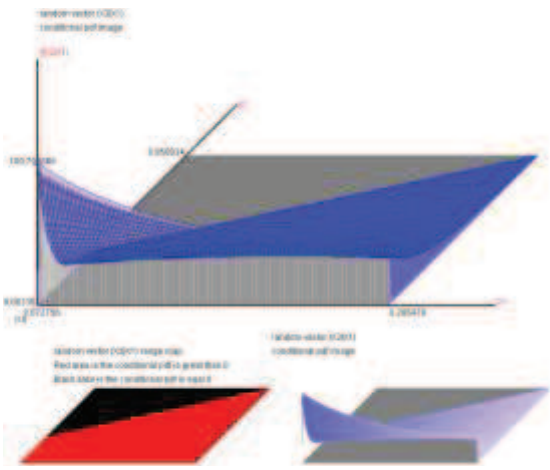
$$f_X(x;\lambda)=\frac{ln\left(\frac{1-\lambda}{\lambda}\right)\lambda^x(1-\lambda)^{1-x}}{1-2\lambda}=\frac{\lambda\, ln\left(\frac{1-\lambda}{\lambda}\right)\lambda^{-1+x}(1-\lambda)^{1-x}}{1-2\lambda}=\frac{\lambda\left(\frac{1-\lambda}{\lambda}\right)^{1-x}ln\left(\frac{1-\lambda}{\lambda}\right)}{1-2\lambda}$$

$$\xrightarrow[\lambda\to 0]{} f_X(x;\mu)=\frac{1}{\mu}exp\left(-\frac{1}{\mu}x\right)=\frac{1}{\frac{\lambda}{2\lambda-1}+ln\left(\frac{1-\lambda}{\lambda}\right)}exp\left(-\frac{x}{\frac{\lambda}{2\lambda-1}+ln\left(\frac{1-\lambda}{\lambda}\right)}\right),$$

$$0<x<1, X\sim exponential\left(\frac{1}{\mu}\right),$$

$$\mu=E(X)=\frac{\lambda}{2\lambda-1}+\frac{1}{2\,tan^{-1}(1-2\lambda)}=\frac{\lambda}{2\lambda-1}+ln\left(\frac{1-\lambda}{\lambda}\right), \lambda\neq\frac{1}{2}.$$

2.The diagram is (X1=μ , f(X2|X1)) and X2~$exponential\left(\frac{1}{\mu}=\frac{1}{E(X)}\right)$,



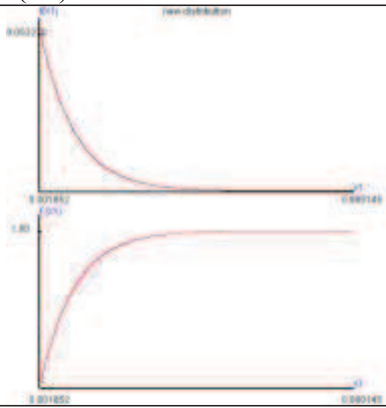
The red area is the range of (X, λ).

$X\sim CB(\lambda=0.01)$, μ =0.20749, $X\sim CB(\lambda=0.000001)$, μ =0.07242,

$X\sim CB(\lambda=0.99)$, μ =0.79260, $X\sim CB(\lambda=0.999999)$, μ =0.92763,

Section 3. The numerical explanations

(3-1) λ =0.0001,

f(x1),F(x1)	Coefficient
	Mathematical Mean: 0.10852 Geometrical Mean : 0.06096 Harmonic Mean : 0.00562 Variance : 0.01170 S.D. : 0.10819 Skewed Coef. : 1.94678 Kurtosis Coef. : 8.36733 MAD : 0.07981 Range : 1.00000 Mid_range : 0.50000 Median : 0.07525 Q1 : 0.03124 Q2 : 0.07525 Q3 : 0.15052 IQR : 0.11927 C.V. : 0.99692

X1~ $f_X(x;\lambda)=C(\lambda)\lambda^x(1-\lambda)^{1-x}, 0\le x\le 1, 0<\lambda<1,$ Continuous Bernoulli,

λ =0.0001,E(X)= 0.10852,

X2~ $f_X(x;\mu)=\frac{1}{\mu}exp\left(-\frac{1}{\mu}x\right), 0<x<1, X\sim exponential\left(\lambda=\frac{1}{\mu}\right),$

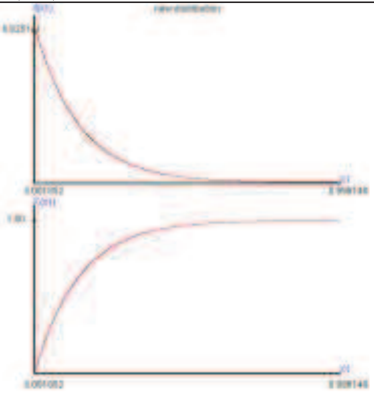
$\frac{1}{\mu}$ =1/E(X)=9.214891264283, SLLN,

E(| X1 distribution - X2 distribution |^2)=0.0000038079
*********** | X1 distribution function - X2 distribution function| ***********
The almost surely limiting theory
E(| X1 distribution function - X2 distribution function|^2)=0.0000000315,
Pr(| X1 distribution function - X2 distribution function|<0.1000000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0500000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0100000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0050000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0010000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0005000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0001000000)= 0.268422,

Note:
SLLN is Strong Large Law of Number, the simulated data number of X1 and X2 is 100,000,000, the amount is closed to the population number.

When
Pr(| X1 distribution function - X2 distribution function|<0.1000000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0500000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0100000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0050000000)= 1.000000,
X1 is approaching to X.

(3-2) λ =0.001,

f(x1),F(x1)	Coefficient
	Mathematical Mean: 0.14380 Geometrical Mean : 0.08108 Harmonic Mean : 0.00698 Variance : 0.01996 S.D. : 0.14129 Skewed Coef. : 1.79554 Kurtosis Coef. : 7.07935 MAD : 0.10538 Range : 1.00000 Mid_range : 0.50000 Median : 0.10022 Q1 : 0.04160 Q2 : 0.10022 Q3 : 0.20030 IQR : 0.15870 C.V. : 0.98257

X1~ $f_X(x;\lambda)=C(\lambda)\lambda^x(1-\lambda)^{1-x}, 0\le x\le 1, 0<\lambda<1,$ Continuous Bernoulli,

λ =0.001,E(X)= 0.14380,

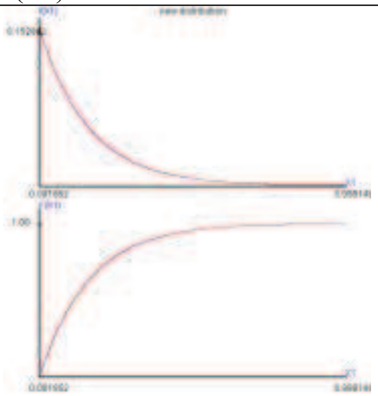
X2~ $f_X(x;\mu)=\frac{1}{\mu}exp\left(-\frac{1}{\mu}x\right), 0<x<1, X\sim exponential\left(\lambda=\frac{1}{\mu}\right),$

μ =1/E(X)= 6.954102920723,

SLLN,

E(| X1 distribution - X2 distribution |^2)=0.0000620743
*********** | X1 distribution function - X2 distribution function| ***********
The almost surely limiting theory
E(| X1 distribution function - X2 distribution function|^2)=0.0000026652,
Pr(| X1 distribution function - X2 distribution function|<0.1000000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0500000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0100000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0050000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0010000000)= 0.266056,
Pr(| X1 distribution function - X2 distribution function|<0.0005000000)= 0.136318,
Pr(| X1 distribution function - X2 distribution function|<0.0001000000)= 0.030575,

(3-3) λ =0.002,

f(x1),F(x1)	Coefficient
	Mathematical Mean: 0.15902 Geometrical Mean : 0.08994 Harmonic Mean : 0.00839 Variance : 0.02392 S.D. : 0.15467 Skewed Coef. : 1.71093 Kurtosis Coef. : 6.48387 MAD : 0.11619 Range : 1.00000 Mid_range : 0.50000 Median : 0.11126 Q1 : 0.04621 Q2 : 0.11126 Q3 : 0.22225 IQR : 0.17603 C.V. : 0.97267

X1~ $f_X(x;\lambda)=C(\lambda)\lambda^x(1-\lambda)^{1-x}, 0 \le x \le 1, 0<\lambda<1,$

Continuous Bernoulli, λ =0.002,E(X)=0.15902,

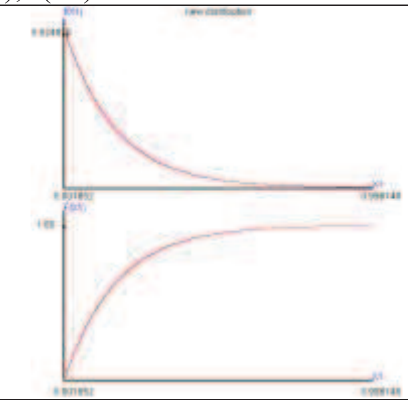
X2~ $f_X(x;\mu)=\frac{1}{\mu}exp\left(-\frac{1}{\mu}x\right), 0<x<1, X \sim exponential\left(\lambda=\frac{1}{\mu}\right),$

μ =1/E(X)= 6.288517167652,

SLLN,

E(| X1 distribution - X2 distribution |^2)=0.0001440610
*********** | X1 distribution function - X2 distribution function| ***********
The almost surely limiting theory
E(| X1 distribution function - X2 distribution function|^2)=0.0000063764,
Pr(| X1 distribution function - X2 distribution function|<0.1000000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0500000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0100000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0050000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0010000000)= 0.167050,
Pr(| X1 distribution function - X2 distribution function|<0.0005000000)= 0.086729,
Pr(| X1 distribution function - X2 distribution function|<0.0001000000)= 0.021327,

(3-4) λ =0.0025,

f(x1),F(x1)	Coefficient
	Mathematical Mean: 0.16446 Geometrical Mean : 0.09316 Harmonic Mean : 0.00767 Variance : 0.02536 S.D. : 0.15925 Skewed Coef. : 1.67815 Kurtosis Coef. : 6.27293 MAD : 0.11999 Range : 1.00000 Mid_range : 0.50000 Median : 0.11532 Q1 : 0.04789 Q2 : 0.11532 Q3 : 0.23020 IQR : 0.18232 C.V. : 0.96833

X1~ $f_X(x;\lambda)=C(\lambda)\lambda^x(1-\lambda)^{1-x}, 0\le x\le 1, 0<\lambda<1,$ Continuous Bernoulli,

λ =0.0025,E(X)=0.16446,

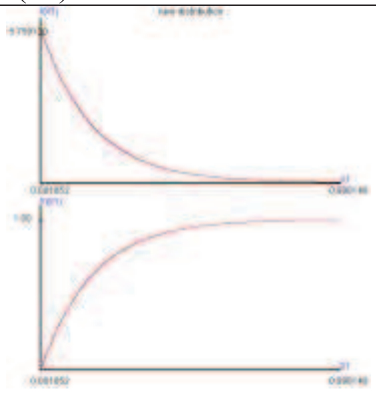
X2~ $f_X(x;\mu)=\frac{1}{\mu}exp\left(-\frac{1}{\mu}x\right), 0<x<1, X\sim exponential\left(\lambda=\frac{1}{\mu}\right),$

μ =1/E(X)= 6.080505898091,

SLLN,

E(| X1 distribution - X2 distribution |^2)=0.0001866097
*********** | X1 distribution function - X2 distribution function| ***********
The almost surely limiting theory
E(| X1 distribution function - X2 distribution function|^2)=0.0000091538,
Pr(| X1 distribution function - X2 distribution function|<0.1000000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0500000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0100000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0050000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0010000000)= 0.134573,
Pr(| X1 distribution function - X2 distribution function|<0.0005000000)= 0.059289,
Pr(| X1 distribution function - X2 distribution function|<0.0001000000)= 0.012352,

(3-5) λ=0.003,

f(x1),F(x1)	Coefficient
	Mathematical Mean: 0.16926 Geometrical Mean : 0.09599 Harmonic Mean : 0.00861 Variance : 0.02665 S.D. : 0.16326 Skewed Coef. : 1.64972 Kurtosis Coef. : 6.09432 MAD : 0.12333 Range : 1.00000 Mid_range : 0.50000 Median : 0.11888 Q1 : 0.04938 Q2 : 0.11888 Q3 : 0.23726 IQR : 0.18788 C.V. : 0.96458

X1~ $f_X(x;\lambda)=C(\lambda)\lambda^x(1-\lambda)^{1-x}, 0 \le x \le 1, 0<\lambda<1,$ Continuous Bernoulli,

λ=0.003,E(X)=0.16926,

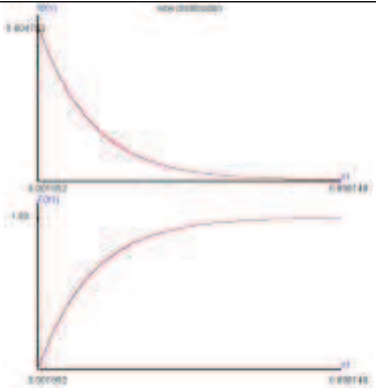
X2~ $f_X(x;\mu)=\frac{1}{\mu}exp\left(-\frac{1}{\mu}x\right), 0<x<1, X \sim exponential\left(\lambda=\frac{1}{\mu}\right),$

μ=1/E(X)= 5.908070424199,

SLLN,

E(| X1 distribution - X2 distribution |^2)=0.0002319264
*********** | X1 distribution function - X2 distribution function| ***********
The almost surely limiting theory
E(| X1 distribution function - X2 distribution function|^2)=0.0000118559,
Pr(| X1 distribution function - X2 distribution function|<0.1000000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0500000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0100000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0050000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0010000000)= 0.116617,
Pr(| X1 distribution function - X2 distribution function|<0.0005000000)= 0.058436,
Pr(| X1 distribution function - X2 distribution function|<0.0001000000)= 0.010514,

(3-6) λ =0.0035,

f(x1),F(x1)	Coefficient
	Mathematical Mean: 0.17349 Geometrical Mean : 0.09852 Harmonic Mean : 0.00912 Variance : 0.02781 S.D. : 0.16676 Skewed Coef. : 1.62462 Kurtosis Coef. : 5.94059 MAD : 0.12627 Range : 1.00000 Mid_range : 0.50000 Median : 0.12204 Q1 : 0.05072 Q2 : 0.12204 Q3 : 0.24350 IQR : 0.19278 C.V. : 0.96117

X1~ $f_X(x;\lambda) = C(\lambda)\lambda^x(1-\lambda)^{1-x}, 0 \le x \le 1, 0 < \lambda < 1,$ Continuous Bernoulli,

λ =0.0035,E(X)= 0.17349,

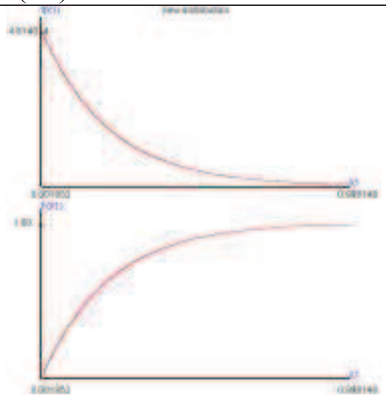
X2~ $f_X(x;\mu) = \frac{1}{\mu}exp\left(-\frac{1}{\mu}x\right), 0 < x < 1, X \sim exponential\left(\lambda = \frac{1}{\mu}\right),$

μ =1/E(X)= 5.764020981036,

SLLN,

E(| X1 distribution - X2 distribution |^2)=0.0002820432
*********** | X1 distribution function - X2 distribution function| ***********
The almost surely limiting theory
E(| X1 distribution function - X2 distribution function|^2)=0.0000133607,
Pr(| X1 distribution function - X2 distribution function|<0.1000000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0500000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0100000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0050000000)= 0.873428,
Pr(| X1 distribution function - X2 distribution function|<0.0010000000)= 0.128933,
Pr(| X1 distribution function - X2 distribution function|<0.0005000000)= 0.064557,
Pr(| X1 distribution function - X2 distribution function|<0.0001000000)= 0.011957,

(3-7) λ =0.01,

f(x1),F(x1)	Coefficient
	Mathematical Mean: 0.20749 Geometrical Mean : 0.11939 Harmonic Mean : 0.01146 Variance : 0.03708 S.D. : 0.19255 Skewed Coef. : 1.41496 Kurtosis Coef. : 4.82711 MAD : 0.14896 Range : 1.00000 Mid_range : 0.50000 Median : 0.14870 Q1 : 0.06188 Q2 : 0.14870 Q3 : 0.29532 IQR : 0.23344 C.V. : 0.92802

X1~ $f_X(x;\lambda)=C(\lambda)\lambda^x(1-\lambda)^{1-x}, 0 \le x \le 1, 0<\lambda<1,$ Continuous Bernoulli,

λ =0.01,E(X)= 0.20749,

X2~ $f_X(x;\mu)=\frac{1}{\mu}exp\left(-\frac{1}{\mu}x\right), 0<x<1, X \sim exponential\left(\lambda=\frac{1}{\mu}\right),$

μ =1/E(X)= 4.819509373946,

SLLN,

E(| X1 distribution - X2 distribution |^2)=0.0010025140
*********** | X1 distribution function - X2 distribution function| ***********
The almost surely limiting theory
E(| X1 distribution function - X2 distribution function|^2)=0.0000705031,
Pr(| X1 distribution function - X2 distribution function|<0.1000000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0500000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0100000000)= 0.642854,
Pr(| X1 distribution function - X2 distribution function|<0.0050000000)= 0.266286,
Pr(| X1 distribution function - X2 distribution function|<0.0010000000)= 0.048885,
Pr(| X1 distribution function - X2 distribution function|<0.0005000000)= 0.024426,
Pr(| X1 distribution function - X2 distribution function|<0.0001000000)= 0.005207,

Section 4. The requirement

(1) $\lambda \le 0.01$,

$$f_X(x;\lambda) = C(\lambda)\lambda^x(1-\lambda)^{1-x}, 0 \le x \le 1, 0 < \lambda < 1,$$

$$\mu = E(X) = \frac{\lambda}{2\lambda - 1} + \frac{1}{2\tan^{-1}(1-2\lambda)} = \frac{\lambda}{2\lambda - 1} + ln\left(\frac{1-\lambda}{\lambda}\right),$$

$$f_X(x;\lambda) \xrightarrow[\lambda \le 0.01]{} f_X(x;\mu) = \frac{1}{\mu} exp\left(-\frac{1}{\mu}x\right), 0 < x < 1, X \sim exponential\left(\frac{1}{\mu}\right)$$

$$F_X(x) \xrightarrow[\lambda \le 0.01]{} 1 - exp\left(-\frac{1}{\mu}x\right),$$

(2) $\lambda \ge 0.99$,

$$f_X(x;\lambda) = C(\lambda)\lambda^x(1-\lambda)^{1-x}, 0 \le x \le 1, 0 < \lambda < 1,$$

$$Y = 1 - X, f_Y(y;\lambda) = C(1-\lambda)(1-\lambda)^y \lambda^{1-y}, 0 \le y \le 1, 0 < \lambda < 1,$$

$$\mu = E(Y) = \frac{1-\lambda}{2(1-\lambda)-1} + ln\left(\frac{\lambda}{1-\lambda}\right) = \frac{1-\lambda}{1-2\lambda} + ln\left(\frac{\lambda}{1-\lambda}\right),$$

$$f_y(y;\lambda) \xrightarrow[1-\lambda \le 0.01]{} f_Y(y;\mu) = \frac{1}{\mu} exp\left(-\frac{1}{\mu}y\right), 0 < y < 1, Y \sim exponential\left(\frac{1}{\mu}\right),$$

$$F_Y(y) \xrightarrow[1-\lambda \le 0.01]{} 1 - exp\left(-\frac{1}{\mu}y\right),$$

$$F_X(x) = P(X \le x) = P(Y \ge 1-x) = 1 - F_Y(1-x) = exp\left(-\frac{1}{\mu}(1-x)\right),$$

Chapter 2 The Continuous Binomial distribution

$X_1, X_2, ..., X_n \overset{iid}{\sim}$ Continuous Bernoulli(λ),

$\sum_{i=1}^{n} X_i$ is Continuous Binomial(n, λ).

Section 1. The pdf of $X = X_1 + X_2 + \ldots + X_n$, $n = 2, 3, 4, 5$

1. n=2

The probability density function,

$$f_{X_1}(x_1; \lambda, n) = C(\lambda)\lambda^{x_1}(1-\lambda)^{1-x_1}, 0 \le x_1 \le 1, 0 < \lambda < 1,$$

$$f_{X_2}(x_2; \lambda, n) = C(\lambda)\lambda^{x_2}(1-\lambda)^{1-x_2}, 0 \le x_2 \le 1, 0 < \lambda < 1,$$

$X_{,1}, X_2$ are independent random variables,

$$f_{X_1,X_2}(x_1, x_2; \lambda, n) = f_{X_1}(x_1; \lambda, n) f_{X_2}(x_2; \lambda, n)$$

$$= (C(\lambda))^2 \lambda^{x_1+x_2}(1-\lambda)^{2-x_1-x_2}, 0 \le x_1 \le, 0 \le x_2 \le 1,$$

$$f_{X_1,X}(x_1, x; \lambda, n) = f_{X_1,X_2}(x_1, x_2 = x - x_1; \lambda, n),$$

$$= (C(\lambda))^2 \lambda^{x}(1-\lambda)^{2-x} \times \frac{\partial(x_1, x_2)}{\partial(x_1, x)}, \frac{\partial(x_1, x_2)}{\partial(x_1, x)} = 1,$$

$$X = X_{,1} + X_2, 0 < x_2 = x - x_1 < 1,$$

$$max(0, x-1) < x_1 < min(1, x), 0 \le x \le 2,$$

$$\begin{cases} 0 < x_1 < x & if\ 0 \le x \le 1, \\ x - 1 < x_1 < 1 & if\ 1 \le x \le 2, \end{cases}$$

$$f_X(x; \lambda, n) = \int_{max(0,x-1)}^{min(1,x)} (C(\lambda))^2 \lambda^{x}(1-\lambda)^{2-x} dx_1$$

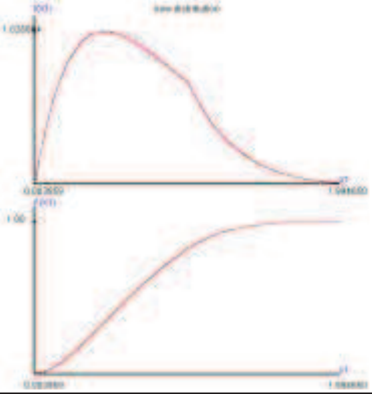
$$\begin{cases} f_X(x; \lambda, n) = (C(\lambda))^2 \lambda^{x}(1-\lambda)^{2-x} \int_0^x 1dx_1 & if\ 0 \le x \le 1, \\ f_X(x; \lambda, n) = (C(\lambda))^2 \lambda^{x}(1-\lambda)^{2-x} \int_{x-1}^1 1dx_1 & if\ 1 \le x \le 2, \end{cases}$$

$$f_X(x; \lambda, n) = \begin{cases} (C(\lambda))^2 x\lambda^{x}(1-\lambda)^{2-x} & if\ 0 \le x \le 1 \\ (C(\lambda))^2 (2-x)\lambda^{x}(1-\lambda)^{2-x} & if\ 1 \le x < 2 \end{cases}$$

for example, $\lambda = \frac{1}{2}$,

$$f_X(x; \lambda, n) = \begin{cases} x & if\ 0 \le x \le 1 \\ 2 - x & if\ 1 \le x \le 2 \end{cases}$$

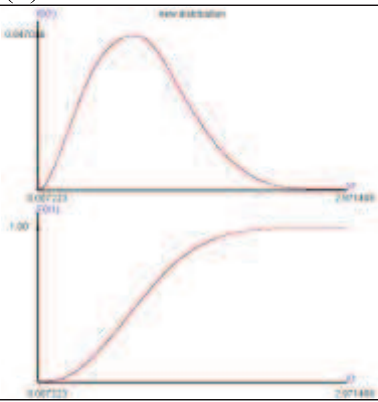
λ =0.1,n=2, $X = X_1 + X_2 + ... + X_n$,

f(x), F(x)	Coefficient
	Mathematical Mean: 0.66038 Geometrical Mean : 0.54178 Harmonic Mean : 0.38075 Variance : 0.13309 S.D. : 0.36481 Skewed Coef. : 0.52557 Kurtosis Coef. : 2.78969 MAD : 0.29911 Range : 1.99819 Mid_range : 0.99925 Median : 0.62012 Q1 : 0.37209 Q2 : 0.62012 Q3 : 0.90821 IQR : 0.53612 C.V. : 0.55243

2. n=3

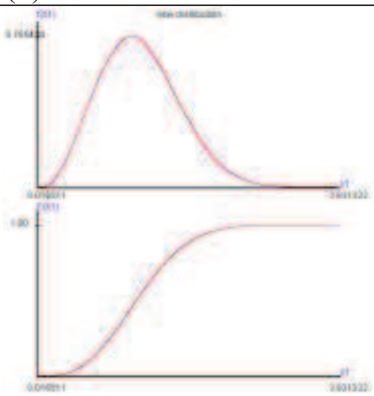
$$f_X(x;\lambda,n) = \begin{cases} (C(\lambda))^3 \dfrac{x^2}{2} \lambda^x (1-\lambda)^{3-x} & if\ 0 \le x \le 1 \\ (C(\lambda))^3 \dfrac{(-2x^2 + 6x - 3)}{2} (2-x) \lambda^x (1-\lambda)^{3-x} & if\ 1 \le x < 2 \\ (C(\lambda))^3 \dfrac{(2-x)^2}{2} \lambda^x (1-\lambda)^{3-x} & if\ 2 \le x \le 3 \end{cases}$$

λ =0.1,n=3, $X = X_1 + X_2 + ... + X_n$

f(x), F(x)	Coefficient
	Mathematical Mean: 0.99053 Geometrical Mean : 0.87677 Harmonic Mean : 0.73594 Variance : 0.19966 S.D. : 0.44683 Skewed Coef. : 0.42949 Kurtosis Coef. : 2.86040 MAD : 0.36187 Range : 2.97520 Mid_range : 1.48932 Median : 0.95720 Q1 : 0.65421 Q2 : 0.95720 Q3 : 1.28357 IQR : 0.62936 C.V. : 0.45110

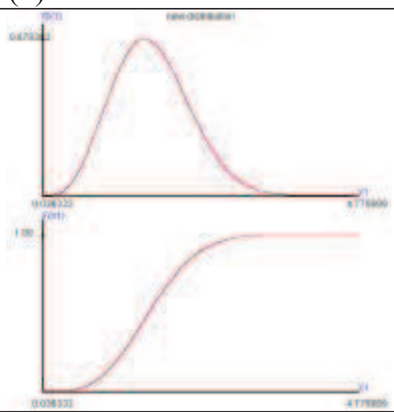
3.n=4,

$$f_X(x;\lambda,n)=\begin{cases}(C(\lambda))^4\,\dfrac{x^3}{6}\lambda^x(1-\lambda)^{4-x} & if\ 0\le x\le 1\\ (C(\lambda))^4\,\dfrac{(-3x^3+12x^2-12x+4)}{6}(2-x)\lambda^x(1-\lambda)^{4-x} & if\ 1\le x<2\\ (C(\lambda))^4\,\dfrac{(3x^3-24x^2+60x-44)}{6}\lambda^x(1-\lambda)^{4-x} & if\ 2\le x\le 3\\ (C(\lambda))^4\,\dfrac{(4-x)^3}{6}\lambda^x(1-\lambda)^{4-x} & if\ 3\le x\le 4\end{cases}$$

f(x), F(x)	Coefficient
	Mathematical Mean: 1.32053 Geometrical Mean : 1.20985 Harmonic Mean : 1.08000 Variance : 0.26608 S.D. : 0.51583 Skewed Coef. : 0.37208 Kurtosis Coef. : 2.89474 MAD : 0.41595 Range : 3.92936 Mid_range : 1.97392 Median : 1.28631 Q1 : 0.94296 Q2 : 1.28631 Q3 : 1.65965 IQR : 0.71668 C.V. : 0.39062

4. n=5,

$$f_X(x;\lambda,n)=\begin{cases}(C(\lambda))^5\,\dfrac{x^4}{24}\lambda^x(1-\lambda)^{5-x} & if\ 0\le x\le 1\\ (C(\lambda))^5\,\dfrac{(-4x^4+20x^3-30x^2+20x-5)}{24}(2-x)\lambda^x(1-\lambda)^{5-x} & if\ 1\le x<2\\ (C(\lambda))^5\,\dfrac{(6x^4-60x^3+210x^2-330x+155)}{24}\lambda^x(1-\lambda)^{5-x} & if\ 2\le x\le 3\\ (C(\lambda))^5\,\dfrac{(-4x^4+60x^3-330x^2+780x-655)}{24}\lambda^x(1-\lambda)^{5-x} & if\ 3\le x\le 4\\ (C(\lambda))^5\,\dfrac{(5-x)^4}{24}\lambda^x(1-\lambda)^{5-x} & if\ 4\le x\le 5\end{cases}$$

f(x), F(x)	Coefficient
	Mathematical Mean: 1.65072 Geometrical Mean : 1.54198 Harmonic Mean : 1.41864 Variance : 0.33267 S.D. : 0.57677 Skewed Coef. : 0.33307 Kurtosis Coef. : 2.91623 MAD : 0.46410 Range : 4.75698 Mid_range : 2.40601 Median : 1.61668 Q1 : 1.23424 Q2 : 1.61668 Q3 : 2.03011 IQR : 0.79587 C.V. : 0.34941

Section 2. The Continuous Binomial distribution

$X \sim$ Continuous Binomial distribution(λ),
The pdf of Continuous Binomial distribution(λ) is

$$f_X(x;\lambda,n) = h(x)(C(\lambda))^n \lambda^x (1-\lambda)^{n-x}, 0 \le x \le n, 0 < \lambda < 1.$$

and $X = \sum_{i=1}^{n} X_i \xrightarrow[n\to\infty]{} Normal\left(E(X) = n\mu, Var(X) = n\sigma^2\right).$

$$\mu = E(X) = \begin{cases} \dfrac{\lambda}{2\lambda-1} + \dfrac{1}{2\tan^{-1}(1-2\lambda)} & if\ \lambda \ne \dfrac{1}{2} \\ \dfrac{1}{2} & if\ \lambda = \dfrac{1}{2} \end{cases}$$

$$\sigma^2 = Var(X) = \begin{cases} \dfrac{(1-\lambda)\lambda}{(1-2\lambda)^2} + \dfrac{1}{\left(2\tan^{-1}(1-2\lambda)\right)^2} & if\ \lambda \ne \dfrac{1}{2} \\ \dfrac{1}{12} & if\ \lambda = \dfrac{1}{2} \end{cases}$$

Note:

$X_1, X_2, \ldots, X_n \overset{iid}{\sim}$ Uniform($\alpha = 0, \beta = 1$),

$X = X_1 + X_2 + \ldots + X_n, h(x)$ is Irwin-hall distribution and parameter n .

Section 3. The simulator of $\sum_{i=1}^{n} X_i$

The Continuous Bernoulli simulated data $x(RND,\lambda)$ when random number= RND and parameter is λ,

$$x(RND,\lambda)=\begin{cases}\dfrac{log_e(RND\times(2\lambda-1)-(\lambda-1))-log_e(1-\lambda)}{log_e\left(\dfrac{\lambda}{1-\lambda}\right)},\lambda\neq\dfrac{1}{2}\\ RND,\lambda=\dfrac{1}{2}\end{cases},$$

(1)The simulation process

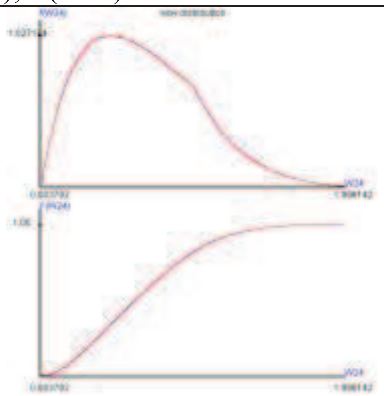
(i) Getting random number, $RND_1, RND_2,....., RND_n$ are independently,

(ii) $x_1(RND_1,\lambda), x_2(RND_2,\lambda),....., x_n(RND_n,\lambda)$

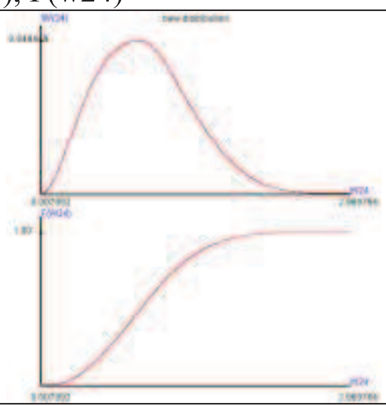
(iii) $x_j=\sum_{i=1}^{n} x_i(RND_i,\lambda)$, j=1,2,…..,100000000,

Repeat (i)~(iii) 100000000 times, the database of simulated data will be gotten.
This database can convert frequency distribution and $E(X)$, $Var(X),\gamma_1(X),\gamma_2(X)$,
This database is approached to Continuous Binomial distribution(λ).

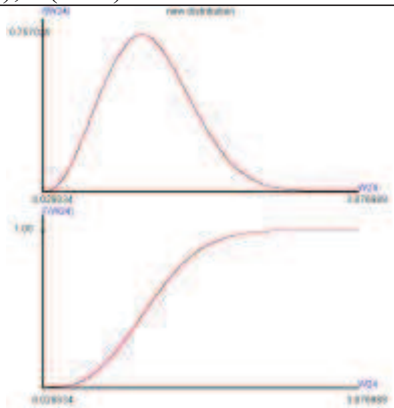
(1)n=2, λ =0.1,W24= $X_1+X_2+....+X_n$,

f(w24), F(w24)	Coefficient
	Mathematical Mean: 0.66035 Geometrical Mean : 0.54183 Harmonic Mean : 0.38106 Variance : 0.13306 S.D. : 0.36478 Skewed Coef. : 0.52586 Kurtosis Coef. : 2.79028 MAD : 0.29908 Range : 1.99976 Mid_range : 0.99997 Median : 0.62011 Q1 : 0.37209 Q2 : 0.62011 Q3 : 0.90807 IQR : 0.53598 C.V. : 0.55240

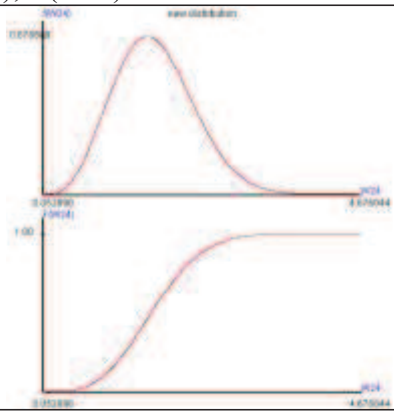
(2)n=3, λ =0.1,W24= $X_1+X_2+....+X_n$,

f(w24), F(w24)	Coefficient
	Mathematical Mean: 0.99087 Geometrical Mean : 0.87710 Harmonic Mean : 0.73638 Variance : 0.19978 S.D. : 0.44696 Skewed Coef. : 0.42939 Kurtosis Coef. : 2.85937 MAD : 0.36198 Range : 2.97278 Mid_range : 1.48888 Median : 0.95754 Q1 : 0.65442 Q2 : 0.95754 Q3 : 1.28397 IQR : 0.62955 C.V. : 0.45108

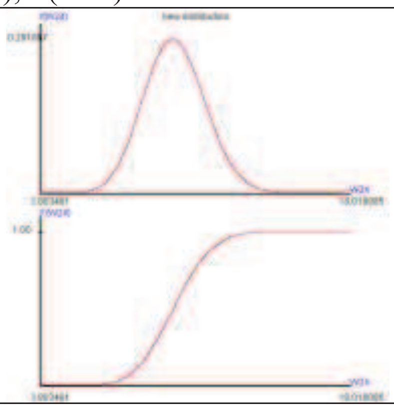
(3)n=4, λ =0.1,W24= $X_1 + X_2 + ... + X_n$,

f(w24), F(w24)	Coefficient	
	Mathematical Mean:	1.32058
	Geometrical Mean :	1.20983
	Harmonic Mean :	1.07988
	Variance :	0.26622
	S.D. :	0.51597
	Skewed Coef. :	0.37193
	Kurtosis Coef. :	2.89496
	MAD :	0.41608
	Range :	3.86326
	Mid_range :	1.95251
	Median :	1.28629
	Q1 :	0.94276
	Q2 :	1.28629
	Q3 :	1.65998
	IQR :	0.71722
	C.V. :	0.39071

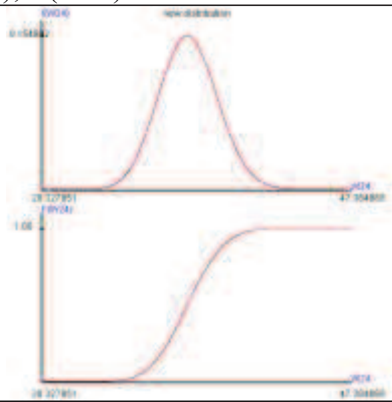
(4)n=5, λ =0.1,W24= $X_1 + X_2 + ... + X_n$,

f(w24), F(w24)	Coefficient	
	Mathematical Mean:	1.65079
	Geometrical Mean :	1.54207
	Harmonic Mean :	1.41874
	Variance :	0.33261
	S.D. :	0.57672
	Skewed Coef. :	0.33301
	Kurtosis Coef. :	2.91701
	MAD :	0.46402
	Range :	4.64033
	Mid_range :	2.36447
	Median :	1.61687
	Q1 :	1.23445
	Q2 :	1.61687
	Q3 :	2.03022
	IQR :	0.79577
	C.V. :	0.34936

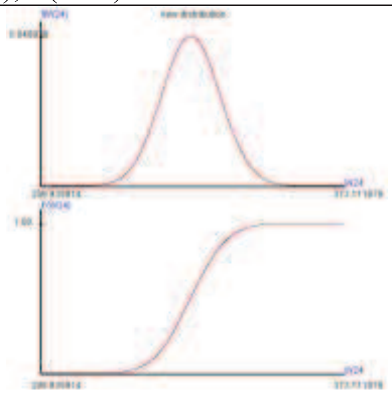
(5)n=30, λ =0.1,W24= $X_1 + X_2 + ... + X_n$,

f(w24), F(w24)	Coefficient	
	Mathematical Mean:	9.90661
	Geometrical Mean :	9.80458
	Harmonic Mean :	9.70068
	Variance :	1.99611
	S.D. :	1.41284
	Skewed Coef. :	0.13588
	Kurtosis Coef. :	2.98513
	MAD :	1.12887
	Range :	14.26745
	Mid_range :	10.91078
	Median :	9.87436
	Q1 :	8.93257
	Q2 :	9.87436
	Q3 :	10.84534
	IQR :	1.91278
	C.V. :	0.14262

(6)n=100, λ =0.1,W24= $X_1 + X_2 + ... + X_n$,

f(w24), F(w24)	Coefficient
	Mathematical Mean: 33.02027 Geometrical Mean : 32.91910 Harmonic Mean : 32.81740 Variance : 6.65598 S.D. : 2.57992 Skewed Coef. : 0.07459 Kurtosis Coef. : 2.99595 MAD : 2.05937 Range : 27.15750 Mid_range : 33.85641 Median : 32.98780 Q1 : 31.25982 Q2 : 32.98780 Q3 : 34.74515 IQR : 3.48533 C.V. : 0.07813

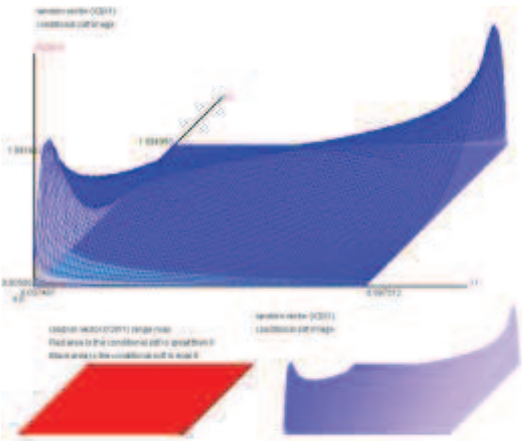
(7)n=1,000, λ =0.1,W24= $X_1 + X_2 + ... + X_n$,

f(w24), F(w24)	Coefficient
	Mathematical Mean: 330.20226 Geometrical Mean : 330.10147 Harmonic Mean : 330.00063 Variance : 66.53806 S.D. : 8.15709 Skewed Coef. : 0.02381 Kurtosis Coef. : 2.99953 MAD : 6.50920 Range : 84.48889 Mid_range : 331.02390 Median : 330.16686 Q1 : 324.67916 Q2 : 330.16686 Q3 : 335.68862 IQR : 11.00946 C.V. : 0.02470

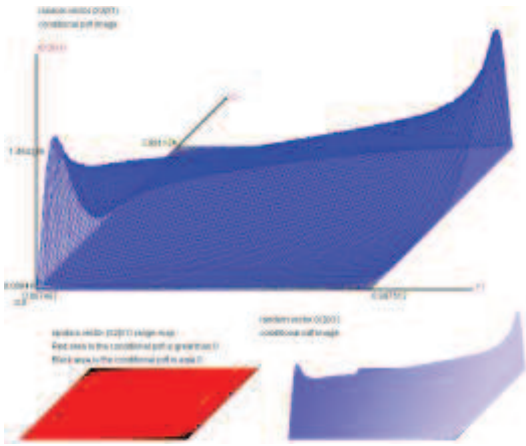
Section 4. $\sum_{i=1}^{n} X_i \xrightarrow[n\to\infty]{} Normal\left(E\left(\sum_{i=1}^{n} X_i\right), Var\left(\sum_{i=1}^{n} X_i\right)\right)$

$X_1, X_2, ..., X_n \overset{iid}{\sim} CB(\lambda)$, $X2=\sum_{i=1}^{n} X_i$, the simulator and transformation can get f(X2|X1=λ), $0<\lambda<1$, the simulated data number=1,000,000,000.
The diagram is (X1=λ, f(X2|X1)).

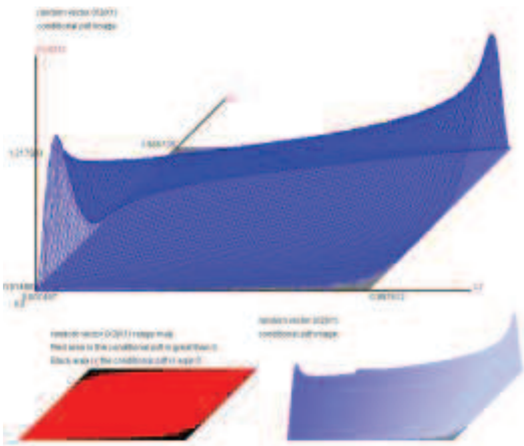
$n=2$,



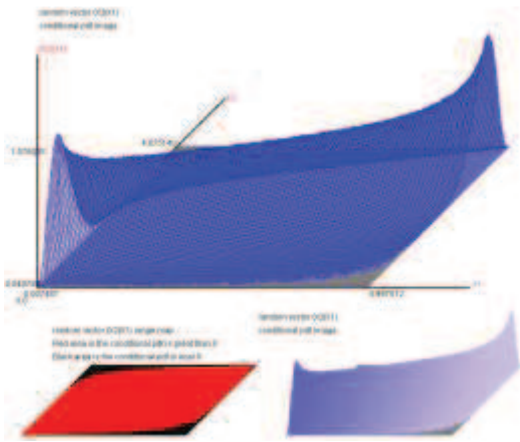
$n=3$,



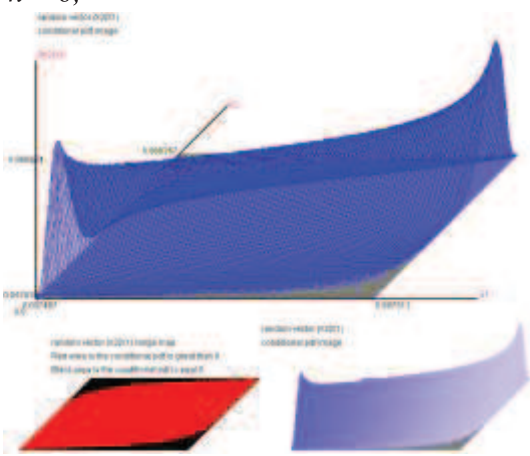
$n=4$,



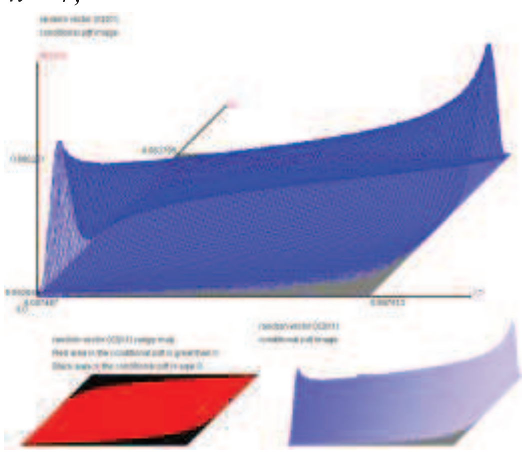
$n=5$,



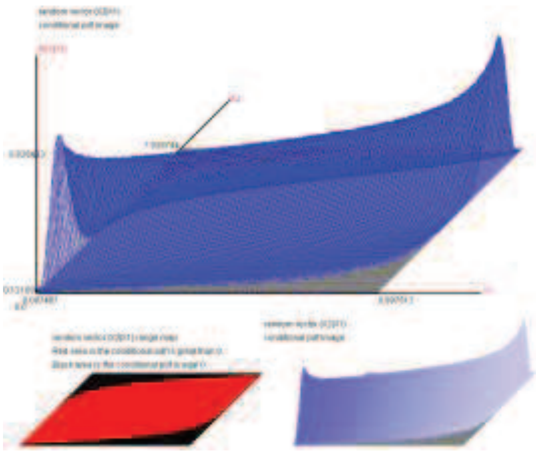
$n=6$,



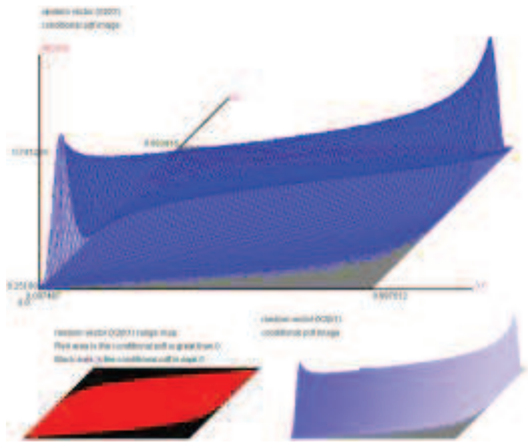
$n=7$,



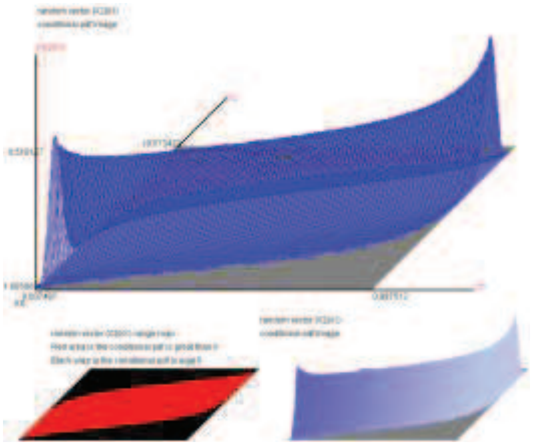
$n = 8,$



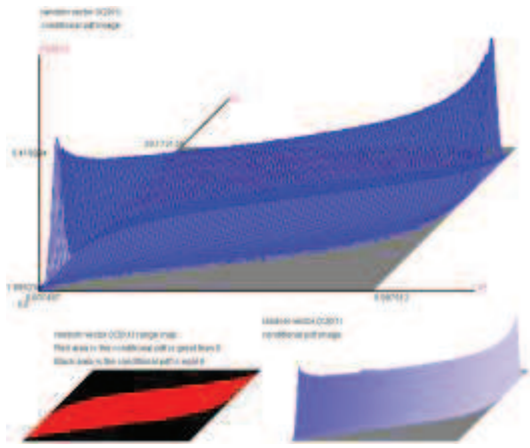
$n = 10,$



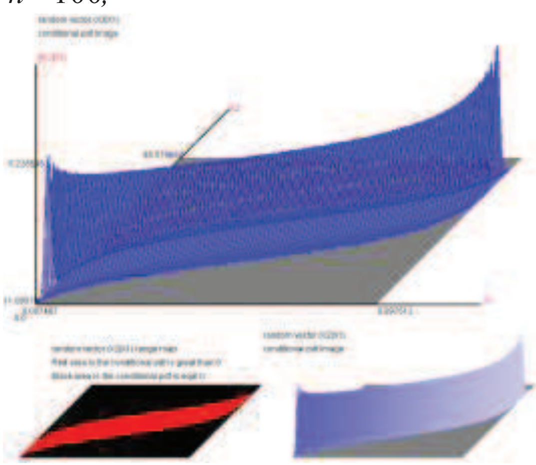
$n = 20,$



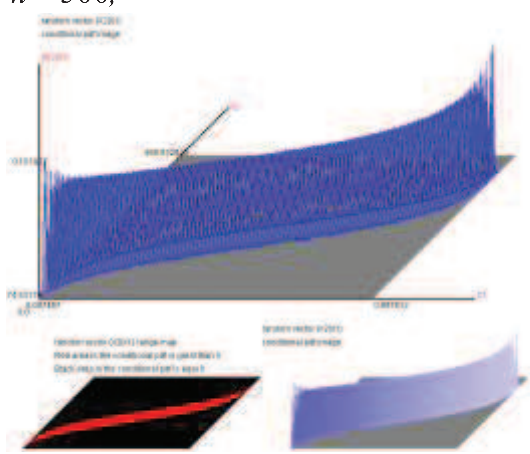
$n = 30,$



$n = 100,$



$n = 500,$



The red area is the range of ($\sum_{i=1}^{n} X_i$, λ).

The λ in $\sum_{i=1}^{n} X_i$ which is changed to the shape parameter to the location parameter when n is very large. When $X_1, X_2, \ldots, X_n \overset{iid}{\sim} CB(\lambda)$ and

$$\sum_{i=1}^{n} X_i \xrightarrow[n\to\infty]{} Normal\left(E\left(\sum_{i=1}^{n} X_i\right), Var\left(\sum_{i=1}^{n} X_i\right)\right).$$

Section 5. How to compute the Continuous Bernoulli distribution

1. $0.01 \le \lambda \le 0.99$

(1) $n \le 5$
The pdf of Continuous Binomial distribution(λ) is

$$f_X(x;\lambda,n) = h(x)(C(\lambda))^n \lambda^x (1-\lambda)^{n-x}, 0 \le x \le n, 0 < \lambda < 1.$$

$h(x)$ is Irwin-hall distribution and parameter n.

(2) $6 \le n \le 10$,

$$f_X(x;\lambda,n) \approx \frac{\sqrt{3}}{\sqrt{n\pi}} exp\left(-\frac{6\left(x-\frac{n}{2}\right)^2}{n} \right)(C(\lambda))^n \lambda^x (1-\lambda)^{n-x}, 0 \le x \le n, 0 < \lambda < 1,$$

$$h(x) \xrightarrow[n\ge 6]{} Normal\left(E(h(x)) = \frac{n}{2}, Var(h(x)) = \frac{n}{12} \right).$$

(3) $n \ge 6+250\times|\lambda-0.5|$, if $0.1 \le \lambda \le 0.9$,
$X \xrightarrow[n\to\infty]{} Normal(n\mu, n\sigma^2)$,

$$\sigma^2 = Var(X) = \begin{cases} \dfrac{(1-\lambda)\lambda}{(1-2\lambda)^2} + \dfrac{1}{\left(2\tan^{-1}(1-2\lambda)\right)^2} & if\ \lambda \ne \dfrac{1}{2} \\ \dfrac{1}{12} & if\ \lambda = \dfrac{1}{2} \end{cases}$$

2. $\lambda \le 0.01$ **or** $\lambda \ge 0.99$,
Please see Chapter 3.

Chapter 3 The Continuous Binomial approaching distribution when λ → 0

Section 1. The Continuous Binomial distribution will approach to Gamma distribution when λ → 0 and λ → 1

1. $\lambda \to 0$

$$X_1, X_2, \ldots, X_n \overset{iid}{\sim} f_X(x;\lambda) = C(\lambda)\lambda^x(1-\lambda)^{1-x}, 0 \le x \le 1, 0 < \lambda < 1,,$$

$$C(\lambda) = \begin{cases} \dfrac{2\,tanh^{-1}(1-2\lambda)}{1-2\lambda}, \lambda \ne \dfrac{1}{2} \\ 2, \lambda = \dfrac{1}{2} \end{cases}$$

$$\mu = E(X) = \frac{\lambda}{2\lambda - 1} + \frac{1}{2\,tan^{-1}(1-2\lambda)} = \frac{\lambda}{2\lambda - 1} + ln\left(\frac{1-\lambda}{\lambda}\right),$$

$X = \sum_{i=1}^{n} X_i$ ~Continuous Binomial distribution(n, λ) $\xrightarrow[\lambda \to 0]{}$ Gamma($\alpha = \text{n}, \beta = \mu$),

$$E(X) = \alpha\beta = n\mu, Var(X) = \alpha\beta^2 = n\mu^2 ,$$

$$\frac{2X}{\mu} = \frac{2\sum_{i=1}^{n} X_i}{\mu} \xrightarrow[\lambda \to 0]{} \chi^2_{2n},$$

In application, $X = \sum_{i=1}^{n} X_i \xrightarrow[\lambda \ge 0.01]{}$ Gamma($\alpha = \text{n}, \beta = \mu$),

$$\frac{2X}{\mu} = \frac{2\sum_{i=1}^{n} X_i}{\mu} \xrightarrow[\lambda \ge 0.01]{} \chi^2_{2n}.$$

2. $\lambda \to 1 (1 - \lambda \to 0)$

$$Y_i = 1 - X_i, f_{Y_i}(y_i;\lambda) = C(1-\lambda)(1-\lambda)^{y_i}\lambda^{1-y_i}, 0 \le y_i \le 1, 0 < \lambda < 1,$$

$$\mu_Y = E(Y_i) = \frac{1-\lambda}{2(1-\lambda)-1} + ln\left(\frac{\lambda}{1-\lambda}\right) = \frac{1-\lambda}{1-2\lambda} + ln\left(\frac{\lambda}{1-\lambda}\right),$$

$n - X = Y = \sum_{i=1}^{n} Y_i$ ~Continuous Binomial distribution(n, λ),

$n - X$ ~Continuous Binomial distribution(n,1- λ)

$n - X \xrightarrow[1-\lambda \to 0]{}$ Gamma($\alpha = \text{n}, \beta = \mu$),

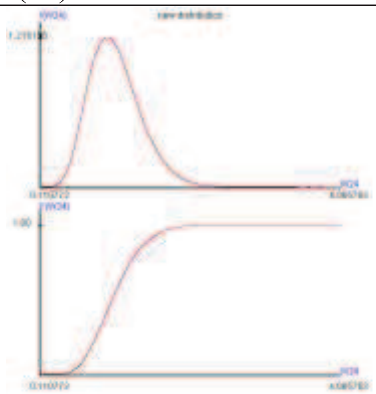
In application, $n - \sum_{i=1}^{n} X_i \xrightarrow[1-\lambda \ge 0.01]{}$ Gamma($\alpha = \text{n}, \beta = \mu_Y$),

Section 2. The numerical explanations

1.X1~ Continuous Binomial distribution(n, λ) when $\lambda \to 0$

(1) λ=0.0001,

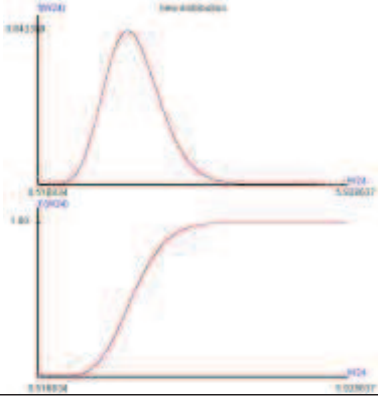
(1-1)n=10, λ=0.0001, W24=X1,

f(x1),F(x1)	Coefficient
	Mathematical Mean: 1.08450 Geometrical Mean : 1.03102 Harmonic Mean : 0.97651 Variance : 0.11678 S.D. : 0.34174 Skewed Coef. : 0.61599 Kurtosis Coef. : 3.53797 MAD : 0.27075 Range : 3.96971 Mid_range : 2.08828 Median : 1.04904 Q1 : 0.83847 Q2 : 1.04904 Q3 : 1.29224 IQR : 0.45377 C.V. : 0.31511

X1~ Continuous Binomial distribution(n=10, λ =0.0001),
X2~ Gamma(α = n=10, $\beta = \mu$), μ =E(X)= 0.10852, SLLN,

E(| X1 distribution - X2 distribution |^2)=0.0000044578
*********** | X1 distribution function - X2 distribution function| ***********
The almost surely limiting theory
E(| X1 distribution function - X2 distribution function|^2)=0.0000000716,
Pr(| X1 distribution function - X2 distribution function|<0.1000000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0500000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0100000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0050000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0010000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0005000000)= 0.957486,
Pr(| X1 distribution function - X2 distribution function|<0.0001000000)= 0.153571,

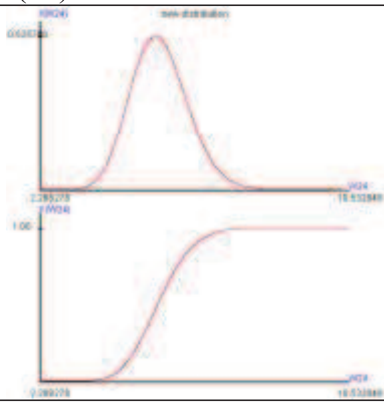
(1-2)n=20, λ=0.0001, W24=X1

f(x1),F(x1)	Coefficient
	Mathematical Mean: 2.16940 Geometrical Mean : 2.11568 Harmonic Mean : 2.06143 Variance : 0.23378 S.D. : 0.48351 Skewed Coef. : 0.43498 Kurtosis Coef. : 3.26888 MAD : 0.38447 Range : 5.43192 Mid_range : 3.22274 Median : 2.13414 Q1 : 1.82665 Q2 : 2.13414 Q3 : 2.47411 IQR : 0.64745 C.V. : 0.22288

X1~ Continuous Binomial distribution(n=20, λ=0.0001),
X2~ Gamma(α=n=10, $\beta=\mu$), μ=E(X)= 0.10852, SLLN,

E(| X1 distribution - X2 distribution |^2)=0.0000075180
*********** | X1 distribution function - X2 distribution function| ***********
The almost surely limiting theory
E(| X1 distribution function - X2 distribution function|^2)=0.0000002894,
Pr(| X1 distribution function - X2 distribution function|<0.1000000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0500000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0100000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0050000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0010000000)= 0.965526,
Pr(| X1 distribution function - X2 distribution function|<0.0005000000)= 0.571179,
Pr(| X1 distribution function - X2 distribution function|<0.0001000000)= 0.105471,

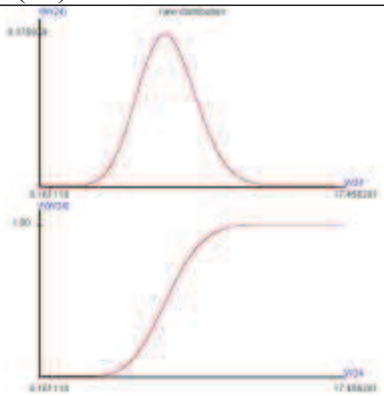
(1-3)n=50, λ=0.0001, W24=X1,

f(x1),F(x1)	Coefficient
	Mathematical Mean: 5.42558 Geometrical Mean : 5.37173 Harmonic Mean : 5.31768 Variance : 0.58486 S.D. : 0.76476 Skewed Coef. : 0.27468 Kurtosis Coef. : 3.10618 MAD : 0.60940 Range : 8.29529 Mid_range : 6.40056 Median : 5.39041 Q1 : 4.89197 Q2 : 5.39041 Q3 : 5.92099 IQR : 1.02901 C.V. : 0.14096

X1~ Continuous Binomial distribution(n=50, λ=0.0001),
X2~ Gamma(α=n=50, $\beta=\mu$), μ=E(X)= 0.10852, SLLN,

E(| X1 distribution - X2 distribution |^2)=0.0000128495
*********** | X1 distribution function - X2 distribution function| ***********
The almost surely limiting theory
E(| X1 distribution function - X2 distribution function|^2)=0.0000002686,
Pr(| X1 distribution function - X2 distribution function|<0.1000000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0500000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0100000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0050000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0010000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0005000000)= 0.575158,
Pr(| X1 distribution function - X2 distribution function|<0.0001000000)= 0.073629,

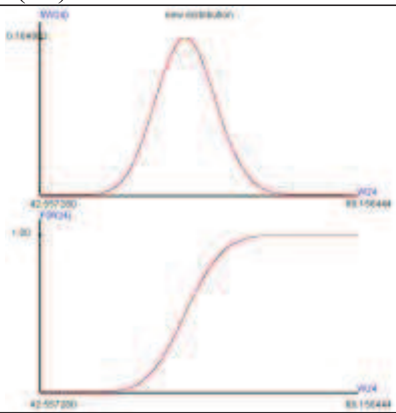
(1-4)n=100, λ=0.0001, W24=X1

f(x1),F(x1)	Coefficient
	Mathematical Mean: 10.85276 Geometrical Mean : 10.79886 Harmonic Mean : 10.74486 Variance : 1.17047 S.D. : 1.08188 Skewed Coef. : 0.19451 Kurtosis Coef. : 3.05396 MAD : 0.86263 Range : 11.33708 Mid_range : 11.80866 Median : 10.81762 Q1 : 10.10506 Q2 : 10.81762 Q3 : 11.56258 IQR : 1.45753 C.V. : 0.09969

X1~ Continuous Binomial distribution(n=100, λ=0.0001),
X2~ Gamma(α = n=100, $\beta = \mu$), μ =E(X)= 0.10852, SLLN,

E(| X1 distribution - X2 distribution |^2)=0.0000163052
*********** | X1 distribution function - X2 distribution function| ***********
The almost surely limiting theory
E(| X1 distribution function - X2 distribution function|^2)=0.0000004500,
Pr(| X1 distribution function - X2 distribution function|<0.1000000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0500000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0100000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0050000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0010000000)= 0.830769,
Pr(| X1 distribution function - X2 distribution function|<0.0005000000)= 0.431600,
Pr(| X1 distribution function - X2 distribution function|<0.0001000000)= 0.107253,

(1-5)n=500, λ=0.0001, W24=X1,

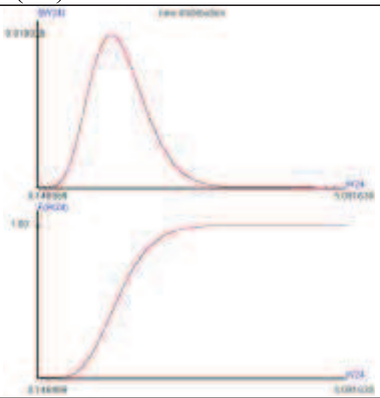
f(x1),F(x1)	Coefficient
	Mathematical Mean: 54.24449 Geometrical Mean : 54.19062 Harmonic Mean : 54.13673 Variance : 5.84492 S.D. : 2.41763 Skewed Coef. : 0.08706 Kurtosis Coef. : 3.00933 MAD : 1.92887 Range : 25.69433 Mid_range : 55.35686 Median : 54.21009 Q1 : 52.59435 Q2 : 54.21009 Q3 : 55.85543 IQR : 3.26107 C.V. : 0.04457

X1~ Continuous Binomial distribution(n=500, λ=0.0001),
X2~ Gamma(α = n=500, $\beta = \mu$), μ =E(X)= 0.10852, SLLN,

E(| X1 distribution - X2 distribution |^2)=0.0004147776
*********** | X1 distribution function - X2 distribution function| ***********
The almost surely limiting theory
E(| X1 distribution function - X2 distribution function|^2)=0.0000042527,
Pr(| X1 distribution function - X2 distribution function|<0.1000000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0500000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0100000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0050000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0010000000)= 0.211993,
Pr(| X1 distribution function - X2 distribution function|<0.0005000000)= 0.110857,
Pr(| X1 distribution function - X2 distribution function|<0.0001000000)= 0.021994,

(2) λ=0.001

(2-1)n=10, λ=0.001, W24=X1

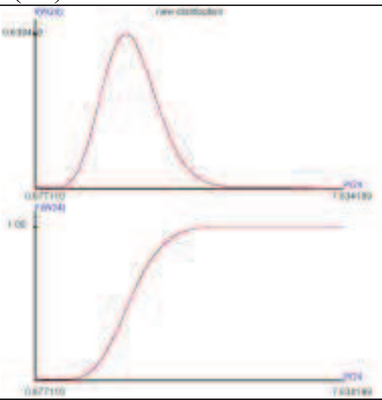
f(x1),F(x1)	Coefficient
	Mathematical Mean: 1.43829 Geometrical Mean : 1.36883 Harmonic Mean : 1.29763 Variance : 0.19976 S.D. : 0.44695 Skewed Coef. : 0.56910 Kurtosis Coef. : 3.41110 MAD : 0.35518 Range : 4.96302 Mid_range : 2.61931 Median : 1.39460 Q1 : 1.11588 Q2 : 1.39460 Q3 : 1.71385 IQR : 0.59796 C.V. : 0.31075

X1~ Continuous Binomial distribution(n=10, λ=0.001),

X2~ Gamma(α =n=10, $\beta = \mu$), μ =E(X)= 0.14380, SLLN,

E(| X1 distribution - X2 distribution |^2)=0.0001101042
*********** | X1 distribution function - X2 distribution function| ***********
The almost surely limiting theory
E(| X1 distribution function - X2 distribution function|^2)=0.0000088666,
Pr(| X1 distribution function - X2 distribution function|<0.1000000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0500000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0100000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0050000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0010000000)= 0.141397,
Pr(| X1 distribution function - X2 distribution function|<0.0005000000)= 0.069878,
Pr(| X1 distribution function - X2 distribution function|<0.0001000000)= 0.010999,

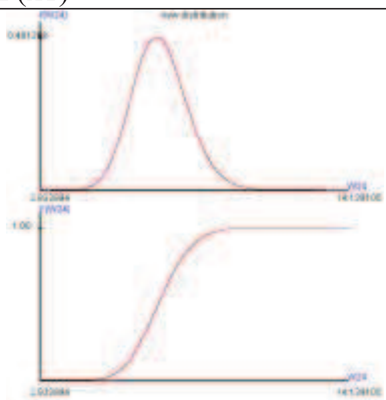
(2-2)n=20, λ=0.001, W24=X1

f(x1),F(x1)	Coefficient
	Mathematical Mean: 2.87650 Geometrical Mean : 2.80702 Harmonic Mean : 2.73663 Variance : 0.39937 S.D. : 0.63196 Skewed Coef. : 0.40117 Kurtosis Coef. : 3.20343 MAD : 0.50332 Range : 7.18369 Mid_range : 4.25565 Median : 2.83360 Q1 : 2.42859 Q2 : 2.83360 Q3 : 3.27802 IQR : 0.84944 C.V. : 0.21970

X1~ Continuous Binomial distribution(n=20, λ=0.001),
X2~ Gamma(α = n=20, $\beta = \mu$), μ =E(X)= 0.14380, SLLN,

E(| X1 distribution - X2 distribution |^2)=0.0001768935
*********** | X1 distribution function - X2 distribution function| ***********
The almost surely limiting theory
E(| X1 distribution function - X2 distribution function|^2)=0.0000097204,
Pr(| X1 distribution function - X2 distribution function|<0.1000000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0500000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0100000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0050000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0010000000)= 0.124178,
Pr(| X1 distribution function - X2 distribution function|<0.0005000000)= 0.060060,
Pr(| X1 distribution function - X2 distribution function|<0.0001000000)= 0.011337,

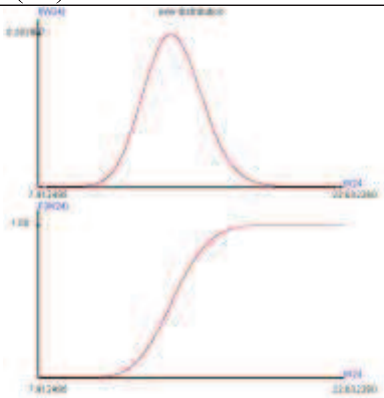
(2-3)n=50, λ=0.001, W24=X1,

f(x1),F(x1)	Coefficient
	Mathematical Mean: 7.19125 Geometrical Mean : 7.12178 Harmonic Mean : 7.05193 Variance : 0.99877 S.D. : 0.99938 Skewed Coef. : 0.25463 Kurtosis Coef. : 3.08267 MAD : 0.79679 Range : 11.34724 Mid_range : 8.48650 Median : 7.14881 Q1 : 6.49478 Q2 : 7.14881 Q3 : 7.84119 IQR : 1.34641 C.V. : 0.13897

X1~ Continuous Binomial distribution(n=50, λ=0.001),
X2~ Gamma(α = n=50, $\beta = \mu$), μ =E(X)= 0.14380, SLLN,

E(| X1 distribution - X2 distribution |^2)=0.0003530485
*********** | X1 distribution function - X2 distribution function| ***********
The almost surely limiting theory
E(| X1 distribution function - X2 distribution function|^2)=0.0000098070,
Pr(| X1 distribution function - X2 distribution function|<0.1000000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0500000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0100000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0050000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0010000000)= 0.133539,
Pr(| X1 distribution function - X2 distribution function|<0.0005000000)= 0.064656,
Pr(| X1 distribution function - X2 distribution function|<0.0001000000)= 0.010464,

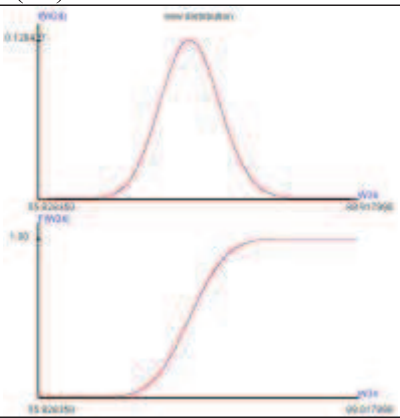
(2-4)n=100, λ=0.001, W24=X1

f(x1),F(x1)	Coefficient
	Mathematical Mean: 14.37988 Geometrical Mean : 14.31044 Harmonic Mean : 14.24082 Variance : 1.99625 S.D. : 1.41289 Skewed Coef. : 0.17845 Kurtosis Coef. : 3.03785 MAD : 1.12702 Range : 14.77449 Mid_range : 15.27238 Median : 14.33753 Q1 : 13.40458 Q2 : 14.33753 Q3 : 15.30941 IQR : 1.90484 C.V. : 0.09825

X1~ Continuous Binomial distribution(n=100, λ=0.001),
X2~ Gamma(α = n=100, $\beta = \mu$), μ =E(X)= 0.14380, SLLN,

E(| X1 distribution - X2 distribution |^2)=0.0003530485
*********** | X1 distribution function - X2 distribution function| ***********
The almost surely limiting theory
E(| X1 distribution function - X2 distribution function|^2)=0.0000098070,
Pr(| X1 distribution function - X2 distribution function|<0.1000000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0500000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0100000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0050000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0010000000)= 0.133539,
Pr(| X1 distribution function - X2 distribution function|<0.0005000000)= 0.064656,
Pr(| X1 distribution function - X2 distribution function|<0.0001000000)= 0.010464,

(2-5)n=500, λ=0.001, W24=X1

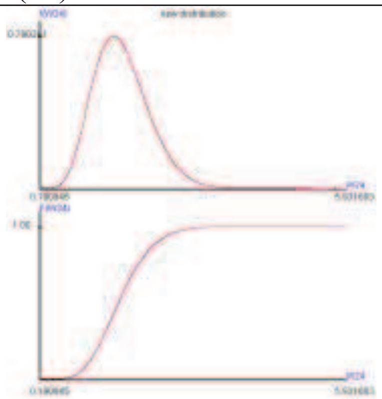
f(x1),F(x1)	Coefficient
	Mathematical Mean: 71.89937 Geometrical Mean : 71.82996 Harmonic Mean : 71.76051 Variance : 9.98064 S.D. : 3.15922 Skewed Coef. : 0.08038 Kurtosis Coef. : 3.00593 MAD : 2.52075 Range : 34.21638 Mid_range : 72.87317 Median : 71.85787 Q1 : 69.74442 Q2 : 71.85787 Q3 : 74.00638 IQR : 4.26195 C.V. : 0.04394

X1~ Continuous Binomial distribution(n=500, λ=0.001),
X2~ Gamma(α = n=500, $\beta = \mu$), μ =E(X)= 0.14380,SLLN,

E(| X1 distribution - X2 distribution |^2)=0.0036260189
*********** | X1 distribution function - X2 distribution function| ***********
The almost surely limiting theory
E(| X1 distribution function - X2 distribution function|^2)=0.0000103136,
Pr(| X1 distribution function - X2 distribution function|<0.1000000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0500000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0100000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0050000000)= 0.993192,
Pr(| X1 distribution function - X2 distribution function|<0.0010000000)= 0.137988,
Pr(| X1 distribution function - X2 distribution function|<0.0005000000)= 0.068562,
Pr(| X1 distribution function - X2 distribution function|<0.0001000000)= 0.012625,

(3) λ=0.003

(3-1)n=10, λ=0.003, W24=X1

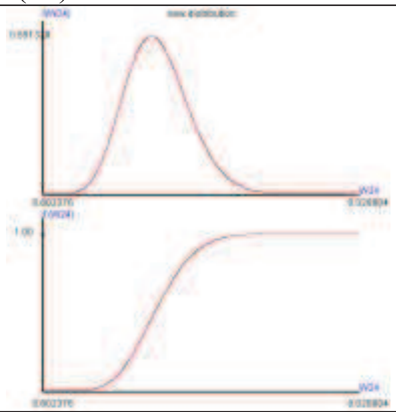
f(x1),F(x1)	Coefficient
	Mathematical Mean: 1.69250 Geometrical Mean : 1.61319 Harmonic Mean : 1.53138 Variance : 0.26654 S.D. : 0.51628 Skewed Coef. : 0.52187 Kurtosis Coef. : 3.30888 MAD : 0.41119 Range : 5.76208 Mid_range : 3.06131 Median : 1.64601 Q1 : 1.32008 Q2 : 1.64601 Q3 : 2.01472 IQR : 0.69465 C.V. : 0.30504

X1~ Continuous Binomial distribution(n=10, λ=0.003),

X2~ Gamma(α=n=10, $\beta=\mu$), μ=E(X)= 0.16926, SLLN,

E(\| X1 distribution - X2 distribution \|^2)=0.0005562091
*********** \| X1 distribution function - X2 distribution function\| ***********
The almost surely limiting theory
E(\| X1 distribution function - X2 distribution function\|^2)=0.0000383465,
Pr(\| X1 distribution function - X2 distribution function\|<0.1000000000)= 1.000000,
Pr(\| X1 distribution function - X2 distribution function\|<0.0500000000)= 1.000000,
Pr(\| X1 distribution function - X2 distribution function\|<0.0100000000)= 1.000000,
Pr(\| X1 distribution function - X2 distribution function\|<0.0050000000)= 0.383958,
Pr(\| X1 distribution function - X2 distribution function\|<0.0010000000)= 0.064553,
Pr(\| X1 distribution function - X2 distribution function\|<0.0005000000)= 0.030347,
Pr(\| X1 distribution function - X2 distribution function\|<0.0001000000)= 0.006239,

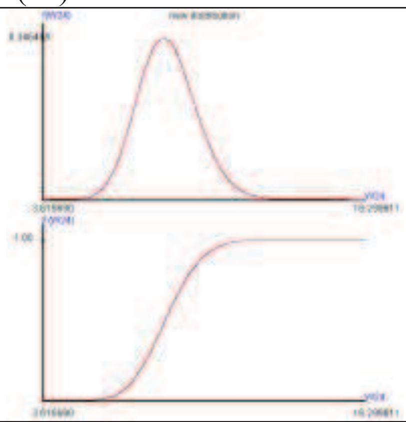
(3-2)n=20, λ=0.003, W24=X1

f(x1),F(x1)	Coefficient
	Mathematical Mean: 3.38623 Geometrical Mean : 3.30709 Harmonic Mean : 3.22666 Variance : 0.53359 S.D. : 0.73047 Skewed Coef. : 0.36873 Kurtosis Coef. : 3.15364 MAD : 0.58235 Range : 7.25329 Mid_range : 4.41559 Median : 3.34059 Q1 : 2.86938 Q2 : 3.34059 Q3 : 3.85363 IQR : 0.98424 C.V. : 0.21572

X1~ Continuous Binomial distribution(n=20, λ=0.003),
X2~ Gamma(α = n=20, $\beta = \mu$), μ =E(X)= 0.16926, SLLN,

E(| X1 distribution - X2 distribution |^2)=0.0008739119
*********** | X1 distribution function - X2 distribution function| ***********
The almost surely limiting theory
E(| X1 distribution function - X2 distribution function|^2)=0.0000429473,
Pr(| X1 distribution function - X2 distribution function|<0.1000000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0500000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0100000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0050000000)= 0.361841,
Pr(| X1 distribution function - X2 distribution function|<0.0010000000)= 0.065340,
Pr(| X1 distribution function - X2 distribution function|<0.0005000000)= 0.031952,
Pr(| X1 distribution function - X2 distribution function|<0.0001000000)= 0.006493,

(3-3)n=50, λ=0.003, W24=X1

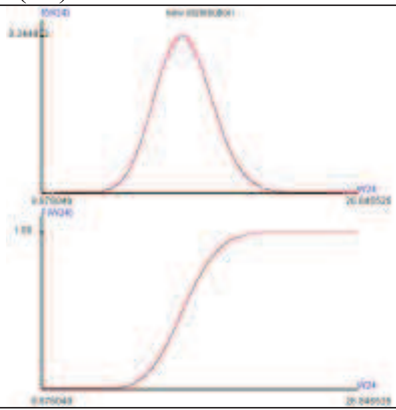
f(x1),F(x1)	Coefficient
	Mathematical Mean: 8.46097 Geometrical Mean : 8.38210 Harmonic Mean : 8.30271 Variance : 1.33200 S.D. : 1.15412 Skewed Coef. : 0.23323 Kurtosis Coef. : 3.06215 MAD : 0.92058 Range : 12.73107 Mid_range : 9.95765 Median : 8.41580 Q1 : 7.65815 Q2 : 8.41580 Q3 : 9.21453 IQR : 1.55638 C.V. : 0.13641

X1~ Continuous Binomial distribution(n=50, λ=0.003),
X2~ Gamma(α = n=50, $\beta = \mu$), μ =E(X)= 0.16926, SLLN,

E(| X1 distribution - X2 distribution |^2)=0.0020768445
*********** | X1 distribution function - X2 distribution function| ***********
The almost surely limiting theory
E(| X1 distribution function - X2 distribution function|^2)=0.0000381470,
Pr(| X1 distribution function - X2 distribution function|<0.1000000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0500000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0100000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0050000000)= 0.368358,
Pr(| X1 distribution function - X2 distribution function|<0.0010000000)= 0.064055,
Pr(| X1 distribution function - X2 distribution function|<0.0005000000)= 0.031095,
Pr(| X1 distribution function - X2 distribution function|<0.0001000000)= 0.006205,

(3-4)n=100, λ=0.003, W24=X1

f(x1),F(x1)



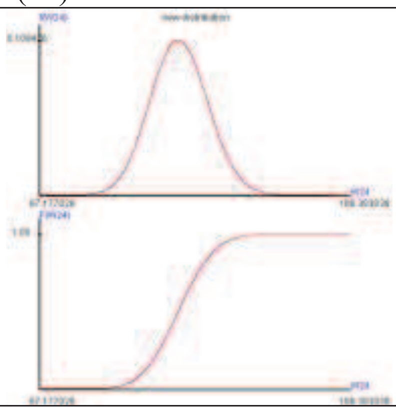
Coefficient		
Mathematical Mean:		16.92539
Geometrical Mean :		16.84660
Harmonic Mean	:	16.76754
Variance	:	2.66469
S.D.	:	1.63239
Skewed Coef.	:	0.16512
Kurtosis Coef.	:	3.03148
MAD	:	1.30215
Range	:	18.03628
Mid_range	:	17.86079
Median	:	16.88013
Q1	:	15.80039
Q2	:	16.88013
Q3	:	18.00158
IQR	:	2.20120
C.V.	:	0.09645

X1~ Continuous Binomial distribution(n=100, λ=0.003),
X2~ Gamma(α = n=100, $\beta = \mu$), μ =E(X)= 0.16926, SLLN,

E(| X1 distribution - X2 distribution |^2)=0.0038626153
*********** | X1 distribution function - X2 distribution function| ***********
The almost surely limiting theory
E(| X1 distribution function - X2 distribution function|^2)=0.0000395243,
Pr(| X1 distribution function - X2 distribution function|<0.1000000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0500000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0100000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0050000000)= 0.359010,
Pr(| X1 distribution function - X2 distribution function|<0.0010000000)= 0.062836,
Pr(| X1 distribution function - X2 distribution function|<0.0005000000)= 0.032454,
Pr(| X1 distribution function - X2 distribution function|<0.0001000000)= 0.006561,

(3-5)n=500, λ=0.003, W24=X1

f(x1),F(x1)



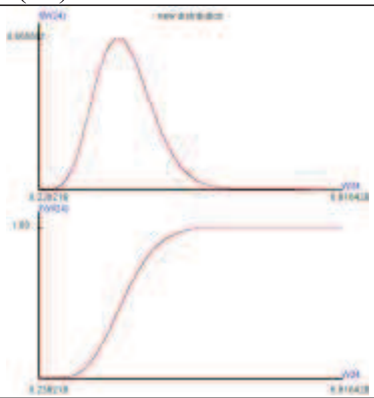
Coefficient		
Mathematical Mean:		84.62584
Geometrical Mean :		84.54710
Harmonic Mean	:	84.46830
Variance	:	13.32514
S.D.	:	3.65036
Skewed Coef.	:	0.07364
Kurtosis Coef.	:	3.00528
MAD	:	2.91271
Range	:	39.35176
Mid_range	:	86.78003
Median	:	84.58104
Q1	:	82.13800
Q2	:	84.58104
Q3	:	87.06427
IQR	:	4.92627
C.V.	:	0.04314

X1~ Continuous Binomial distribution(n=500, λ=0.003),
X2~ Gamma(α = n=500, $\beta = \mu$), μ =E(X)= 0.16926, SLLN,

E(| X1 distribution - X2 distribution |^2)=0.0189925637
*********** | X1 distribution function - X2 distribution function| ***********
The almost surely limiting theory
E(| X1 distribution function - X2 distribution function|^2)=0.0000401060,
Pr(| X1 distribution function - X2 distribution function|<0.1000000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0500000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0100000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0050000000)= 0.358669,
Pr(| X1 distribution function - X2 distribution function|<0.0010000000)= 0.065129,
Pr(| X1 distribution function - X2 distribution function|<0.0005000000)= 0.031695,
Pr(| X1 distribution function - X2 distribution function|<0.0001000000)= 0.005538,

(4) λ=0.01

(4-1)n=10, λ=0.01, W24=X1

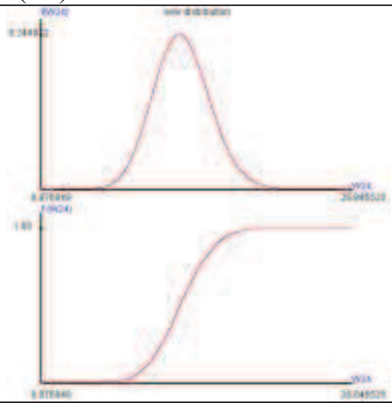
f(x1),F(x1)	Coefficient
	Mathematical Mean: 2.07436 Geometrical Mean : 1.98344 Harmonic Mean : 1.88874 Variance : 0.37063 S.D. : 0.60879 Skewed Coef. : 0.44778 Kurtosis Coef. : 3.18245 MAD : 0.48607 Range : 6.59664 Mid_range : 3.52432 Median : 2.02736 Q1 : 1.63674 Q2 : 2.02736 Q3 : 2.46082 IQR : 0.82408 C.V. : 0.29348

X1~ Continuous Binomial distribution(n=10, λ=0.01),

X2~ Gamma(α = n=10, $\beta = \mu$), μ =E(X)= 0.20749, SLLN,

E(| X1 distribution - X2 distribution |^2)=0.0029471275
*********** | X1 distribution function - X2 distribution function| ***********
The almost surely limiting theory
E(| X1 distribution function - X2 distribution function|^2)=0.0001755437,
Pr(| X1 distribution function - X2 distribution function|<0.1000000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0500000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0100000000)= 0.354756,
Pr(| X1 distribution function - X2 distribution function|<0.0050000000)= 0.159047,
Pr(| X1 distribution function - X2 distribution function|<0.0010000000)= 0.029541,
Pr(| X1 distribution function - X2 distribution function|<0.0005000000)= 0.014637,
Pr(| X1 distribution function - X2 distribution function|<0.0001000000)= 0.002881,

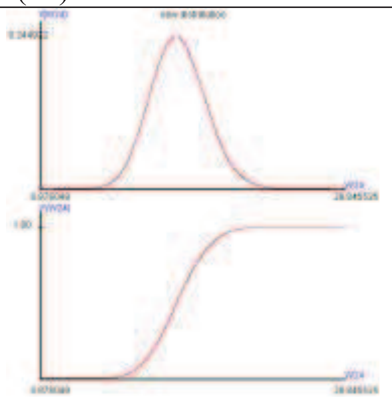
(4-2)n=20, λ=0.01, W24=X1

f(x1),F(x1)	Coefficient		
	Mathematical Mean:		4.15018
	Geometrical Mean :		4.05996
	Harmonic Mean	:	3.96786
	Variance	:	0.74173
	S.D.	:	0.86124
	Skewed Coef.	:	0.31638
	Kurtosis Coef.	:	3.08986
	MAD	:	0.68739
	Range	:	9.64723
	Mid_range	:	5.70726
	Median	:	4.10398
	Q1	:	3.54324
	Q2	:	4.10398
	Q3	:	4.70687
	IQR	:	1.16363
	C.V.	:	0.20752

X1~ Continuous Binomial distribution(n=20, λ=0.01),
X2~ Gamma(α = n=20, $\beta = \mu$), μ =E(X)= 0.20749, SLLN,

E(| X1 distribution - X2 distribution |^2)=0.0052423447
*********** | X1 distribution function - X2 distribution function| ***********
The almost surely limiting theory
E(| X1 distribution function - X2 distribution function|^2)=0.0001770199,
Pr(| X1 distribution function - X2 distribution function|<0.1000000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0500000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0100000000)= 0.348392,
Pr(| X1 distribution function - X2 distribution function|<0.0050000000)= 0.158827,
Pr(| X1 distribution function - X2 distribution function|<0.0010000000)= 0.028705,
Pr(| X1 distribution function - X2 distribution function|<0.0005000000)= 0.013863,
Pr(| X1 distribution function - X2 distribution function|<0.0001000000)= 0.002793,

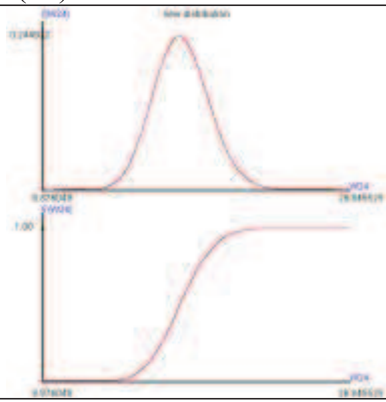
(4-3)n=50, λ=0.01, W24=X1

f(x1),F(x1)	Coefficient		
	Mathematical Mean:		10.37061
	Geometrical Mean :		10.28095
	Harmonic Mean	:	10.19054
	Variance	:	1.85247
	S.D.	:	1.36105
	Skewed Coef.	:	0.20067
	Kurtosis Coef.	:	3.03662
	MAD	:	1.08613
	Range	:	15.08996
	Mid_range	:	11.83881
	Median	:	10.32502
	Q1	:	9.42692
	Q2	:	10.32502
	Q3	:	11.26431
	IQR	:	1.83739
	C.V.	:	0.13124

X1~ Continuous Binomial distribution(n=50, λ=0.01),
X2~ Gamma(α = n=50, $\beta = \mu$), μ =E(X)= 0.20749, SLLN,

E(| X1 distribution - X2 distribution |^2)=0.0118881291
*********** | X1 distribution function - X2 distribution function| ***********
The almost surely limiting theory
E(| X1 distribution function - X2 distribution function|^2)=0.0001667659,
Pr(| X1 distribution function - X2 distribution function|<0.1000000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0500000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0100000000)= 0.351099,
Pr(| X1 distribution function - X2 distribution function|<0.0050000000)= 0.155998,
Pr(| X1 distribution function - X2 distribution function|<0.0010000000)= 0.028809,
Pr(| X1 distribution function - X2 distribution function|<0.0005000000)= 0.014183,
Pr(| X1 distribution function - X2 distribution function|<0.0001000000)= 0.003115,

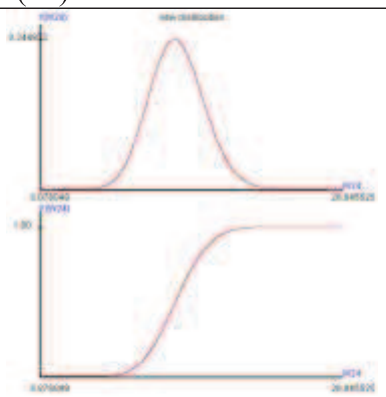
(4-4)n=100, λ=0.01, W24=X1

f(x1),F(x1)	Coefficient
	Mathematical Mean: 20.75082 Geometrical Mean : 20.66126 Harmonic Mean : 20.57132 Variance : 3.70960 S.D. : 1.92603 Skewed Coef. : 0.14167 Kurtosis Coef. : 3.01805 MAD : 1.53687 Range : 20.36842 Mid_range : 21.90591 Median : 20.70517 Q1 : 19.42607 Q2 : 20.70517 Q3 : 22.02534 IQR : 2.59927 C.V. : 0.09282

X1~ Continuous Binomial distribution(n=100, λ=0.01),
X2~ Gamma(α=n=100, $\beta=\mu$), μ=E(X)= 0.20749, SLLN,

E(| X1 distribution - X2 distribution |^2)=0.0234820303
*********** | X1 distribution function - X2 distribution function| ***********
The almost surely limiting theory
E(| X1 distribution function - X2 distribution function|^2)=0.0001720030,
Pr(| X1 distribution function - X2 distribution function|<0.1000000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0500000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0100000000)= 0.345888,
Pr(| X1 distribution function - X2 distribution function|<0.0050000000)= 0.158210,
Pr(| X1 distribution function - X2 distribution function|<0.0010000000)= 0.029653,
Pr(| X1 distribution function - X2 distribution function|<0.0005000000)= 0.015006,
Pr(| X1 distribution function - X2 distribution function|<0.0001000000)= 0.002896,

(4-5)n=500, λ=0.01, W24=X1

f(x1),F(x1)	Coefficient
	Mathematical Mean: 103.70756 Geometrical Mean : 103.61821 Harmonic Mean : 103.52878 Variance : 18.52618 S.D. : 4.30420 Skewed Coef. : 0.06344 Kurtosis Coef. : 3.00210 MAD : 3.43455 Range : 48.04510 Mid_range : 105.57974 Median : 103.66335 Q1 : 100.77818 Q2 : 103.66335 Q3 : 106.58542 IQR : 5.80724 C.V. : 0.04150

X1~ Continuous Binomial distribution(n=500, λ=0.01),
X2~ Gamma(α=n=500, $\beta=\mu$), μ=E(X)= 0.20749, SLLN,

E(| X1 distribution - X2 distribution |^2)=0.1159039475
*********** | X1 distribution function - X2 distribution function| ***********
The almost surely limiting theory
E(| X1 distribution function - X2 distribution function|^2)=0.0001724137,
Pr(| X1 distribution function - X2 distribution function|<0.1000000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0500000000)= 1.000000,
Pr(| X1 distribution function - X2 distribution function|<0.0100000000)= 0.351585,
Pr(| X1 distribution function - X2 distribution function|<0.0050000000)= 0.161648,
Pr(| X1 distribution function - X2 distribution function|<0.0010000000)= 0.030630,
Pr(| X1 distribution function - X2 distribution function|<0.0005000000)= 0.015277,
Pr(| X1 distribution function - X2 distribution function|<0.0001000000)= 0.002858,

2. The sampling distribution of Continuous Bernoulli(n, λ) when 0.000001 ≤ λ ≤ 0.01

$$X_1, X_2, \ldots, X_n \overset{iid}{\sim} f_X(x;\lambda) = C(\lambda)\lambda^x(1-\lambda)^{1-x}, 0 \le x \le 1, 0 < \lambda < 1, \mu = E(X),$$

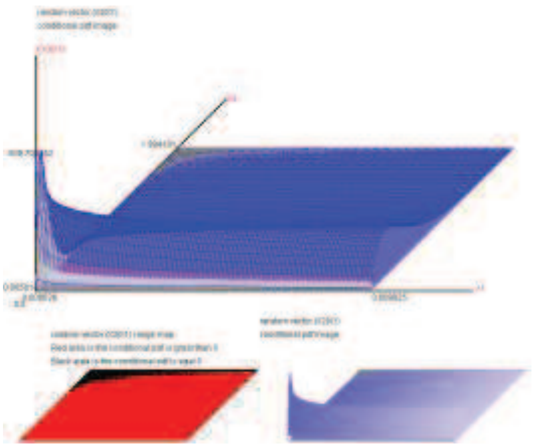
$$C(\lambda) = \begin{cases} \dfrac{2\tanh^{-1}(1-2\lambda)}{1-2\lambda}, \lambda \ne \dfrac{1}{2} \\ 2, \lambda = \dfrac{1}{2} \end{cases}$$

$$\mu = E(X) = \frac{\lambda}{2\lambda - 1} + \ln\left(\frac{1-\lambda}{\lambda}\right), \lambda \ne \frac{1}{2},$$

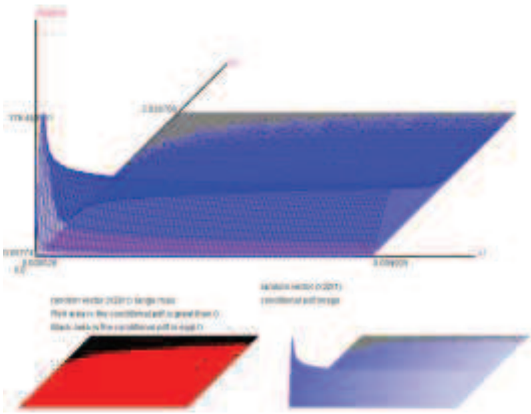
$X = \sum_{i=1}^{n} X_i$ ~Continuous Binomial distribution(n, λ) $\xrightarrow[\lambda \to 0]{}$ Gamma(α = n, $\beta = \mu$),

$f\left(X2 = \sum_{i=1}^{n} X_i \middle| X1 = \lambda\right)$ =?, $0.000001 \le \lambda \le 0.01$

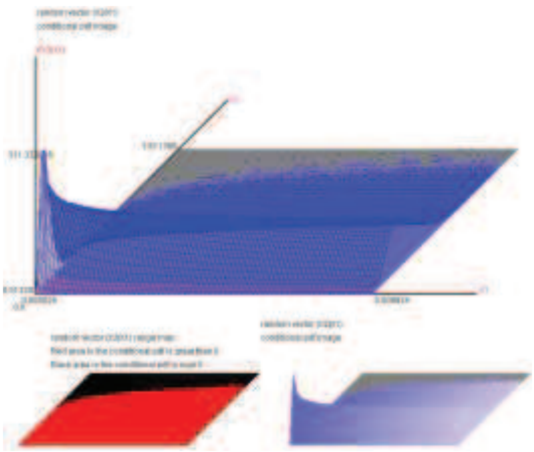
(2-1)n=2,



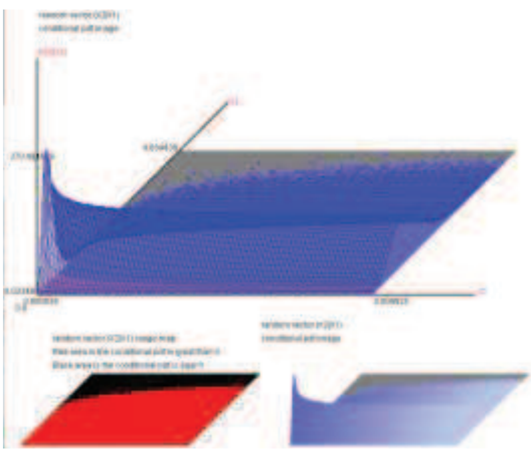
(2-2)n=3,



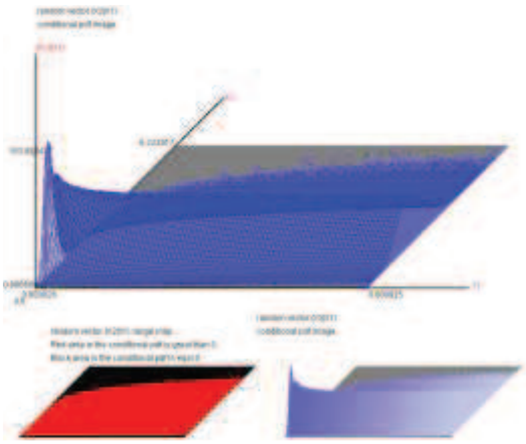
(2-3)n=4,



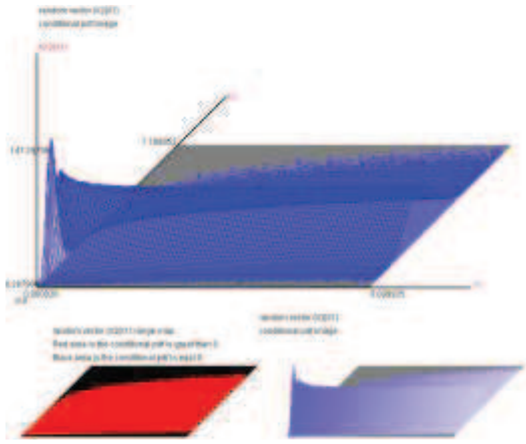
(2-4)n=5,



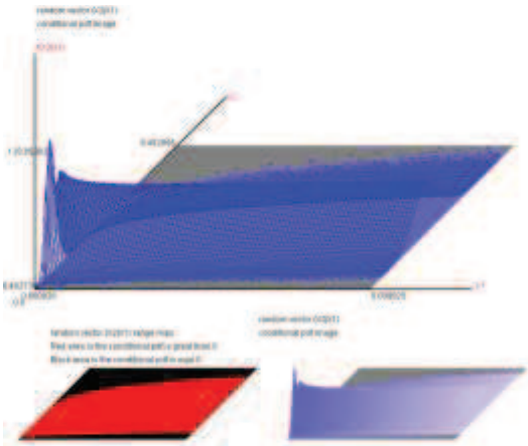
(2-5)n=10,



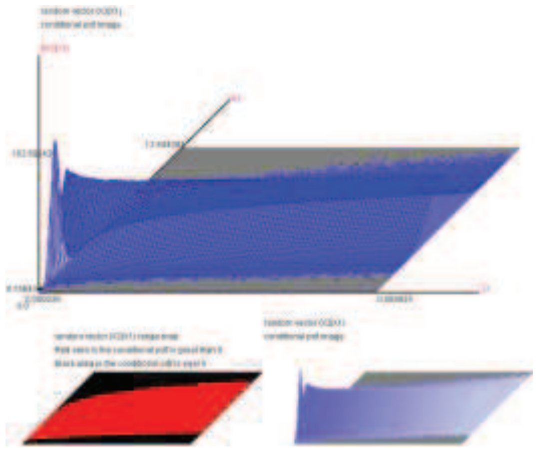
(2-6)n=15,



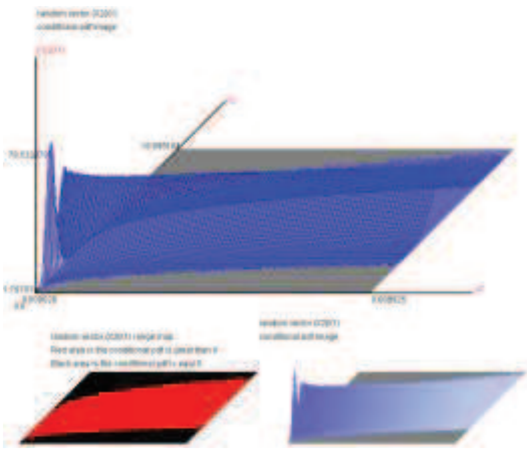
(2-7)n=20,



(2-8)n=30,



(2-9)n=50,



(2-10)n=100,