An electromagnetic deuteron model

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Abstract

This paper shows how one can use potentials to build up a spin-zero model of the deuteron. The spin-zero model consists of a proton, and another proton plus an electron which combine in an electrically neutral particle which we refer to as the neutron. We treat all particles as spin-zero particles because we assume their *magnetic* moment is zero. As such, it may complement Paolo Di Sia's model of the nucleus (2018), which we give due attention. In contrast to Di Sia, we think of neutrons – or the electron cloud that surrounds the proton inside – as *electric* dipoles.

The model does so by interpretating Yukawa's potential function as a dipole potential. Instead of predefining the range parameter *a*, we calculate it from the equilibrium condition (equal but opposite magnitudes of the Coulomb and nuclear forces). We find a very acceptable value of about 2.88 fm for *a*, and find an equally acceptable value for the distance between the positively charged center of the neutron and the center of the electron cloud which, in a deuteron nucleus, must shift it center of charge towards the proton so as to ensure stability – not unlike the sharing of valence electrons in chemical bonds.

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Introduction

This is a 'for fun' paper showing how one can use potentials to build up a spin-zero model of the deuteron. The spin-zero model consists of a proton, and another proton plus an electron which combine in an electrically neutral particle which we refer to as the neutron. We treat all particles as spin-zero particles because we assume their *magnetic* moment is zero. Hence, magnetic forces do not come into play, but we will – later, much later – want to combine with Paolo Di Sia's model of the nucleus, which we will mention in this introduction.[1] However, we will assume the neutron has an *electric* dipole moment.

Paolo Di Sia's model: nuclear force as magnetic force

Paolo Di Sia (2018) showed the presumed nuclear attraction between nucleons might be explained in terms of the interaction between the *magnetic* moments of nucleons. This magnetic moment arises from a presumed internal current in one nucleon, which will generate a magnetic field whose strength is given by the Biot-Savart law, which is illustrated below[2].

$$\boldsymbol{B}(1) = -\frac{1}{4\pi\varepsilon_0 c^2}\int \frac{I(\boldsymbol{e_{12}} \times d\boldsymbol{s_2})}{r_{12}^2}$$

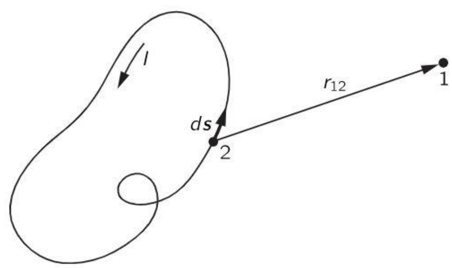

Figure 1: The magnetic field from a current is obtained from an integral around the circuit

Di Sia assumes the current itself is electrically neutral (the nucleons have spin but zero charge) and there is, therefore, no electrostatic field. Hence, if we place a charge q at point 1, the q***E*** term of the Lorentz force F = q***E*** + q(***v***×***B***) charge is zero and the charge will experience a magnetic force q(***v***×***B***) only.

However, Di Sia does not put a charge in the field: he puts another zero-charge nucleon with spin (magnetic moment) and, therefore, he models the *magnetic* force between two zero-charge nucleons only. The numerical example which he provides is for nucleons with an approximate size of 0.5 fm (we take this to be the *radius* of the current loop) which are separated by the typical interproton distance

[1] Paolo Di Sia, *A solution to the 80-year-old problem of nuclear attraction*, October 2018.

[2] We borrow the illustration (and formula) from Feynman's Lectures (II-14-7). The reader should check the physical dimensions and the nature of the force field. Current is measured in C/s, so we get C·m/s·m^2 = C/m·s in the integrand. The physical proportionality constant has an added c^2, so its dimension is [1/4πε$_0$]/[m^2/s^2] = (N·m^2/C^2)·(s^2/m^2 = N·s/C^2. Multiplying both yields the (C/m·s)·(N·s/C^2) = (N/C)·(s/m) dimension we would expect to see. The s/m factor reflects the orthogonality of the $\mathbf{e}_{12}$×d$\mathbf{s_2}$ vector cross product to both $\mathbf{e}_{12}$ as well as d$\mathbf{s_2}$ (you can also think of the orthogonality of the $\mathbf{F}_{\text{magnetic}}$ = q(***v***×***B***) force vector cross product with the ***v*** and ***B*** vectors. The $\mathbf{e}_{12}$× *operator* effectively corresponds to a rotation by 90 degrees, which corresponds to the s/m dimension or, defining suitable conventions for the sign, a multiplication by the imaginary unit *i*. The *c**B*** = $\mathbf{e}_{r'}$×***E*** cross-product, which we analyzed as part of Feynman's analysis of electromagnetic wave propagation may therefore also be written as *c**B*** = *i**E***.

(about 2 fm), which corresponds to the usual *range* parameter in Yukawa's formula for the nuclear potential.

Interestingly, Di Sia also considers the *phase* of currents, which may effectively be in or out of phase and conveniently calculates *energy* levels for the magnetic binding so we can immediately compare with relevant values for nuclear binding energies. For the mentioned values (0.5 and 2 fm) he gets an energy range between 3.97 KeV and 0.127 MeV (the latter value assumes in-phase currents). While this is, without any doubt, significant, it is only 5% of the 2.2 MeV energy difference between the deuteron nucleus (about 1875.613 MeV) and its two constituents (neutron and proton) in their unbound state (939.565 MeV + 938.272 MeV = 1,877.837 MeV).

The values get (much) better when changing the parameters (nucleon size and internucleon distance) significantly (2-3 MeV) and, better still, considering *paired* nucleons creating dipoles acting on other paired nucleons (values up to 5 MeV), but we would wish the author would offer a model of a nuclear lattice showing how currents and nucleon pairing actually works in 3D space.

In short, we like the model but we are hungry for more detail. Di Sia models full-spin nucleons without a charge, and thinks of a nuclear lattice as a lattice of *magnetic* dipoles. In this paper, we want to do the opposite: we want to develop a model for charged nucleons without spin, and a small nuclear lattice based on the concept of an *electric* dipole. The basic idea is this: we think of deuteron as consisting of two protons and a *deep* electron[3] and we, therefore, have a classical three-body problem, for which no general closed-form solution exists.[4]

What is the idea here?

Special-case solutions for three-body problems exist, and one of these special cases is the case where two of the three bodies are very massive relative to the third one: think of the Earth–Moon–Sun system, for example. Our deuteron model will basically be the same: we think of one of the two protons being closely bound with the electron. However, we will not think of its magnetic moment but of its *electric* dipole moment resulting from two opposite charges being *very* close and creating an (electrostatic) dipole field.

Of course, the informed reader will immediately object that a neutron has a significant magnetic moment – about -9.66×10^{-27} $A\cdot m^2$, to be precise[5] – but, in contrast, it has no noticeable electric dipole moment. In fact, now that we are here, we should quickly give you the (approximate) value of the *magnetic* moments of all these particles so you can check what adds up and what does not:

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[3] We are grateful to Andrew Meulenberg (Meulenberg and Paillet, *Highly relativistic electrons and the Dirac equation*, 2019-2020) for having drawn our attention to the work on deep electron orbitals. In fact, we write this paper as input for his forthcoming paper, which should generalize the concept and show full consistency with all of classical electromagnetic theory.

[4] We may refer the reader to the Wikipedia article on the three-body problem, which we find to be intuitive and well written.

[5] We prefer the $A\cdot m^2$ unit to the equivalent J/T tesla because it reminds us of the $\mu = I\cdot A$ formula for the magnetic moment of a current loop, with I the current of the loop and A the *surface* of the loop. The minus sign gives us the direction of the magnetic moment when applying the usual right-hand rule. We should, of course, relate this to the *angular momentum* (spin) of the particle but we will not do this here. For a short overview of how this works, see our paper on the radius and magnetic moment of electrons and protons. As for the n = p + e model, see our paper on proton and neutron reactions.

Electron (e^-)	$- \ 9.28\times10^{-24}$ A·m^2
muon-electron μ^-)	$- \ 0.45\times10^{-27}$ A·m^2
proton (p^+)	$+ 14.11\times10^{-27}$ A·m^2
neutron (n^0)	$- \ 9.66\times10^{-27}$ A·m^2
deuteron (D^+)	$+ \ 4.33\times10^{-27}$ A·m^2

As you can see, the magnetic moments of the proton and neutron add up – but very approximately only – to the magnetic moment of the deuteron, but the magnetic moments of the electron and the proton do *not* add up to that of the neutron: we have a *very* different order of magnitude for the electron mm (961 times the proton mm, and 658 times the neutron mm), and thinking of it the neutron-electron as a muon-electron does not help either. Hence, the neutron electron must be a very different beast than the electron we are used to. We might come back to that later.[6]

What about the electron *electric dipole moment* (oft abbreviated as EDM)? Is our model dead even before we developed it? Maybe. Maybe not. Experiments do measure a EDM, but it is very weak, and successive experiments have pushed the *upper* limit down to an order of magnitude of 10^{-26} q_e·m. The discussion is bound up with discussions on symmetry-breaking in Nature[7] and other complicated matters which are of no interest to us here because they would only confuse us even more. The point is: if the electron cloud around the proton in the neutron is spherically symmetric, then the neutron EDM should, effectively, be zero. So why would we even pursue the hypothesis? Because we are thinking of a deep electron orbital that is part of a deuteron nucleus and we, therefore, think the presence of the (other) proton will result in it pulling the blanket towards it. To be honest, the most important reason why we want to present our model is because *some* force must be holding two protons together in the deuteron nucleus.[8]

Indeed, two protons do normally not bind in any stable way. A diproton – the nucleus of ^{2}He isotope – is extremely unstable: a neutron is needed to glue them together in a more stable configuration: ^{3}He. So, yes, we do think of a neutron as consisting of a proton and a *deep* (nuclear) electron which binds both. It is an idea which Rutherford thought of when he first hypothesized the existence of neutrons and which would explain proton-neutron reactions as well as the instability of the neutron outside of the nucleus.

Does that make sense? Maybe not. We will soon see when we try to put all of this into equations, which we will do soon enough. However, let us first present the basic formulas for an electric dipole field.

[6] We offer a comprehensive analysis of particle spin and magnetic moment – based on the ring current model which, in essence, is an application of Wheeler's mass without mass model – in [our paper on classical quantum physics](#).

[7] You can *google* various popular science articles on this, but the Wikipedia article on the nEDM is a good starting point.

[8] While we will first assume genuine nuclear charges besides electric charges, we will soon substitute this hypothesis for a simpler one: we think the presence of the proton results in a shift of the neutron electron cloud, thereby creating an *electric* dipole moment with a potential which diminishes with the (radial) distance as $1/r^2$, as opposed to the electrostatic $1/r$ (Coulomb) potential.

An electric dipole moment is measured by multiplying the magnitude of the two opposite charges ($\pm$ q) by the distance that separates them (d) and the electrostatic potential it generates is approximated by the following formula[9]:

$$\phi(r) = \frac{1}{4\pi\epsilon_0}\frac{p \cdot \cos\theta}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{\mathrm{qd} \cdot \cos\theta}{r^2}$$

This formula tells us the magnitude goes down with the *square* of the distance (as opposed to the $1/r$ function for electrostatic potential[10]) and also depends on the angle between the axis of the dipole (the line of charges) and the radius vector.[11] Also note the dipole moment p·cosθ = q·d·cosθ factor versus the charge q in the numerator of the potential function for a *single* charge which, for the convenience of the reader, we reproduce below:

$$\mathrm{V}(r) = -\frac{1}{4\pi\varepsilon_0}\frac{\mathrm{q}}{r}$$

OK. We are almost done with our introduction to the real thing. Before we dive into it, let us quickly give you the basic data on energy, radii, and other observables.

Energies and radii

The electron-proton scattering experiment by the PRad team at Jefferson Lab measured the root mean square (rms) charge radius of the proton as $r_p = 0.831 \pm 0.007_{stat} \pm 0.012_{syst}$ fm.[12] The root mean square radius of a neutron, as calculated from electron scattering experiments, was also measured to be around 0.8 fm.

The results from electron scattering experiments are precise enough but a straightforward interpretation is not easy because the electron-proton interaction is an interaction between charged particles and will therefore involve the Coulomb force mainly. In contrast, electron-neutron interactions

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[9] See: Feynman's Lectures, II-6-2 (the electric dipole). The formula is derived from (1) a calculation of the two (opposite) potentials (this yields the $1/r$ function), (2) expanding the $(d/2)^2$ term in the distance calculation using the binomial expansion, (3) neglecting all terms with higher power than d^2 (because d is supposed to be very small), and then (4) adding the two simplified potential functions (or, because of the opposite sign, *subtracting* them). Assuming the distance d is measured along the *z*-axis, this yields the formula below, which can easily be rewritten as the formula we use above:

$$\phi(r) = \frac{1}{4\pi\epsilon_0}\frac{z}{r^3}\mathrm{qd}$$

[10] For the convenience of the reader, the electrostatic potential from a single charge is equal to:

$$\mathrm{V}(r) = -\frac{1}{4\pi\varepsilon_0}\frac{\mathrm{q}}{r}$$

[11] For a succinct analysis, see: Feynman's *Lectures*, Vol. II, Chapter 6 ([*The Electric Field in Various Circumstances*]()).

[12] See: https://www.jlab.org/news/releases/new-measurement-yields-smaller-proton-radius. A root mean square statistic involves (1) the squaring of measurements before adding and dividing by the number of observations before (2) taking the square root again to calculate the *mean* value. The approach ensures positive and negative deviations from a zero point do not cancel out while summing, but gives large values more weight. The mean of 1, 2 and 3 is $\frac{1+2+3}{3} = 6$, but the rms calculation yields $\sqrt{\frac{1^2+2^2+3^2}{3}} \approx 2.16$. The arithmetic mean is, therefore, referred to as an *unbiased* estimator. Experimenters obviously take care of these things and we should, therefore, not worry about it.

are (mainly) magnetic. Hence, such experiments use *form factors* which are then used to interpret the scattering data and *calculate* what we would rather refer to as the radius of effective *interference*. We copy a slide from a presentation on form factors which we rather liked[13] because it shows Rutherford's model – a neutron combines a proton and a nuclear electron – is actually being used to calculate the neutron form factor. It should be noted that the 1961 measurements of Hofstadter and Rudolf Mössbauer[14] then yielded a neutron *charge* radius $R_c = 0$ and a *magnetic* radius equal to $R_m = 0.76$ fm.

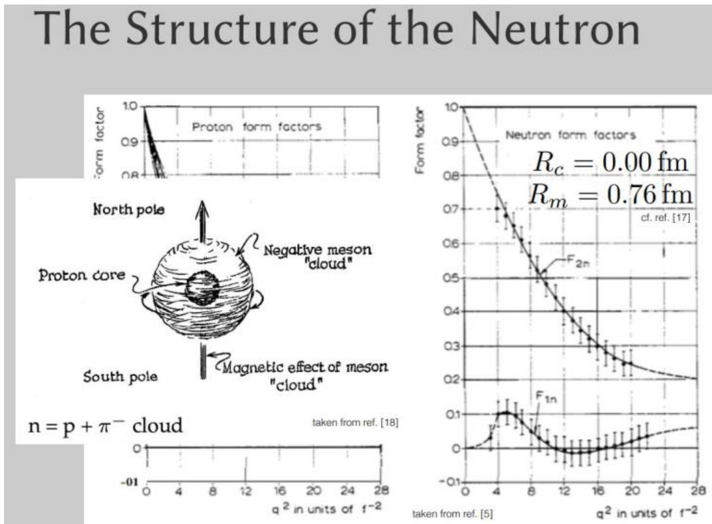

Figure 2: The neutron model and the concepts of charge and magnetic radius (R_c and R_m)

The neutron model shows a proton surrounded by a meson cloud: n = p + π^-. The negative meson (π^-) is unstable and disintegrates into a muon-electron while emitting a neutrino: $\pi^- \to \mu^- + \nu_\mu$. The energy of a meson is 139.570 MeV, which far exceeds the energy difference between a neutron and a proton (939.565 MeV – 938.272 MeV ≈1.3 MeV). The (rest) energy of the muon-electron is about 105.66 MeV, which is about 207 times the electron (rest) energy (0.511 MeV). We, therefore, think the neutron consists of a proton and an electron rather than a proton and a π^- or μ^-. Why? Because one can better account for the 1.3 MeV energy difference between a free neutron and a free proton by a particle whose energy is 0.511 MeV only.

Of course, the electron accounts for about 40% of the energy difference only but a (free) neutron is unstable. A free neutron effectively disintegrates into a proton and an electron[15]: $n^0 \to p^+ + e^- + \nu_e$. The

[13] Christoph Schweiger, *The electron-scattering method, and its applications to the structure of nuclei and nucleons*, 8 January 2016. Schweiger took these illustrations from Robert Hofstadter's 1961 Nobel Prize Lecture, which has the same title. We find Schweiger's added neutron model and the $R_c = 0$ and $R_m = 0.76$ fm formulas very didactic, however.

[14] Schweiger took the illustrations from Robert Hofstadter's 1961 Nobel Prize Lecture, which has the same title. We laud Schweiger for adding the neutron model and the $R_c = 0$ and $R_m = 0.76$ fm formulas, which we find very didactic.

[15] It takes an extraordinarily long time to do so, however: its mean lifetime is $\tau = 879.6 \pm 0.8$ s, so that is a rather amazing 14.96 minutes! Assuming exponential decay, this means we are still left with a fraction of $1/e \approx 0.37$ out of the initial number of neutrons after 15 minutes!

presence of a neutrino suggests the binding energy is nuclear rather than electromagnetic.[16] As we are going to model the nuclear force as a force of attraction between *like* charges (as opposed to the electrostatic Coulomb force, which causes like charge to *repel* each other), we are fine explaining the remaining 60% as *positive* nuclear binding energy between the proton and the electron. *Negative* nuclear binding energy is illustrated by the 2.2 MeV energy difference between the deuteron nucleus (about 1875.613 MeV) and its two constituents (939.565 MeV + 938.272 MeV = 1,877.837 MeV).

OK. Let us get into the meat of the matter now—literally. 😊 What is the charge radius of deuteron again? Right. About 2.1 fm.[17] 😊

Yukawa's nuclear potential and force

Potential, potential energy, fields, and inverse square/cube force laws

To get an intuitive grasp of the *nature* of the potential and force functions, it is probably good to review the basic conventions and definitions:

1. The scalar potential is the potential *energy* of a charge, say Q, per *unit charge*: V(*r*) = U(*r*)/Q. This explains the rather subtle difference between the [$-kq_e^2/r$] and [$-kq_e/r$] dimensions: N·m versus N·m/C. however, because we will be talking *unit* charges only, we will not distinguish between potential and potential *energy* functions.

2. The field is, likewise, nothing but the force *per unit charge* and, hence, it is expressed in N/C. For the magnetic field (***B*** or ***F*** = q***v***×***B***), we have an added s/m factor because of the geometry of the magnetic field and force.[18]

3. The (electrostatic) force (or, when thinking in terms of the unit charge (see remark 1), the *force field*) is the (negative of the) ***gradient*** of the (scalar) potential U or ϕ:

$$\boldsymbol{E} = \left(E_x, E_y, E_z\right) = -\nabla \mathrm{U} = -\left(\frac{\partial \mathrm{U}}{\partial x}, \frac{\partial \mathrm{U}}{\partial y}, \frac{\partial \mathrm{U}}{\partial z}\right)$$

This is quite wonderful: we have a scalar function U(***x***) or U(***r***) from which we can derive the electric field and, therefore, the electrostatic force ***F*** = q·***E*** on a charge. We have three *components* here, which we may write as three equations: $E_x = -\partial \mathrm{U}/\partial x$, $E_x = -\partial \mathrm{U}/\partial y$, $E_x = -\partial \mathrm{U}/\partial z$. The minus sign depends on the nature of the force: the Coulomb force causes like charges to *repel* each other. We will assume a nuclear force which *attracts* like charges in order to explain the mutual (static) attraction of the two protons in the deuteron nucleus.

[16] All these reactions respect conservation of charge, energy as well as linear *and* angular momentum (spin). Adding the overbar or not for the neutrino is a matter of convention: we believe neutrinos and antineutrinos differ in their angular momentum (spin) only because they do not carry charge. To be precise, we think neutrinos are the photons of the nuclear force: just like photons in electromagnetic interactions, they account for the nickel-and-dime in the energy, momentum, and spin conservation equations. As lightlike particles carrying no charge, they must travel at the speed of light, *exactly*. We, therefore, any experiments which supposedly measure neutrino rest mass are erroneous.

[17] For a discussion, see: O.J. Hernandez et. al., *The deuteron-radius puzzle is alive: A new analysis of nuclear structure uncertainties*, . Note this is a radius but it is, of course, also a measure of the internucleon distance.

[18] See the remarks on physical dimensions in footnote 2.

4. The magnetic force (or, when thinking in terms of the unit charge once more (see remark 1), the *force field*) is the ***curl*** of the (vector) potential ***A***:

$$\boldsymbol{B} = \left(E_x, E_y, E_z\right) = \nabla \times \boldsymbol{A} = (\frac{\partial \mathrm{U}}{\partial x}, \frac{\partial \mathrm{U}}{\partial y}, \frac{\partial \mathrm{U}}{\partial z}) \times \left(E_x, E_y, E_z\right) = \left((\nabla \times \boldsymbol{A})_x, (\nabla \times \boldsymbol{A})_y, (\nabla \times \boldsymbol{A})_z\right)$$

We again have three components and, therefore, three *equations* here—one for each of the components (B_x, B_y, and B_z) of ***B***:

$$(\nabla \times \boldsymbol{A})_x = \nabla_x A_y - \nabla_y A_x = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}$$

$$(\nabla \times \boldsymbol{A})_y = \nabla_y A_z - \nabla_z A_y = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}$$

$$(\nabla \times \boldsymbol{A})_z = \nabla_z A_x - \nabla_x A_z = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}$$

One can see that an analysis with both ***E*** and ***B*** vectors is somewhat more difficult because of the simultaneous presence of two vector fields.

5. The Lorenz gauge connects both the scalar and vector potentials:

$$\nabla \cdot \boldsymbol{A} = -\frac{1}{c^2}\frac{\partial \phi}{\partial t}$$

For a time-independent scalar potential, which is what we are going to model, the Lorentz gauge will be zero ($\nabla \cdot \boldsymbol{A} = 0$) because the time derivative is zero: $\partial\phi/\partial t = 0 \Leftrightarrow \nabla \cdot \boldsymbol{A} = 0$.[19] The ***B*** field, therefore, vanishes.

If anything, this shows that modeling electric *and* magnetic fields simultaneously might, after all, *not* be so difficult: all we need to do, is to model the time dependence of the scalar potential ($\partial\phi/\partial t$) and – through the Lorenz gauge – we then get the time-dependent ***B*** field for free, so to speak! What is the connection here? Maxwell's equations, of course! Why? Because Maxwell's equations connect the ***E*** and ***B*** fields through (relatively) simple vector algebra—the theorems of Gauss and Stokes, basically![20]

6. The electrostatic potential decreases with $1/r$. A dipole field, however, follows an inverse square law and, therefore, decreases with $1/r^2$. The respective force or force field will, therefore, follow an inverse square and inverse *cube* law, respectively. We need to watch the signs here: the Coulomb force should *repel* the two protons, while the nuclear force should *attract* them. The forces will, therefore, have opposite signs.

$$\mathrm{F_{coulomb}} \sim -\nabla\left(\frac{1}{r}\right) = -\nabla(r^{-1}) = -\left(\frac{\partial(r^{-1})}{\partial x}, \frac{\partial(r^{-1})}{\partial y}, \frac{\partial(r^{-1})}{\partial z}\right) = \frac{1}{r^2}\left(\frac{\partial r}{\partial x}, \frac{\partial r}{\partial x}, \frac{\partial r}{\partial x}\right)$$

[19] The Lorentz gauge does not refer to the Dutch physicist H.A. Lorentz but to the Danish physicist Ludvig Valentin Lorenz. The reader should not think we have a *choice* here: the Lorentz gauge is one and the same for time-dependent and time-independent fields, but it vanishes with time-independent fields (electromagnetostatics).

[20] This should probably be the next development of our deuteron model, then!

$$F_{\text{nuclear}} \sim \nabla\left(\frac{1}{r^2}\right) = \nabla(r^{-2}) = \left(\frac{\partial(r^{-2})}{\partial x}, \frac{\partial(r^{-2})}{\partial y}, \frac{\partial(r^{-2})}{\partial z}\right) = -\frac{2}{r^3}\left(\frac{\partial r}{\partial x}, \frac{\partial r}{\partial x}, \frac{\partial r}{\partial x}\right)$$

These are the potentials and forces we need to ensure a neatly separated near and far electronuclear field (**Figure 3**). The nuclear force will be larger than the Coulomb for $r \leq 2$ and, vice versa, smaller for $r \geq 2$. Note, however, that the potentials equal each other not at $r = 2$ but at $r = 1$.

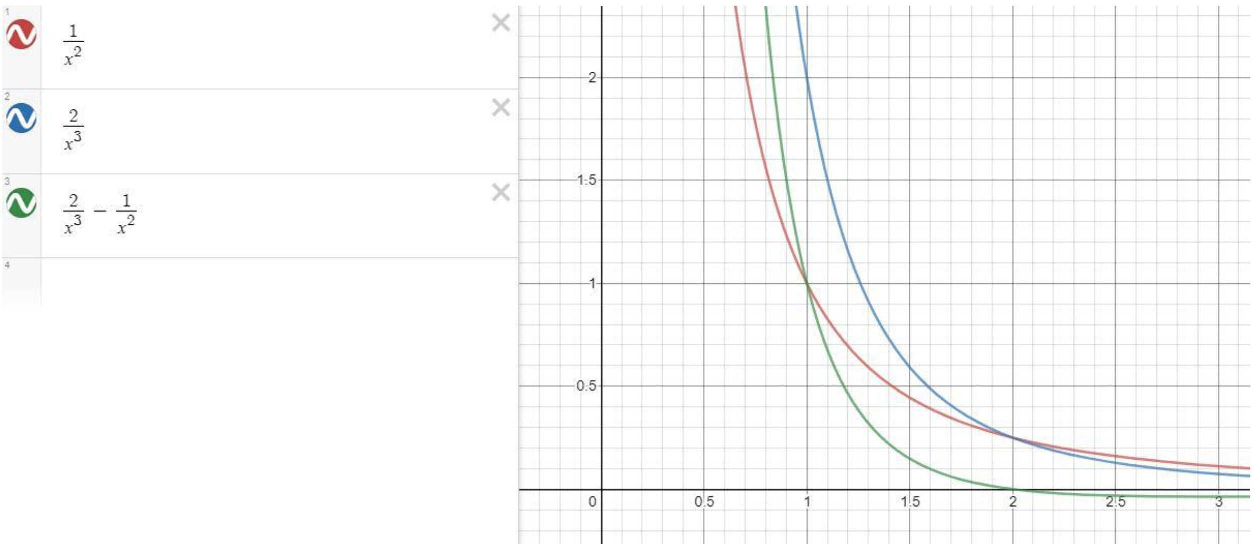

Figure 3: Opposing inverse square and inverse cube force laws

	Forces	**Potentials**
Near field: $F_C \leq F_N$	$\frac{1}{r^2} \leq \frac{2}{r^3} \Leftrightarrow r \leq 2$	$\frac{1}{r} \leq \frac{1}{r^2} \Leftrightarrow r \leq 1$
Far field: $F_C \geq F_N$	$\frac{1}{r^2} \geq \frac{2}{r^3} \Leftrightarrow r \geq 2$	$\frac{1}{r} \geq \frac{2}{r^2} \Leftrightarrow r \geq 1$

So what *is* the assumption here, *really*? The assumption is that the neutron electron cloud does not shield the proton from the electrostatic force between the two protons in the deuteron. However, because the proton also pulls the neutron electron blanket towards itself, so to speak, an electric dipole is created, whose potential follows an inverse square law. The proton, therefore, experiences an inverse *cube* force field which, within the region $r \leq 2$, is sufficiently strong to counter the electrostatic Coulomb force.

Yukawa's potential

The Yukawa potential is usually written as follows[21]:

$$\mathrm{U}(r) = -\frac{\mathrm{g}_{\mathrm{N}}^2}{4\pi\upsilon_0}\frac{e^{-\frac{r}{a}}}{r}$$

To make sure you understand what Yukawa tried to model, we remind you of the formula for the electrostatic (Coulomb) potential:

$$\mathrm{V}(r) = -\frac{\mathrm{q}_{\mathrm{e}}^2}{4\pi\varepsilon_0}\frac{1}{r}$$

However, Yukawa's potential function cannot do the trick: one never gets a nuclear force field that is *stronger* than the Coulomb force: the $1/r$ and $e^{-r/a}/r$ functions never cross, regardless what value we use for the range parameter a. Hence, the associated force functions do never cross either![22] Hence, we must boldly decide to re-write the Yukawa potential as:

$$\mathrm{U}(r) = -\frac{\mathrm{g}_{\mathrm{N}}^2}{4\pi\upsilon_0}\frac{e^{-\frac{r}{a}}}{r^2}$$

The a is a *range* parameter. Of course, we could also just think of it as some kind of natural distance unit, which implies we would measure all distances in units of a. According to Aitchison and Hey, we should use a value around 2.1 fm, which is about the size of deuteron, i.e. the nucleus of deuterium, which consists of a proton and a neutron bound together. However, that does not seem to much sense if we think of $e^{-r/a}$ as the distance d in the dipole moment p = qd: the order of magnitude is then more like 0.1 or 0.2 fm—or, say, 0.5 fm at most! Why? Because these are the distances that come out of actual experiments, and they are also the values which Di Sia used to get relevant nuclear binding energies—i.e. energy values in the MeV range. Have a look at the dipole potential formula once more:

$$\phi(r) = \frac{1}{4\pi\epsilon_0}\frac{\mathrm{qd}\cdot\cos\theta}{r^2}$$

Now think: what can we say about a right now? The $e^{-r/a}$ in Yukawa's formula must, somehow, correspond to the d in the dipole formula, but we do not know how, *exactly*.

[…]

So. Right. Nothing much. Just note a cannot be zero (we should avoid divisions by 0) and that the factor becomes the $e^{-r/a} = e^{-r/a} = 1/e \approx 0.37$ factor we encountered when discussing the exponential decay of

[21] The Wikipedia article uses a mass factor but we prefer the original formula given in Aitchison and Hey's *Gauge Theories in Particle Physics* (2013). It is a widely used textbook in advanced courses and, hence, we will use it as a reference point. We may also refer the reader to Feynman's remarks on it (II-28-6, the nuclear force field) because these are online and free. Note that we write the potential as $V(r) = -kq_e^2/r^2$ rather than as $V(r) = -kq_e/r^2$. We mentioned the subtle difference between potential and potential *energy* already, and that we would not always respect these subtleties ourselves! We are in good company here, however, because Aitchison and Hey do the same!

[22] See the presentation and development of the Yukawa function in our previous paper: *Moving charges, electromagnetic waves, radiation, and near and far fields*, December 2020. For a more authoritative graph of the problem, see Fig. 28-6 in Feynman's Lectures (II-28-6).

neutrons[23] for $a = e \approx 2.71$—which, from a mathematical viewpoint, would seem to a more suitable *fit* for a—if we have to define one, that is, which we do not! Indeed, unlike Aitchison and Hey and others writing on the topic, we will not want to predefine a. We repeat:

You should just think of it as a natural distance unit—so that is the unit in which we assume r to be measured! **The goal is, therefore, to find not only *r* but also *a*!**

Let us look at the *structure* of these two formulas once more. They are exactly the same, except for (1) the $e^{-r/a}$ *function* (which substitutes the distance between the two charges in the dipole potential formula by a variable) and (2) the υ_0 factor, which is usually forgotten. We explained the former ($e^{-r/a}$ as d), so let us now look at the latter (the proportionality constant with the *nuclear* permittivity υ_0).

The nuclear permittivity factor υ_0

We think we need it – for the time being, at least – to ensure the *physical* dimension of both sides of the equation is the same. It is, therefore, similar to the physical dimension of the electric constant ε_0[24]: instead of $C^2/N{\cdot}m^2$, we write: $[\upsilon_0] = Y^2/N{\cdot}m^2$. It does the same trick as ε_0 for the electrostatic Coulomb or dipole potential – it gives us a U(*r*) expressed in *joule* or N·m – and it would, therefore, probably be a mistake to leave it out. Indeed, we started off by saying that the idea of a nucleon charge is something new: we associate some potential with it. However, we should not think of it as electrostatic charge. We have no positive or negative charge, for example: all nucleons – positive, negative, or neutral[25] – share the same charge and should attract each other by the same (strong) force.

We, therefore, think we should define some new *unit* for it. We thought of the *Einstein*, but that name is used for some other unit already.[26] In my previous papers on the topic of the Yukawa potential[27], we suggested the *Yukawa* but I now think there is too much association between that name and the presumed unit of the Yukawa potential.[28] We, therefore, propose the *dirac*.[29] However, for reasons of consistency we will continue to use the charge symbol we used in previous papers: Y.[30]

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[23] See footnote 15.

[24] Note that ε_0 for $\varepsilon_0 = q_e^2/2\alpha hc$ since the revision to the 2019 revision of SI units here, which we think of as being very significant—more significant than CERN's experiments on testing the quark hypothesis or the Higgs field.

[25] Negative? We only have neutrons and protons, don't we? Yes, but we can *imagine* anti-atoms and, hence, anti-protons. Protons and anti-protons will annihilate each other, but two anti-protons should stick together by the same nuclear force.

[26] Believe it or not, but the *Einstein* is defined as a one *mole* ($6.022{\times}10^{23}$) of photons. It is used, for example, when discussing photosynthesis: we can then define the flux of light – or the flux of photons, to be precise – in terms of *x* micro-einsteins per second per square meter. For more information, see the Wikipedia article on the Einstein as a unit: https://en.wikipedia.org/wiki/Einstein_(unit). If we would truly want to honor Einstein, I would suggest we re-define the Einstein as the unit of charge of the nucleon.

[27] See: *The nature of Yukawa's force and charge* and *Who needs Yukawa's wave equation?* (June 2019). Our treatment here is a shortened and revised version of *Neutrinos as the photons of the strong force* (October 2019). The main revision consists of the use of g_N and q_e instead of g_N^2 and q_e^2 in the potential and force formulas.

[28] The Wikipedia article on the Yukawa potential associates the 1/m unit with it, but that makes no sense whatsoever to us.

[29] We note that Dirac's colleagues at Cambridge seem to have defined the *dirac* as 'one word per hour' but we think there is no scope for confusion here.

[30] This matches the *upsilon* (υ) – it is not a μ (mu)! – we use for the proportionality factor.

OK. Now that we have fixed the dimensions, what *numerical* value should we take for υ_0? We have no idea, but now that we are discussing these things in very much detail, should we wonder about the 4π factor? Do we need it? It is common to both potentials (and to the forces, which we calculate in a minute) because it is the 4π factor in the formulas for the surface *area* ($4\pi r^2$) and the *volume* ($4\pi r^3$) of a sphere.[31] Feynman often substitutes $q_e^2/4\pi\varepsilon_0$ by e^2, which is a unit with a strange but exceedingly simple physical dimension: the C^2 in the numerator and denominator cancel out and we are left with $N{\cdot}m^2$ only, which is great because we need the force to be expressed in *newton*, of course![32] So we will do the same here and we hope the reader will be able to distinguish e^2 and e (Euler's number). Writing $g_N^2/4\pi\upsilon_0$ as N^2 (again, we hope this causes not too much confusion in the mind of the reader!), we get the following formulas now:

$$\mathrm{U}(r) = -\frac{\mathrm{N}^2 \cdot e^{-\frac{r}{a}}}{r^2}$$

$$\mathrm{V}(r) = -\frac{\mathrm{e}^2}{r}$$

Force and force range calculations

If we have a potential, we can calculate the force. In fact, we *should* calculate the force, because we should not be thinking in terms of terms of equating potentials here but in terms of *equating forces*.[33] To do this, we should use this force formula:

$$\mathrm{F_N} = \mathrm{F_C} \Leftrightarrow \mathrm{F_N} - \mathrm{F_C} = 0 \Leftrightarrow -\frac{\mathrm{dU}(r)}{\mathrm{d}r} + \frac{\mathrm{dV}(r)}{\mathrm{d}r} \Leftrightarrow \frac{\mathrm{dU}(r)}{\mathrm{d}r} = \frac{\mathrm{dV}(r)}{\mathrm{d}r}$$

Let us think about the minus signs here. The forces should be opposite, right? Right, but the *magnitudes* should be the same and the formula takes care of that. Because we should really keep our wits with us here, let us remind ourselves of what we are trying to do here. We are thinking of two protons here, and these two protons carry an electric charge (q_e) as well as what we vaguely referred to as a nuclear charge (g_N). The electric charge pushes them away from each other, but the nucleon charge pulls them together. At some in-between point, the two forces should be equal but opposite. So we should find some value for a force – expressed in *newton*. Hence, yes, we may assume that a force is a force, even if we know it *acts* on two different unit charges: q_e versus g_N. We express one in *Coulomb* units, and the other in this new unit: the *dirac*. Sounds good? Let us go through the calculations, then.

The Coulomb force is easy to calculate:

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[31] Gauss' Law can be expressed in integral or differential form and these spherical surface area and volume formulas pop up when you go from one to the other. Hence, you should not think of this 4π factor as something weird: it is typical of spherically symmetric fields.

[32] See, for example, Feynman's calculation of the Bohr radius (a) using the $p{\cdot}a = h$ relation—a rather precise expression of the Uncertainty Principle, that is! Note that we will effectively get force formulas – both for the Coulomb as well as for the nuclear force – with $1/r^2$ in the denominator, so we get something expressed in *newton* alright!

[33] The $1/r$ and $e^{-r/a}/r$ functions do not cross anyway, so we should not try to equate them. In fact, the $1/r^2$ and $(r/a +1){\cdot}e^{-r/a}\,]/r^2$ do not cross either! Note that we can compare forces only because the nucleon carries both electric charge *as well as nuclear charge*. The associated fields are, therefore, different: newton/coulomb versus newton/dirac, to be precise.

$$\mathrm{F_C} = \frac{\mathrm{dV}}{\mathrm{d}r} = \frac{\mathrm{d}\left(-\frac{\mathrm{e}^2}{r}\right)}{\mathrm{d}r} = -\mathrm{e}^2\frac{\mathrm{d}\left(\frac{1}{r}\right)}{\mathrm{d}r} = \frac{\mathrm{e}^2}{r^2}$$

This is just Coulomb's Law, of course!

The calculation of the nuclear force is somewhat more complicated because of the $e^{-r/a}$ factor[34]:

$$\mathrm{F_N} = \frac{\mathrm{dU}}{\mathrm{d}r} = \frac{\mathrm{d}\left(-\frac{\mathrm{N}^2\cdot e^{-\frac{r}{a}}}{r^2}\right)}{\mathrm{d}r} = \mathrm{N}^2\cdot\frac{\mathrm{d}\left(-\frac{e^{-\frac{r}{a}}}{r^2}\right)}{\mathrm{d}r} = \mathrm{N}^2\cdot\left(-\frac{\mathrm{d}\left(e^{-\frac{r}{a}}\right)}{\mathrm{d}r}\cdot r^{-2} - e^{-\frac{r}{a}}\cdot\frac{\mathrm{d}(r^{-2})}{\mathrm{d}r}\right)$$
$$= \mathrm{N}^2\cdot\left(\frac{1}{a}\cdot e^{-\frac{r}{a}}\cdot r^{-2} + 2\cdot e^{-\frac{r}{a}}\cdot r^{-3}\right) = \mathrm{N}^2\cdot\frac{\frac{r}{a}\cdot e^{-\frac{r}{a}} + 2\cdot e^{-\frac{r}{a}}}{r^3} = -\frac{\mathrm{N}^2\cdot(\frac{r}{a}+2)\cdot e^{-\frac{r}{a}}}{r^3}$$

This gives us the condition for the nuclear and electrostatic forces to be equal but opposite:

$$\mathrm{F_C} = \mathrm{F_N} \Leftrightarrow \frac{\mathrm{e}^2}{r^2} = \frac{\mathrm{N}^2\cdot\left(\frac{r}{a}+2\right)\cdot e^{-\frac{r}{a}}}{r^3} \Leftrightarrow \frac{\mathrm{e}^2}{\mathrm{N}^2} = \frac{1}{r}\left(\frac{r}{a}+2\right)\cdot e^{-\frac{r}{a}} = \left(1+\frac{2}{r}\right)\cdot e^{-\frac{r}{a}}$$

What can we do with that? We know what e^2 is (we know what the electron charge is and we can, therefore, calculate it), but what about N^2? We have one equation and two unknowns here, so we cannot calculate anything, right? Should we convert back to q_e and g_N? Not sure, but let us see if we get something more meaningful:

$$\mathrm{F_C} = \mathrm{F_N} \Leftrightarrow \frac{\frac{\mathrm{q_e^2}}{4\pi\varepsilon_0}}{\frac{\mathrm{g_N^2}}{4\pi\upsilon_0}} = \frac{\mathrm{q_e^2}}{\mathrm{g_N^2}}\cdot\frac{\upsilon_0}{\varepsilon_0} = \left(1+\frac{2}{r}\right)\cdot e^{-\frac{r}{a}}$$

Eureka![35] We know *r* must be *equal* to *a* if the two forces are equal, right? Right. And let us cheat now: we know that is the case when $r = 2$. Let us also forget about the *dirac* and the nuclear permittivity factor (that was just to make you think), so we can equate $g_N^2/4\pi\upsilon_0$ and $q_e^2/4\pi\varepsilon_0$. The condition above should then become trivial to solve for *a*. Solve for *a*? Yes. That is actually want we should do: remember we did not want to predefine *a*? But we should get some value for it, right? Right:

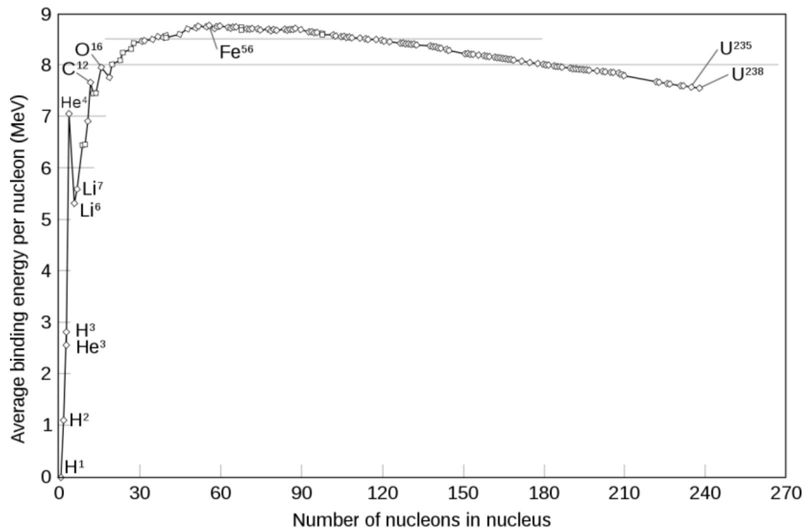
$$\frac{\frac{\mathrm{q_e^2}}{4\pi\varepsilon_0}}{\frac{\mathrm{g_N^2}}{4\pi\upsilon_0}} = 1 = \left(1+\frac{2}{r}\right)\cdot e^{-\frac{r}{a}} = 2\cdot e^{-\frac{2}{a}} \Leftrightarrow e^{\frac{2}{a}} = 2 \Leftrightarrow \frac{2}{a} = \ln(2) \Leftrightarrow a = \frac{2}{\ln(2)} \approx 2.885\ \mathrm{fm}$$

Does this make sense as an interproton distance? We think it does! Indeed, the model offers a pretty plausible explanation of why typical nuclei – ranging from the deuteron nucleus which we studied here

[34] We need to take the derivative of a quotient of two functions here. Needless to say, we invite the reader to carefully check all logic and double-check the calculations and – if needed – to email us their remarks and/or corrections.

[35] Archimedes is said to have exclaimed this in the bathtub when he found a way to distinguish fake from real gold for the tyrant who paid him, but Scientific American thinks the story is fake news. We think Archimedes must have had several *aha* moments. We do too.

to more massive elements – should be stable, although the graph below shows the formula for nuclear binding energies is not as straightforward as you might think![36]



Of course, from the formula that we have found, we can also calculate the distance d between the neutron proton-electron charges. It must be equal to:

$$\mathrm{d} = e^{-\frac{r}{a}} = e^{-\frac{2\cdot\ln(2)}{2}} = \frac{1}{e^{\ln(2)}} = 0.5\ \mathrm{fm}$$

Does this make sense as the distance between the electron and the proton inside of the neutron? We think it does! In fact, we get very similar values to the internucleon and current radius values which work in Di Sia's analysis. We, therefore, consider our problem solved!

Conclusion

We showed how one can use potentials to build up a spin-zero model of the deuteron. As such, it may complement Paolo Di Sia's model of the nucleus (2018), which we gave due attention. In contrast to Di Sia, we think of neutrons – or the electron cloud that surrounds the proton inside – as *electric* dipoles. We did so by interpreting Yukawa's potential function as a dipole potential. Instead of predefining the range parameter *a*, we calculate it from the equilibrium condition (equal but opposite magnitudes of the Coulomb and nuclear forces): we found a very acceptable value of about 2.88 fm for *a*, and an equally acceptable value for the distance between the positively charged center of the neutron and the center of the electron cloud which, in a deuteron nucleus, must shift it center of charge towards the proton so as to ensure stability – not unlike the sharing of valence electrons in chemical bonds.

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[36] We borrow the graph from the Wikipedia article on Fe-56, which is probably the most stable element in the periodic table. The same article tells us the size of the iron nucleus is about 4 proton diameters wide—so that is a *lot* more than our calculated 2.88 fm! Of course, there are more practical explanations for this than our theoretical deuteron model. We will not go into the various formulas for calculating nuclear binding energy but LibreText offers a nice explanation of nuclear (in)stability using magic numbers: we warmly recommend the read!