The ∞-manifolds
The ∞-bundles

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Abstract

We introduce the ∞-manifolds and the ∞-bundles which are spaces of dimension the cardinality of the continuum.

1 The classical tensor calculus

For a differential manifold M [B][K], it is possible to make a tensor calculus [A][BG][S] with tensor products of the tangent and cotangent spaces. We tensorize the spaces and introduce local coordinates (x_i). A tensor is then an expression like:

$$R^i_{jkl}$$

It is possible to transform the tensor under coordinates changes $\tilde{x}_j$ by the matrix:

$$\frac{\partial \tilde{x}_i}{\partial x_j}$$

We obtain new expressions, for example:

$$\tilde{A}^i = \sum_j A^j \frac{\partial \tilde{x}_i}{\partial x_j}$$

2 The ∞-manifolds

It is possible to make a tensor calculus when the index of the tensor is continuous instead of being discreet. For example, is x^t are the local coordinates; the tensor A^t transforms under the change of coordinates $\tilde{x}^{t'}$, according to:

$$\tilde{A}^t = \int_{-\infty}^{+\infty} A^{t'} (\frac{\partial \tilde{x}^t}{\partial x^{t'}}) dt'$$

We have the coherence rule for the change of coordinates:

$$\int_{-\infty}^{+\infty} (\frac{\partial x^t}{\partial \tilde{x}^{t'}})(\frac{\partial \tilde{x}^{t'}}{\partial x^{t''}}) dt' = \delta(t - t'')$$

With δ, the Dirac function. If $\tilde{x}^t = x^t$, we obtain the equation:

$$\int_{-\infty}^{+\infty} \delta(t-t')\delta(t'-t'')dt' = \delta(t-t'')$$

The basic space is the Fréchet space of Schwartz functions [M] (smooth real functions with polynomial decreasing at infinity of the functions and all their derivatives). So that we have:

$$x^t(f) = f(t) = \delta(t)(f)$$

The functions over this space are functionals over the smooth Schwartz functions. A functional $\mathcal{F}$ is derivable if the following limit exists:

$$\lim_{\epsilon \to 0} \frac{\mathcal{F}(g+\epsilon h) - \mathcal{F}(g)}{\epsilon} = d\mathcal{F}_g(h)$$

and if the differential is a distribution over the Schwartz space. The functional $\mathcal{F}$ is smooth if we can infinitely iterate the differentials. The derivations are identified with the Schwartz functions and we have:

$$X\mathcal{F}(g) = d\mathcal{F}_g(X)$$

The differential of a functional is:

$$d\mathcal{F} = \int_{-\infty}^{+\infty} \frac{\partial \mathcal{F}}{\partial x^t} dx^t dt$$

We have, under a change of coordinates:

$$\frac{\partial \mathcal{F}}{\partial \tilde{x}^t} = \int_{-\infty}^{+\infty} (\frac{\partial \mathcal{F}}{\partial x^{t'}})(\frac{\partial x^{t'}}{\partial \tilde{x}^t})dt'$$

The metric g is a 2-tensor such that:

$$g(X,Y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g_{tt'} X^t Y^{t'} dt dt'$$

The metric is a riemannian metric [J] if the quadratic form is definite positiv. The inverse of the metric $g_{tt'}$ is $g^{tt'}$ such that:

$$\int_{-\infty}^{+\infty} g_{tt'} g^{t't''} dt' = \delta(t-t'')$$

Definition:
The manifolds which are modeled over the Schwartz space are called the ∞-manifods.

3 The ∞-bundles

Definition:
The ∞-bundles over an ∞-manifold M are projectiv modules over the ring of smooth functionals of M.

The connections over an ∞-bundle are defined by the fact that they are linear and the Leibniz condition:

$$\nabla_X(\mathcal{F}.s) = X\mathcal{F}.s + \mathcal{F}.\nabla_X(s)$$

with $\mathcal{F}$ a smooth functional over M, and s an element of the ∞-bundle. The Levi-Civita connection can be defined by the condition of zero torsion and that it conserves the riemannian metric.

References

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