Velocity and acceleration vectors on second order curves

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Annotation

The movement of a material point along curves of the second order is represented by the differential equation. The projections on the coordinate axes of the first and second derivatives are calculated. The direction of the velocity and acceleration vectors is determined.

Keywords

derivatives; vectors; coordinates; velocity; acceleration.

The second law states that the rate of change of momentum of a body over time is directly proportional to the force applied, and occurs in the same direction as the applied force.

$$Q = m\ddot{x} \tag{1}$$

The applied force causes acceleration. Conversely, acceleration is impossible without force.

Nonlinear motion is described by differential equations.

To compose differential equations, we represent the force acting on a point in a fixed Cartesian coordinate system.

$$m\ddot{x} = -Qcos\big(\varphi(t)\big) \tag{2}$$
$$m\ddot{y} = -Qsin\big(\varphi(t)\big) \tag{3}$$

Из (3)

$$Q = \frac{-m\ddot{x}}{cos\big(\varphi(t)\big)} \tag{4}$$

Let us substitute equation (4) into equation (3)

$$\ddot{y} = \frac{\ddot{x}}{cos\big(\varphi(t)\big)} sin\big(\varphi(t)\big) \tag{5}$$

The point coordinates can be represented as the function of angle of deflection $\varphi(t)$

and radius *r(t)* , Figure 1.

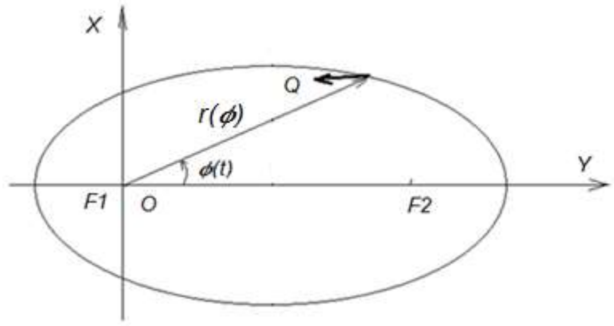


Figure 1

$$x = r(\varphi(t))\cdot cos(\varphi(t)) \tag{6}$$
$$y = r(\varphi(t))\cdot sin(\varphi(t)) \tag{7}$$
$$r(\varphi(t)) = \frac{p}{1-e*cos(\varphi(t))} \tag{8}$$

Let us calculate the first and second time derivative From equations (6), (7), (8). Let the second time derivative be put in the equation (5) and move everything to the left side.

$$\dot{x} = \frac{d}{dt}\left(\frac{p}{1-e*cos(\varphi(t))}cos(\varphi(t))\right) = -\frac{p*cos(\varphi(t))*e*sin(\varphi(t))*\frac{d}{dt}\varphi(t)}{\left(1-e*cos(\varphi(t))\right)^2} - \frac{p*sin(\varphi(t))*\frac{d}{dt}\varphi(t)}{1-e*cos(\varphi(t))} = -\frac{b^2*sin(\varphi(t))*\frac{d}{dt}\varphi(t)}{a*\left(1-2*e*cos(\varphi(t))+e^2*cos(\varphi(t))^2\right)} \tag{9}$$

$$\dot{y} = \frac{d}{dt}\left(\frac{p}{1-e*cos(\varphi(t))}sin(\varphi(t))\right) = -\frac{p*e*sin(\varphi(t))^2*\frac{d}{dt}\varphi(t)}{\left(1-e*cos(\varphi(t))\right)^2} + \frac{p*cos(\varphi(t))*\frac{d}{dt}\varphi(t)}{1-e*cos(\varphi(t))} = \frac{b^2*\frac{d}{dt}\varphi(t)*\left(-e+cos(\varphi(t))\right)}{a*\left(1-2*e*cos(\varphi(t))+e^2*cos(\varphi(t))^2\right)} \tag{10}$$

$$\ddot{x} = \frac{2*p*e^2*cos(\varphi(t))*sin(\varphi(t))^2*\left(\frac{d}{dt}\varphi(t)\right)^2}{\left(1-e*cos(\varphi(t))\right)^3} + \frac{2*p*e*sin(\varphi(t))^2*\left(\frac{d}{dt}\varphi(t)\right)^2}{\left(1-e*cos(\varphi(t))\right)^2} - \frac{p*e*cos(\varphi(t))^2*\left(\frac{d}{dt}\varphi(t)\right)^2}{\left(1-e*cos(\varphi(t))\right)^2} - \frac{p*e*cos(\varphi(t))*sin(\varphi(t))*\frac{d^2}{dt^2}\varphi(t)}{\left(1-e*cos(\varphi(t))\right)^2} - \frac{p*cos(\varphi(t))*\frac{d^2}{dt^2}\varphi(t)}{1-e*cos(\varphi(t))} - \frac{p*sin(\varphi(t))*\frac{d^2}{dt^2}\varphi(t)}{1-e*cos(\varphi(t))} \tag{11}$$

$$\ddot{y} = \frac{2*p*e^2*sin(\varphi(t))^3*\left(\frac{d}{dt}\varphi(t)\right)^2}{(1-e*cos(\varphi(t)))^3} - \frac{3*p*e*sin(\varphi(t))*cos(\varphi(t))*\left(\frac{d}{dt}\varphi(t)\right)^2}{(1-e*cos(\varphi(t)))^2} -$$

$$\frac{p*e*sin(\varphi(t))^2*\frac{d^2}{dt^2}\varphi(t)}{(1-e*cos(\varphi(t)))^2} - \frac{p*sin(\varphi(t))*\left(\frac{d}{dt}\varphi(t)\right)^2}{1-e*cos(\varphi(t))} + \frac{p*cos(\varphi(t))*\frac{d^2}{dt^2}\varphi(t)}{1-e*cos(\varphi(t))} \quad (12)$$

$$\frac{d^2}{dt^2}\varphi(t) = \frac{2*e*sin(\varphi(t))*\left(\frac{d}{dt}\varphi(t)\right)^2}{1-e*cos(\varphi(t))} \quad (13)$$

or

$$\ddot{\varphi} = \frac{2*e*\sin(\varphi)*\dot{\varphi}^2}{1-e*\cos(\varphi)} \quad (14)$$

Equation (14) is differential equation of second-order for conic section with respect to the focus. Different values of the eccentricity will lead into a different shape of the curve.

By solving equation (14), we obtain the values of angular velocity ($\dot{\varphi}$) and angular acceleration ($\ddot{\varphi}$).

Equations (9) to (12) are the first and second coordinate derivatives.

A circle is a special case of an ellipse. Eccentricity is zero. Angular acceleration

$$\frac{d^2}{dt^2}\varphi(t) = 0 \quad (15)$$

From equations (14), (15) it follows that only motion along a circle has a constant acceleration.

$$\frac{d^2}{dt^2}\varphi(t) \neq 0 \quad (16)$$

We have two options for calculating linear velocity (v) and acceleration ($\dot{v}$):

1.

$$v = \sqrt{\dot{x}^2 + \dot{y}^2} \quad (17)$$

$$\dot{v} = \sqrt{\ddot{x}^2 + \ddot{y}^2} \quad (18)$$

2.

$$v = r * \dot{\varphi} \quad (19)$$

$$\dot{v} = r * \sqrt{\dot{\varphi}^4 + \ddot{\varphi}^2} \quad (20)$$

Option 1 allows you to get the direction of the vectors of linear velocity and acceleration.

Let's set the number of conditional days to walk the perimeter of the ellipse. From equation (14) we obtain the angles equal to the sectoral velocity. Substitute the values of the angles in equations (6), (7), determine the coordinates of the points on the ellipse.

We obtain the angular parameters by a numerical method, the program Winkel100MGU_ab_read_from_file.exe [1]. The calculation results are written to the ellpi.txt file [1], in the format of Table 1.

N	t	$\varphi(t)$	$\dot{\varphi}$	$\ddot{\varphi}$
1				
2				
3				
4				

t - time, $\varphi(t)$ - angle, $\dot{\varphi}$ - angular velocity, $\ddot{\varphi}$- angular acceleration.

Table 1

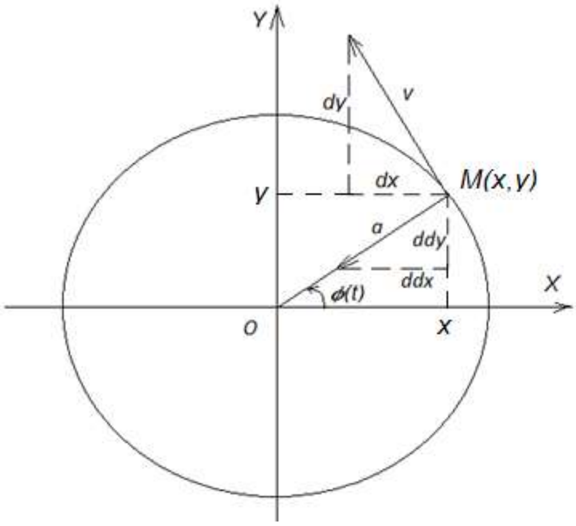
Linear parameters: velocity and acceleration projections on the X, Y axis, equations (9) to (12), velocity vector length, equations (8), (9), acceleration vector length, equations (10), (11), calculated by the program Linear_acceleration_vector.exe. The calculation results are written to the calc_results.txt file [1], in the format of Table 2.

N	r	v_φ	v_{xy}	$\dot{v}_\varphi$	$\dot{v}_{xy}$	angle	dotx	doty
1								
2								
3								
4								

r - polar radius, v_φ - linear speed according to formula (19), v_{xy} - linear speed according to formulas (17), $\dot{v}_\varphi$ - linear acceleration according to formulas (20), $\dot{v}_{xy}$ - linear acceleration according to formulas (19), *angle* - angle between vectors of linear velocity and acceleration, *dotx* is the coordinate of the intersection point of the acceleration vector and the X-axis, *doty* is the coordinate of the intersection of the acceleration vector and the Y-axis.

Table 2

Graphical representation of vectors of velocities and accelerations.



v - speed, *a* - acceleration, *dx, dy, ddx, ddy* - first and second derivatives along the coordinate axes.

Figure 2

Coordinates of the origin of the vectors of speed and acceleration, points of the original ellipse *(x, y)*.

Velocity vector end coordinates

$$(dx+x,\ dy+y) \tag{21}$$

Acceleration vector end coordinates

$$(ddx-dx-x,\ ddy-dy-y) \tag{22}$$

Set in the program equal to the semiaxes of the ellipse $(a = b)$. This is a circle, a special case of an ellipse. The results are shown in Figures 3, 4. Calculation results are written to files calc_results_05_05_20.txt, calc_results_05_05_80.txt [1].

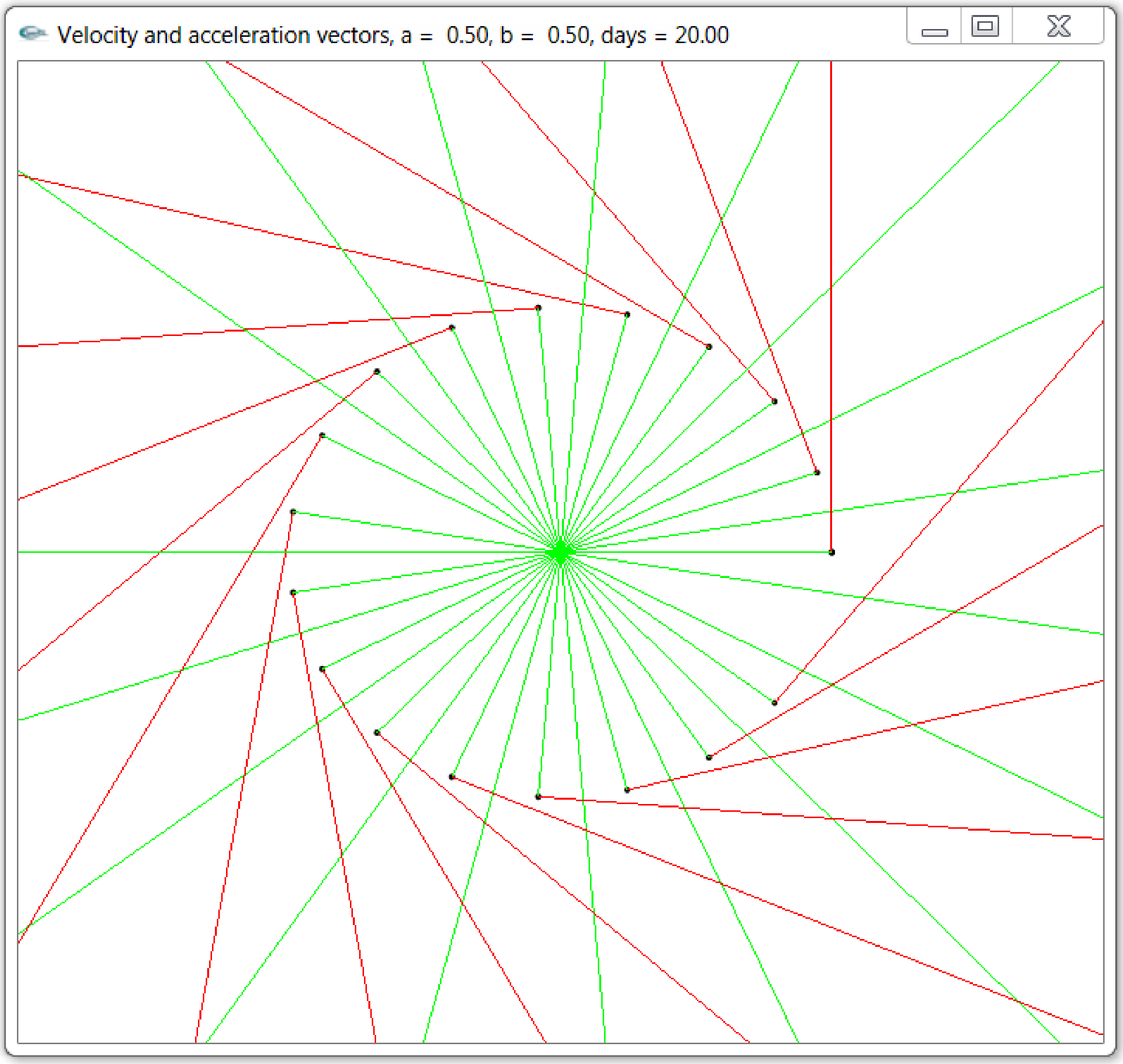


Figure 3

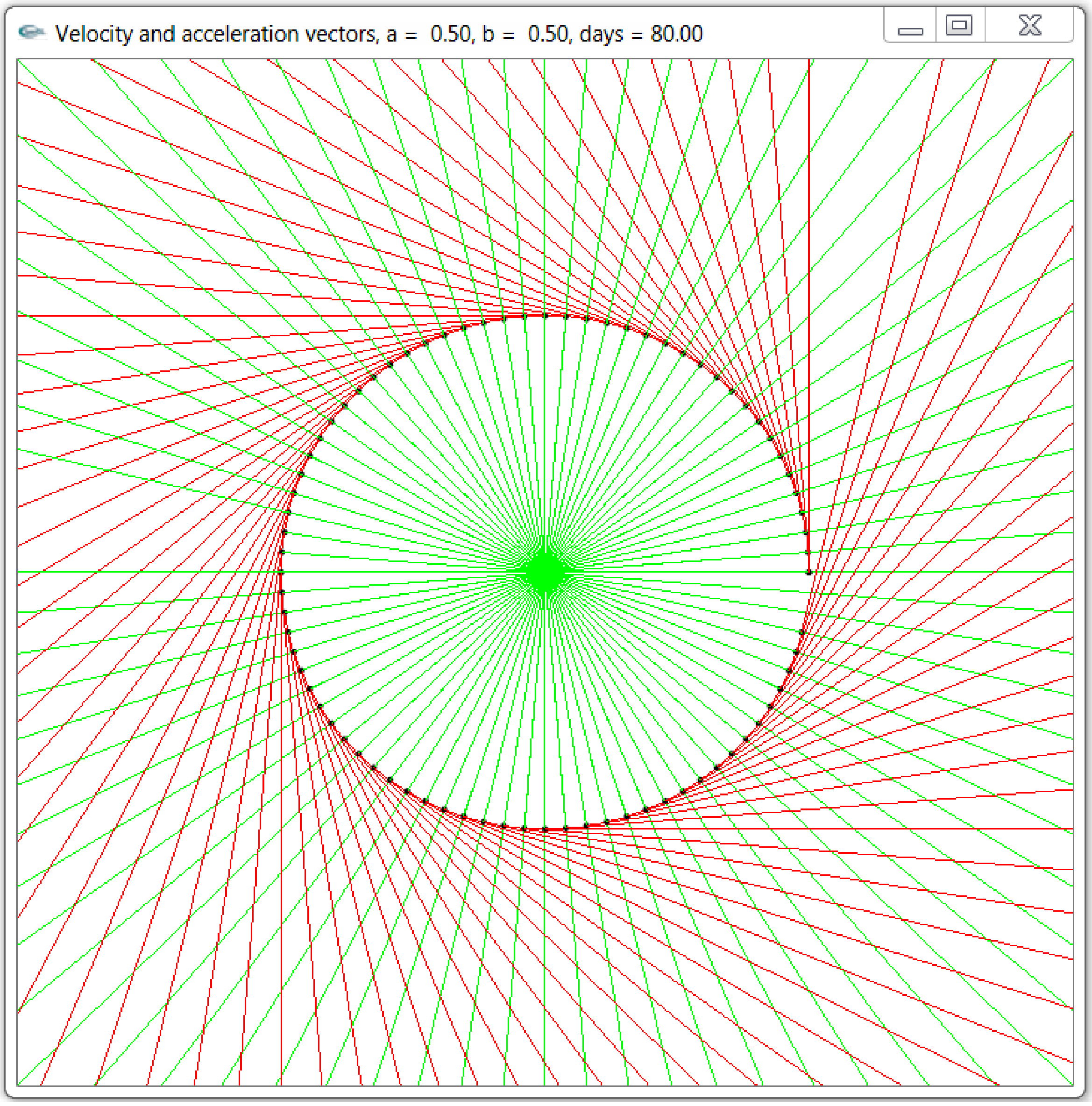


Figure 4

It is known that with uniform motion along a circle, the velocity and acceleration are constant, the velocity is directed tangentially, the acceleration is perpendicular to the velocity. From the figures and tables, we see that the calculated values coincide with the theoretical. Velocity vectors are tangent. The acceleration vectors are normal and intersect at the center of the circle. Change the input parameters. Let us set unequal values of the semiaxes of the ellipse $(a \neq b, a > b)$, Figures 5, 6. Calculation results are written to files _results_05_045_20.txt, calc_results_05_045_80.txt [1].

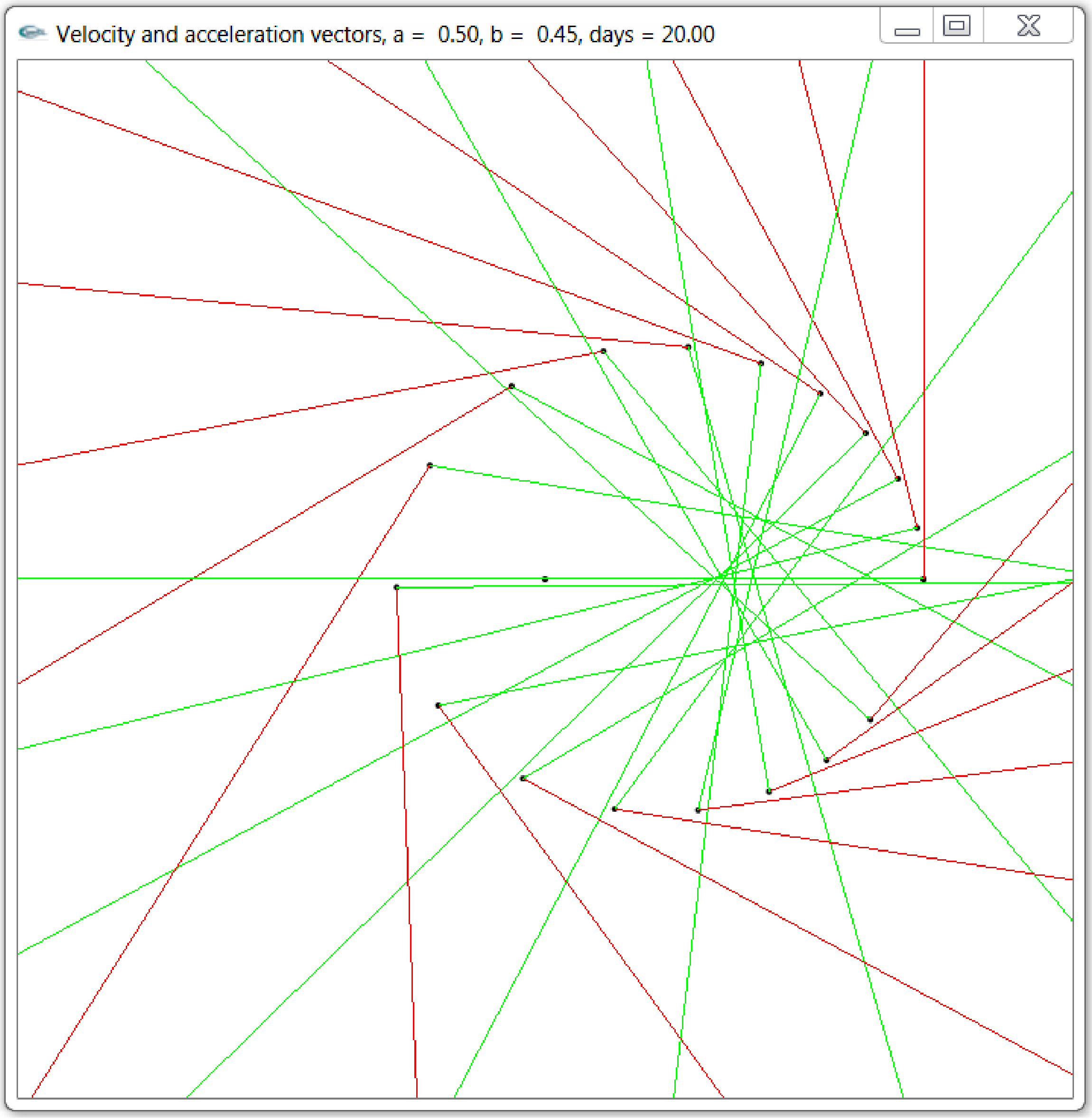


Figure 5

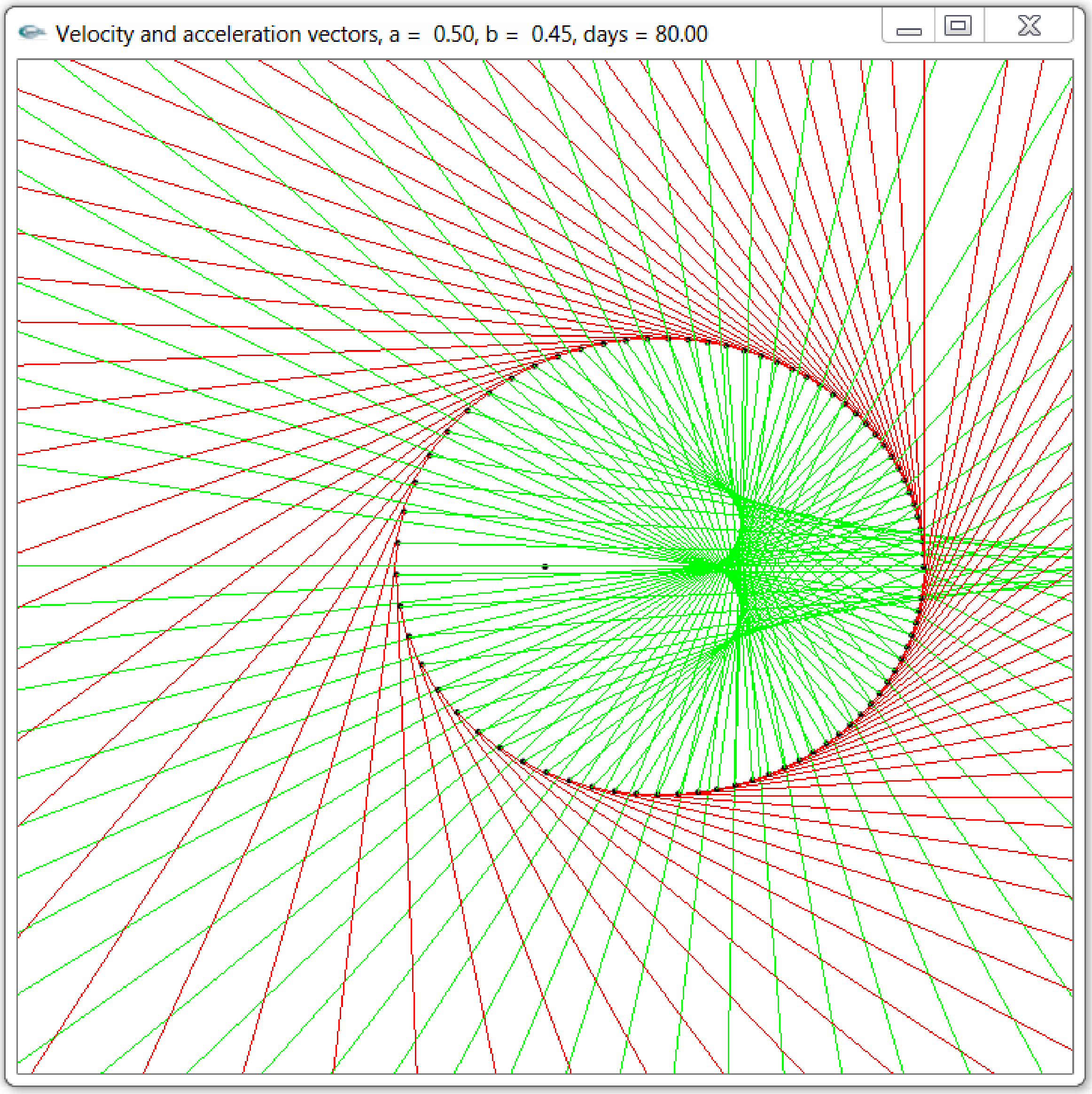


Figure 6

From Figures 5, 6 we see that the acceleration vectors do not have a common intersection point. All lines of acceleration vectors intersect with the axes of the ellipse. Determine the points of intersection of the acceleration vectors with the X axis, Figures 8, 9.

The point of intersection of the velocity vector is determined by the rule of intersection of two segments, Figure 7.

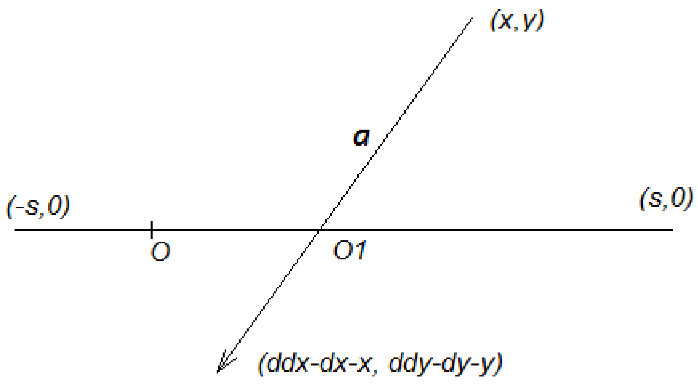


Figure 7.

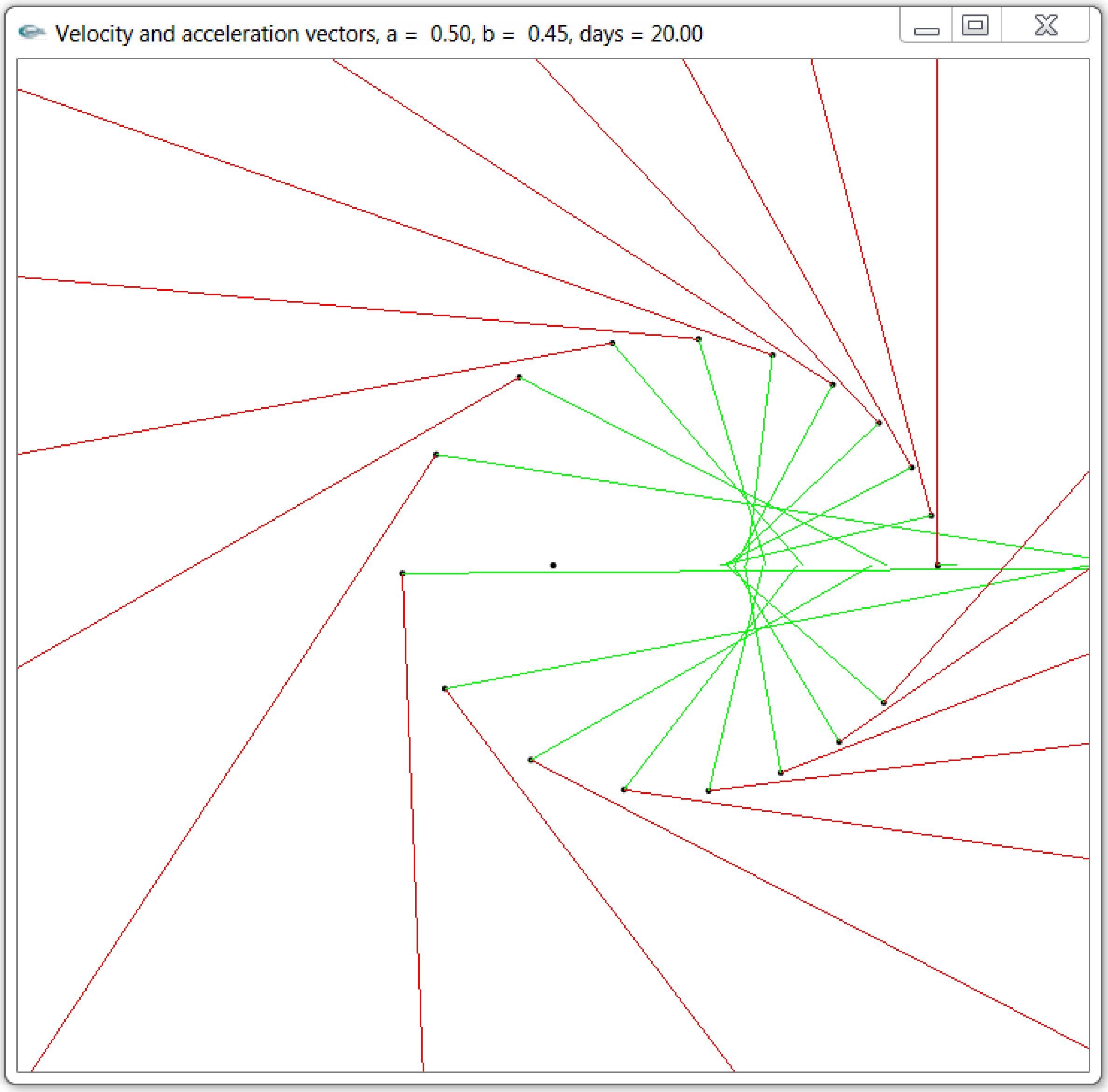


Figure 8.

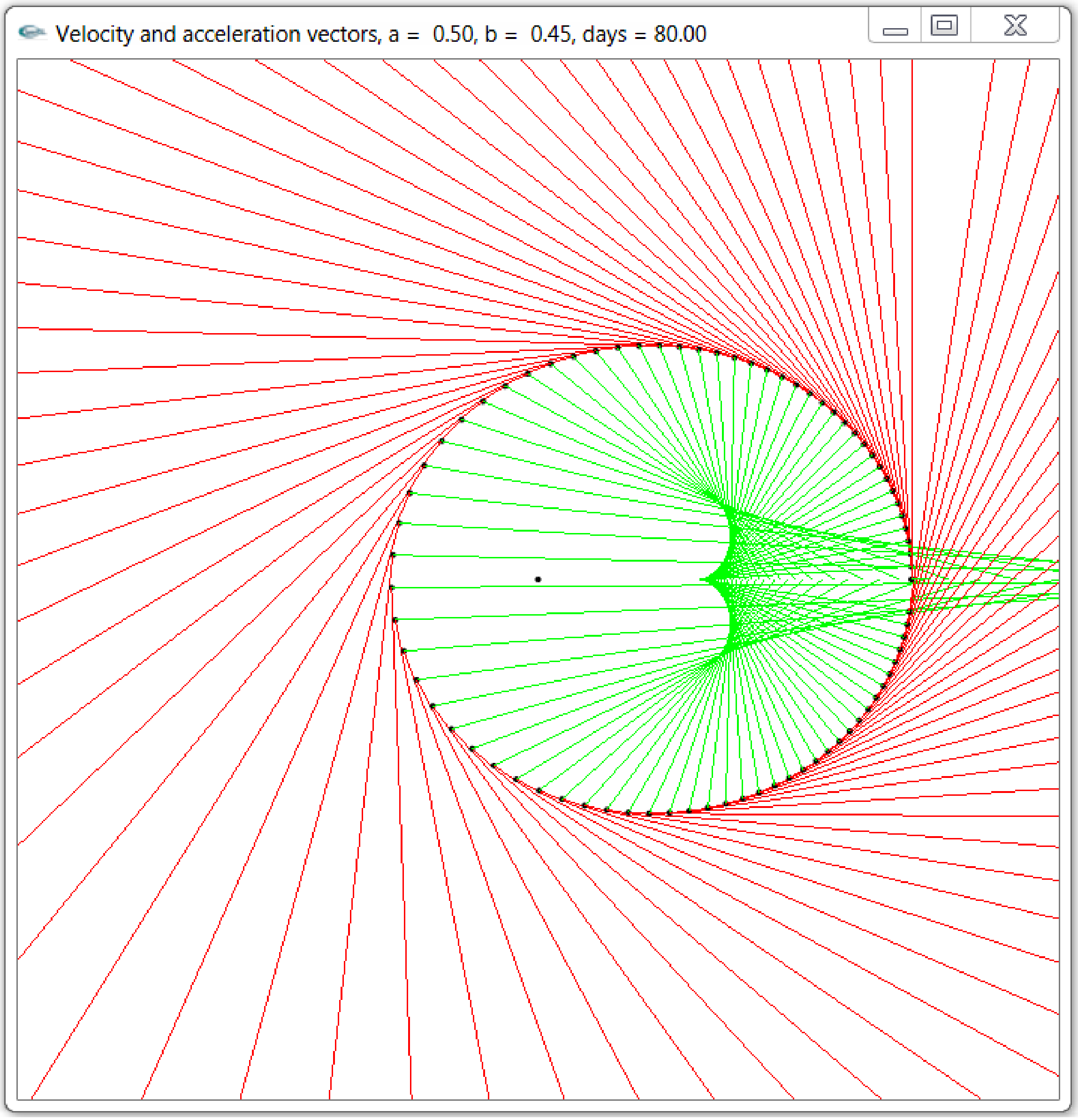


Figure 9

In previous graphs, vector lengths were displayed in absolute values.

Let's find the maximum values of speed and acceleration. Divide all values by the corresponding maximum and display the ellipse, speed and acceleration relative to the common center in one figure 10.

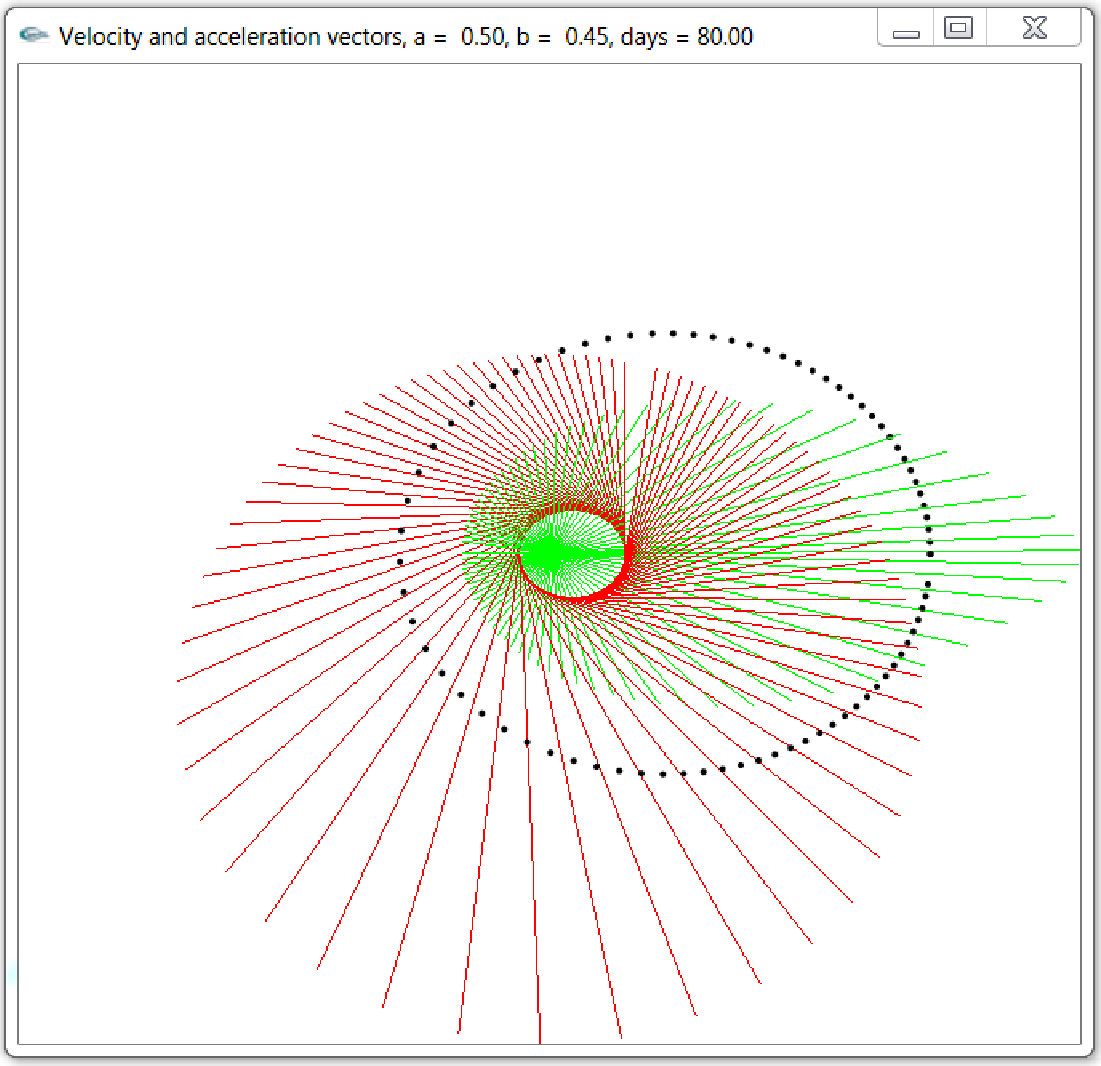


Figure 10

Data received by the program Linear_acceleration_vector.exe [1].

Tangential and normal acceleration

When moving along a curve, the acceleration is decomposed into tangential ($\boldsymbol{a}_\tau$) and normal ($\boldsymbol{a}_n$), Figure 11. The value $\boldsymbol{a}_\tau$ and $\boldsymbol{a}_n$ is not known.

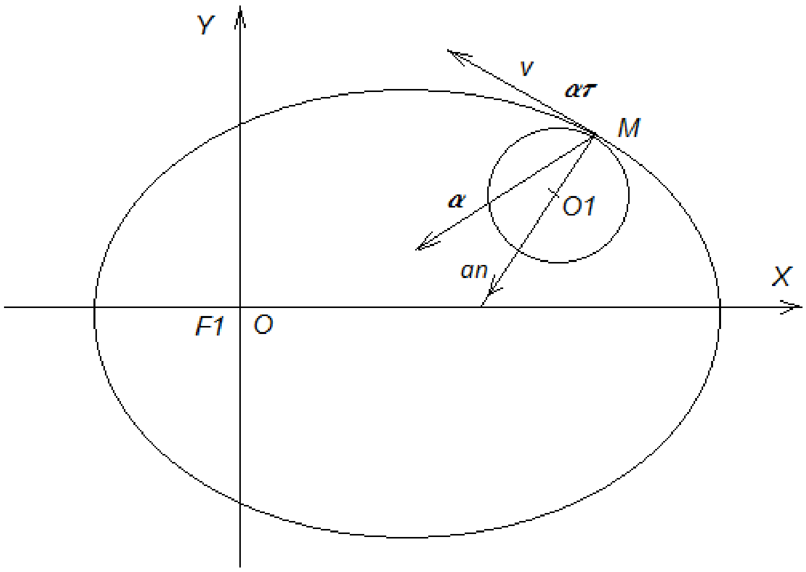


Figure 11

The equation of the tangent to the *curve y = f (x)* at the point *M(x0, y0)* has the form:

$$y - y(x_0) = y'(x_0)(x - x_0) \tag{23}$$

The equation of the normal to the curve *y (x)* at the point *M(x0, y0)* has the form:

$$y - y(x_0) = \frac{1}{y'(x_0)}(x - x_0) \tag{24}$$

Let us write down the equation of the normal to the ellipse

$$\frac{x^2}{a^2} + \frac{y(x)^2}{b^2} = 1 \tag{25}$$

Let's differentiate

$$\frac{2x}{a^2} + \frac{2y(x)\frac{d}{dx}y(x)}{b^2} = 0 \tag{26}$$

$$\frac{d}{dx}y(x) = -\frac{b^2 x}{a^2 y(x)} \tag{27}$$

$$y - y(x_0) = -\frac{a^2 y(x)}{b^2 x}(x - x_0) \tag{29}$$

Let's set the values of the major, minor axis of the ellipse and the number of conditional days of traversing the perimeter of the ellipse. Calculate the coordinates of the normals. Origin coordinates at points on the ellipse. The coordinates of the end of the normals are calculated. Set $x = 0$ and define $y_i(0)$, $(i = 0, 1, 2, ..., days)$. Using formula (29), we carry out calculations, the program Normal_vector.exe [1]. Calculation results are written to a file calc_resultsN.txt [1], in table format 3.

N	x	y	n_x	n_y	t_x	t_y	*angle*
1							
2							
3							
4							

x is the X coordinate, y is the Y coordinate, n_x is the normal to the X axis, n_y is the normal to the Y axis, tx is the tangent coordinate to the X axis, t_y is the tangent coordinate to the Y axis, *angle* is the angle between tangent and normal.

Table 3

Figures 12, 13, 14 show the corresponding normal vectors.

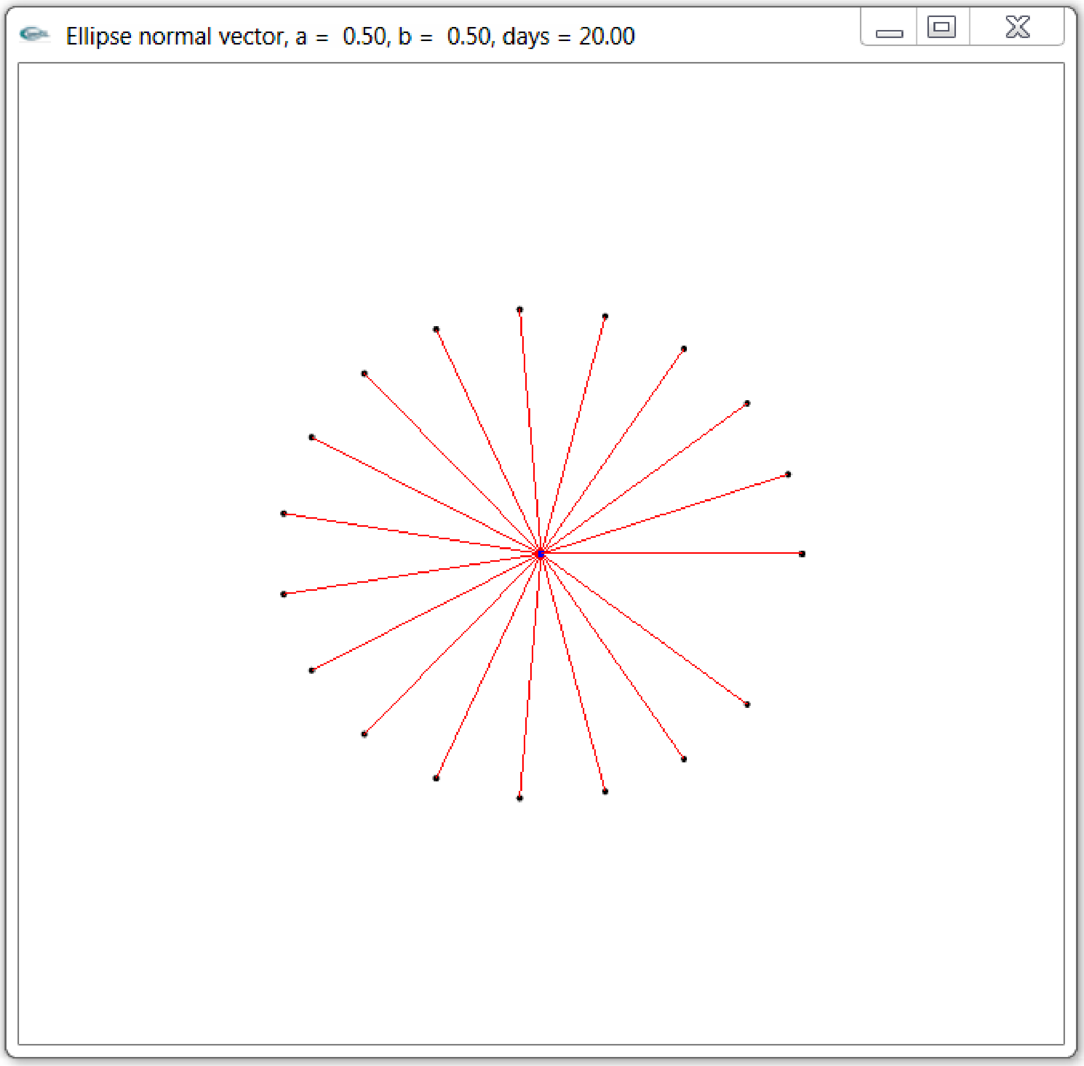


Figure 12

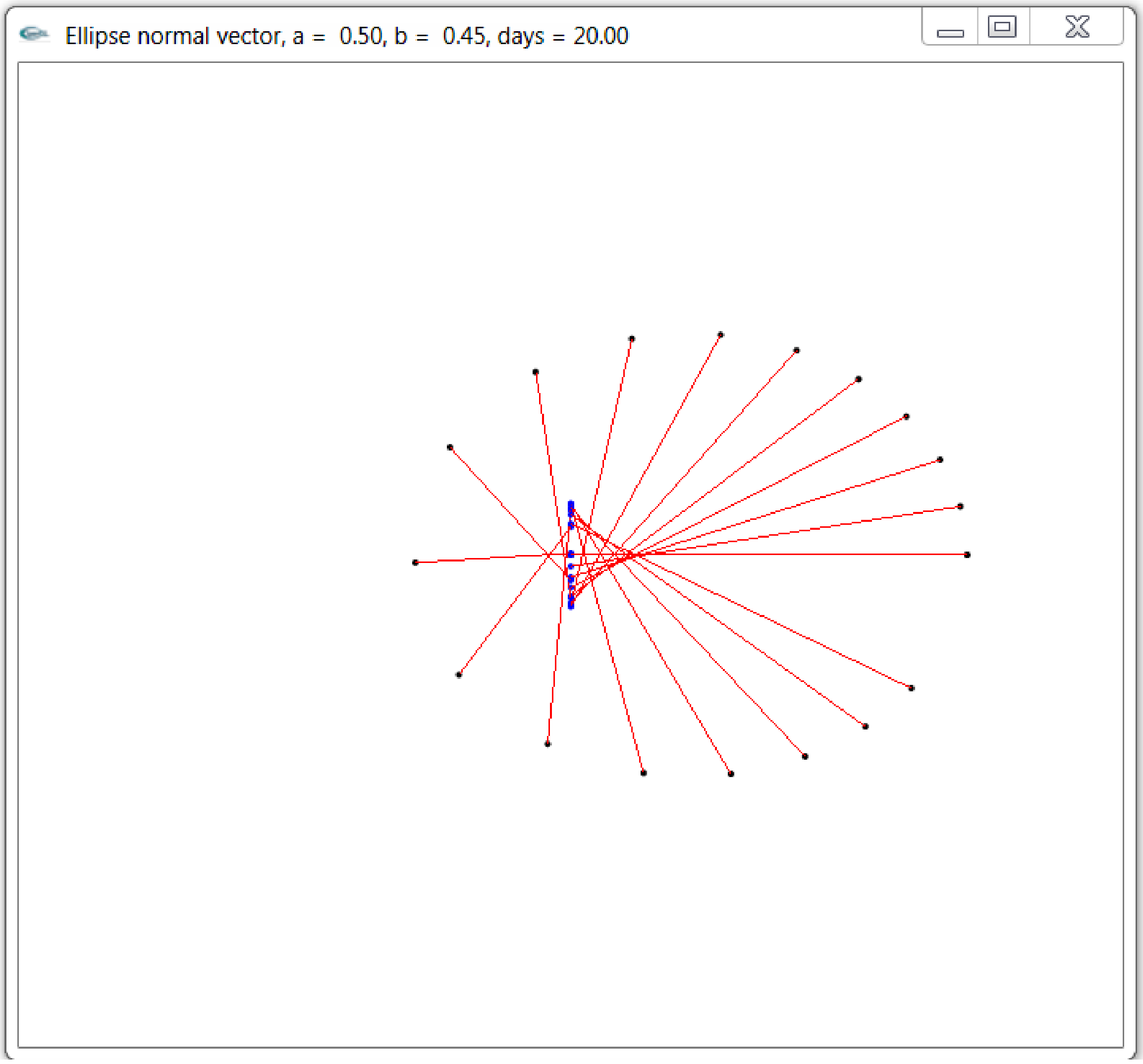


Figure 13

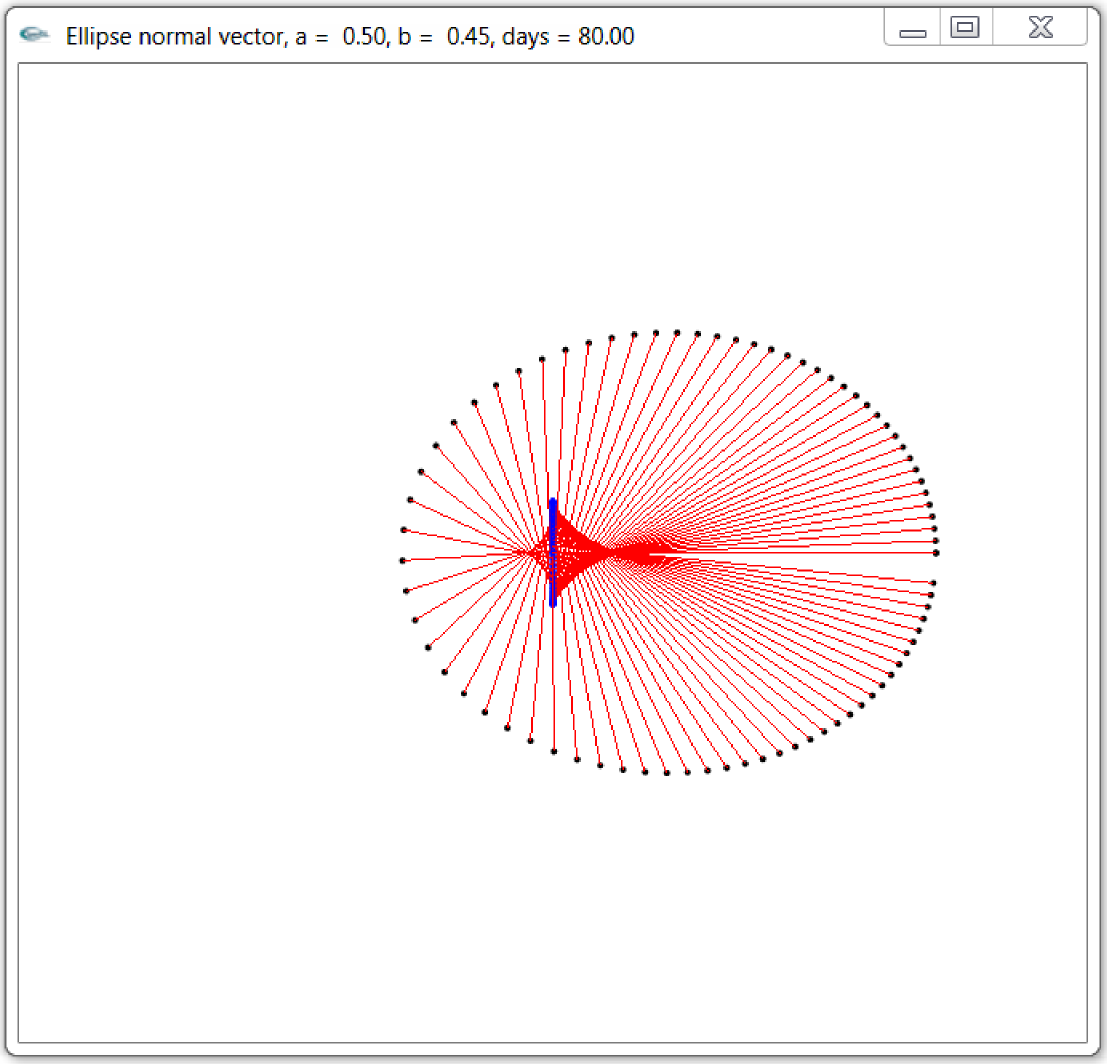


Figure 14

For comparison, we present a drawing of the full acceleration vectors. The acceleration vector is trimmed to the point of intersection with the X-axis, Figure 15.

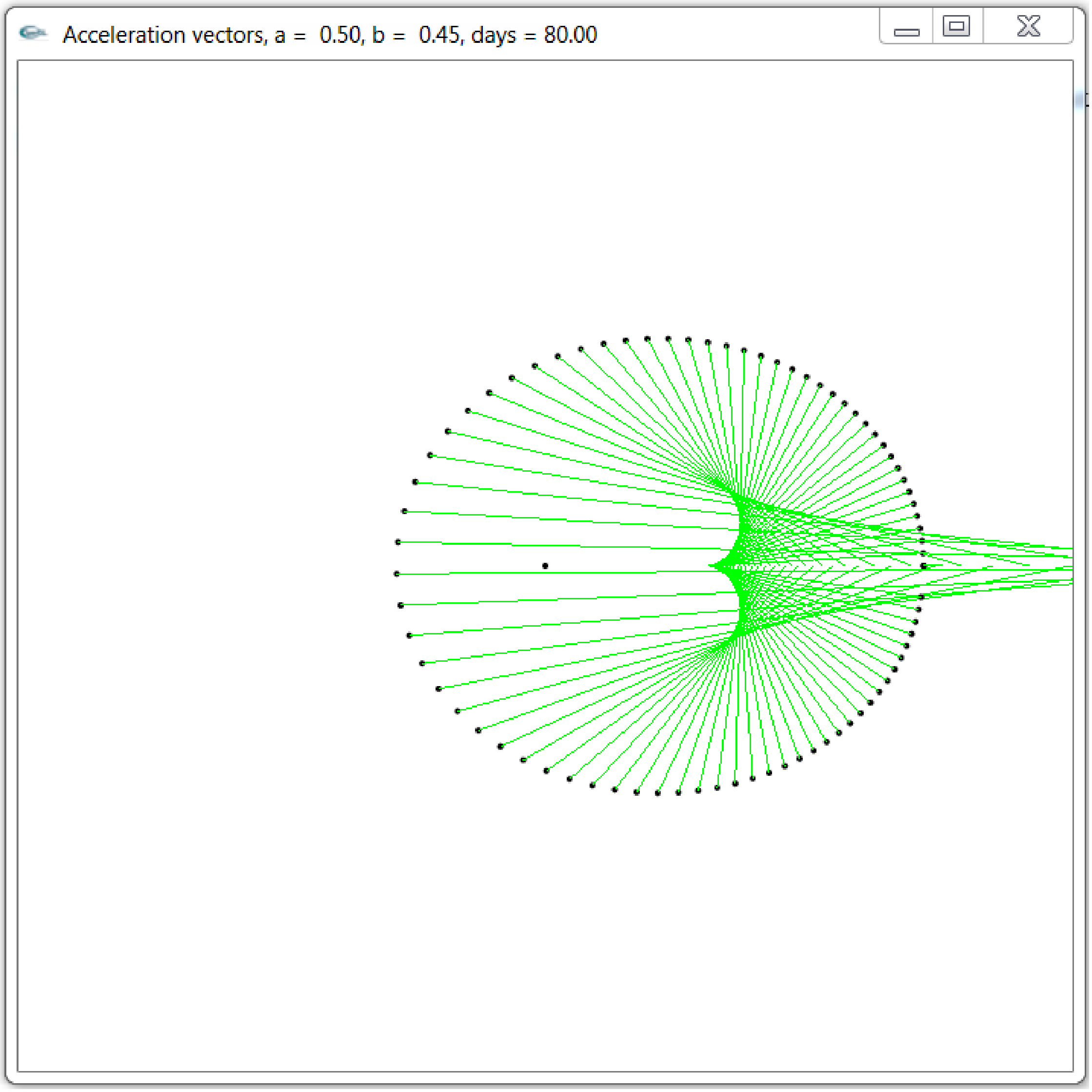


Figure 15

Let us define the resultant of the acceleration vectors.

When adding vectors, the corresponding coordinates are added. The ellipse is symmetrical about the large axis, so when you add coordinates along the Y axis, we get zero. The origin is aligned with the focus, therefore, when adding the coordinates along the X axis, we get an offset, Figure 16.

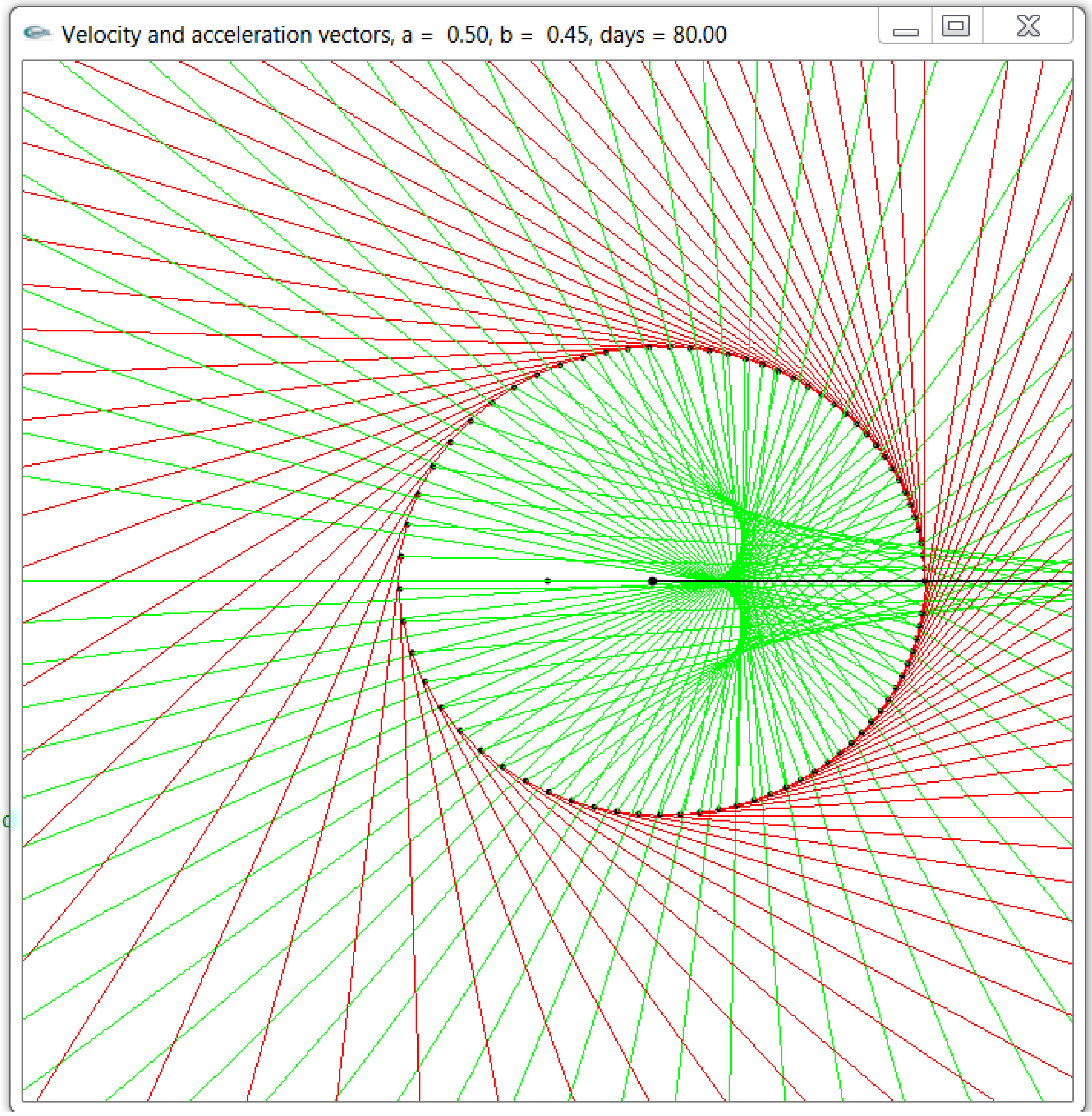


Figure 16

Figure 17 shows the vectors of speed and acceleration of the planet Mercury. Calculation results are written to a file calc_results_Mercury.txt [1].

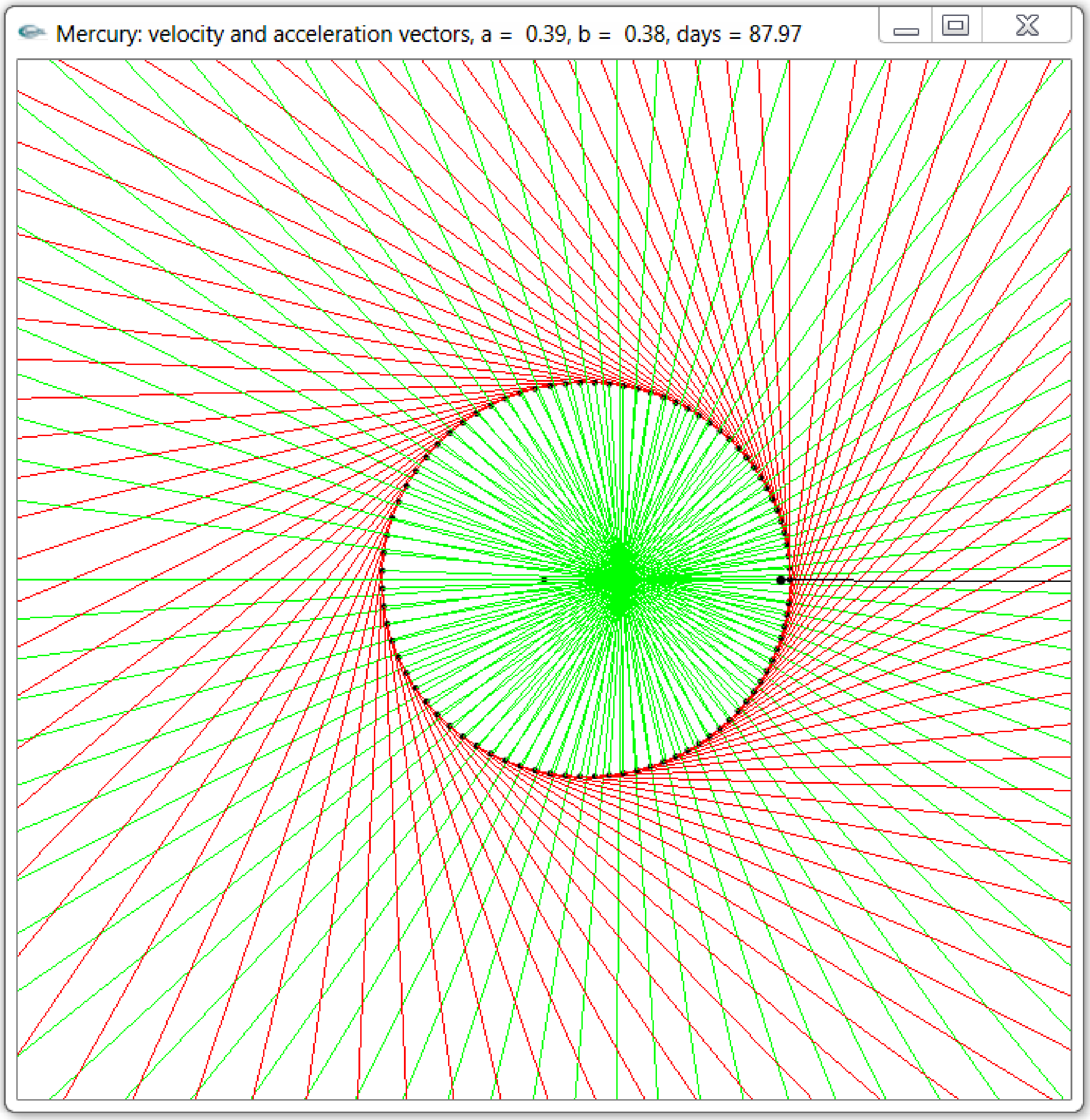


Figure 17

Move a point around the center of the ellipse

Compatible origin with ellipse center, Figure 18.

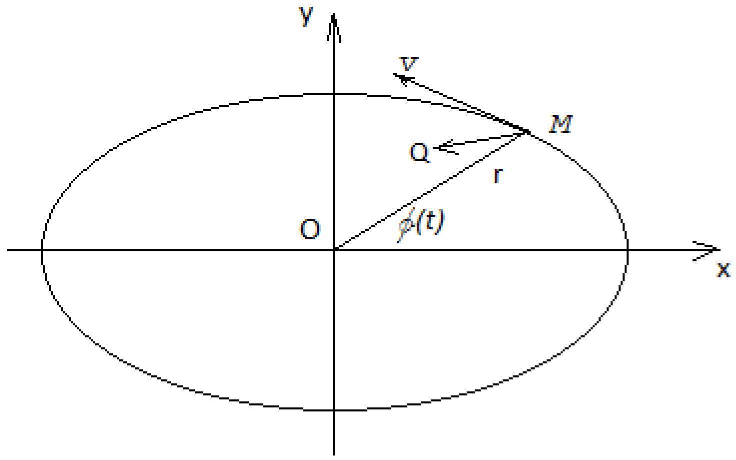


Figure 18

Parametric radius formula

$$r(\varphi(t)) = \frac{b}{\sqrt{1-e^2cos^2\varphi(t)}} \quad (30)$$

Repeat the reasoning of the coordinate center option in focus. We get the following formulas:

$$\dot{x} = \frac{d}{dt}\left(\frac{bcos(\varphi(t))}{\sqrt{1-e^2cos^2\varphi(t)}}\right) = -\frac{bsin(\varphi(t))\frac{d}{dt}\varphi(t)}{\sqrt{1-e^2cos^2\varphi(t)}} - \frac{bcos^2\varphi(t)e^2sin(\varphi(t))\frac{d}{dt}\varphi(t)}{(1-e^2cos^2\varphi(t))(1-e^2cos^2\varphi(t))^{3/2}} \quad (31)$$

$$\dot{y} = \frac{d}{dt}\left(\frac{bsin(\varphi(t))}{\sqrt{1-e^2cos^2\varphi(t)}}\right) = -\frac{bcos(\varphi(t))\frac{d}{dt}\varphi(t)}{\sqrt{1-e^2cos^2\varphi(t)}} - \frac{bsin^2\varphi(t)e^2cos(\varphi(t))\frac{d}{dt}\varphi(t)}{(1-e^2cos^2\varphi(t))(1-e^2cos^2\varphi(t))^{3/2}} \quad (32)$$

$$\ddot{x} = \frac{d^2}{dt}\left(\frac{bcos(\varphi(t))}{\sqrt{1-e^2cos^2\varphi(t)}}\right) = -\frac{bcos(\varphi(t))(\frac{d}{dt}\varphi(t))^2}{\sqrt{1-e^2cos^2\varphi(t)}} - \frac{bsin(\varphi(t))\frac{d^2}{dt}\varphi(t)}{\sqrt{1-e^2cos^2\varphi(t)}} + \frac{3bsin^2\varphi(t)e^2cos(\varphi(t))(\frac{d}{dt}\varphi(t))^2}{(1-e^2cos^2\varphi(t))^{3/2}} + \frac{3bcos^3\varphi(t)e^4sin^2(\varphi(t))(\frac{d}{dt}\varphi(t))^2}{(1-e^2cos^2\varphi(t))^{5/2}} - \frac{bcos^3\varphi(t)e^2(\frac{d}{dt}\varphi(t))^2}{(1-e^2cos^2\varphi(t))^{3/2}} - \frac{bcos^2\varphi(t)e^2sin(\varphi(t))\frac{d^2}{dt}\varphi(t)}{(1-e^2cos^2\varphi(t))^{3/2}} \quad (33)$$

$$\ddot{y} = \frac{d^2}{dt}\left(\frac{bsin(\varphi(t))}{\sqrt{1-e^2cos^2\varphi(t)}}\right) = -\frac{bsin(\varphi(t))(\frac{d}{dt}\varphi(t))^2}{\sqrt{1-e^2cos^2\varphi(t)}} + \frac{bcos(\varphi(t))\frac{d^2}{dt}\varphi(t)}{\sqrt{1-e^2cos^2\varphi(t)}} - \frac{3bcos^2\varphi(t)e^2sin(\varphi(t))(\frac{d}{dt}\varphi(t))^2}{(1-e^2cos^2\varphi(t))^{3/2}} + \frac{3bsin^3\varphi(t)e^4cos^2(\varphi(t))(\frac{d}{dt}\varphi(t))^2}{(1-e^2cos^2\varphi(t))^{5/2}} + \frac{bsin^3\varphi(t)e^2(\frac{d}{dt}\varphi(t))^2}{(1-e^2cos^2\varphi(t))^{3/2}} - \frac{bsin^2\varphi(t)e^2cos(\varphi(t))\frac{d^2}{dt}\varphi(t)}{(1-e^2cos^2\varphi(t))^{3/2}} \quad (34)$$

Let's substitute equations (30), (31) in equation (5), transfer everything to the left and solve

$$\frac{b(2*e^2cos(\varphi(t))sin(\varphi(t))\left(\frac{d}{dt}\varphi(t)\right)^2-\frac{d^2}{dt}\varphi(t)+\frac{d^2}{dt}\varphi(t)e^2cos^2\varphi(t))}{(1-e^2cos^2\varphi(t))^{3/2}cos(\varphi(t))}=0 \tag{35}$$

$$\frac{d^2}{dt^2}\varphi(t)=\frac{2*e^2*cos(\varphi(t))*sin(\varphi(t))*\left(\frac{d}{dt}\varphi(t)\right)^2}{1-e^2*cos(\varphi(t))^2} \tag{36}$$

$$\ddot{\varphi}=\frac{2*e^2*cos(\varphi)*sin(\varphi)*\dot{\varphi}^2}{1-e^2*cos(\varphi)^2} \tag{37}$$

Calculate the coordinates of the velocity and acceleration vectors, program Linear_acceleration_vector_center.exe. Calculation results in files ellpi.txt, calc_results_05_045_20.txt, calc_results_05_045_80.txt [1]. Graphically displayed in Figures 19, 20.

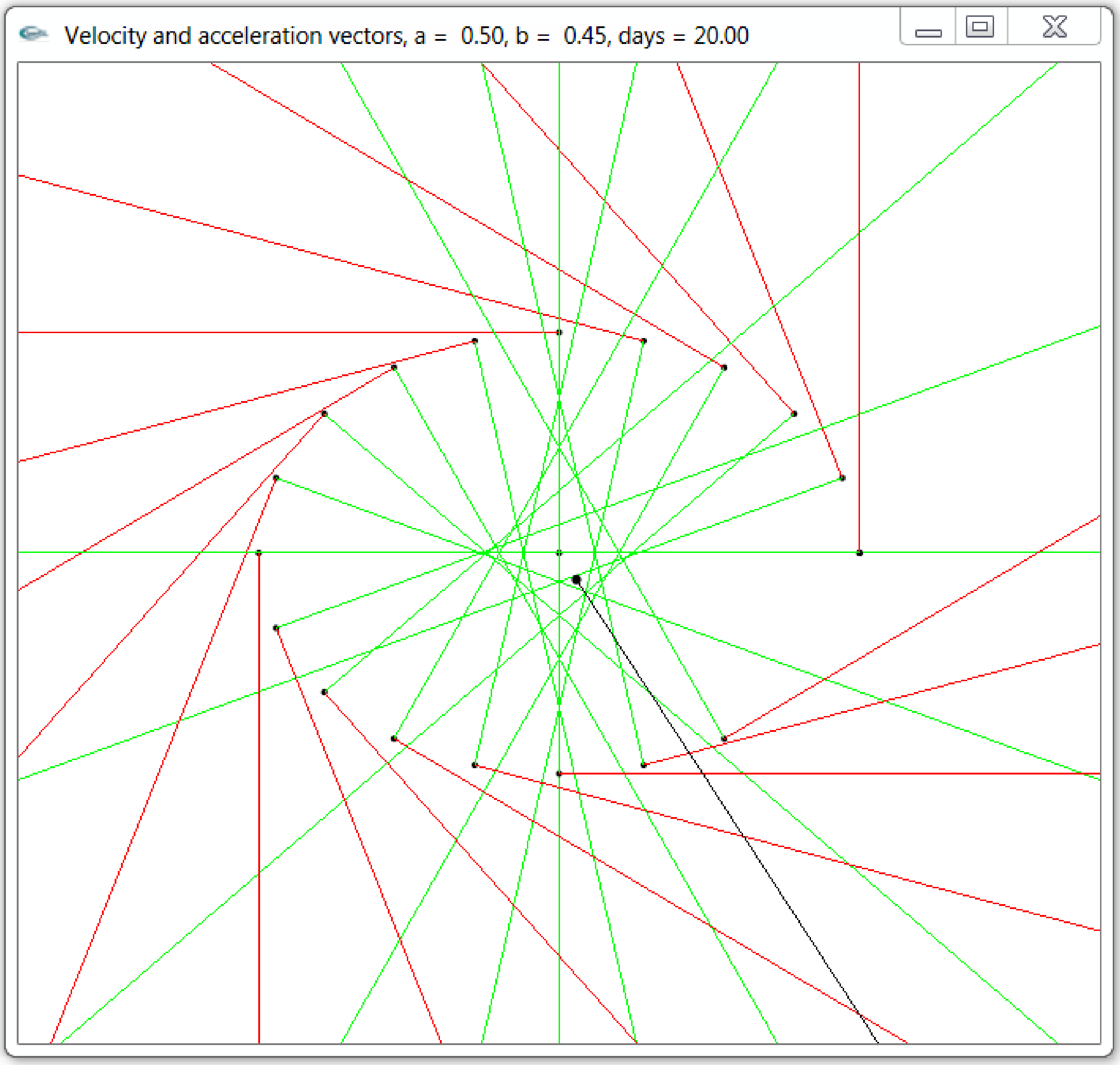


Figure 19

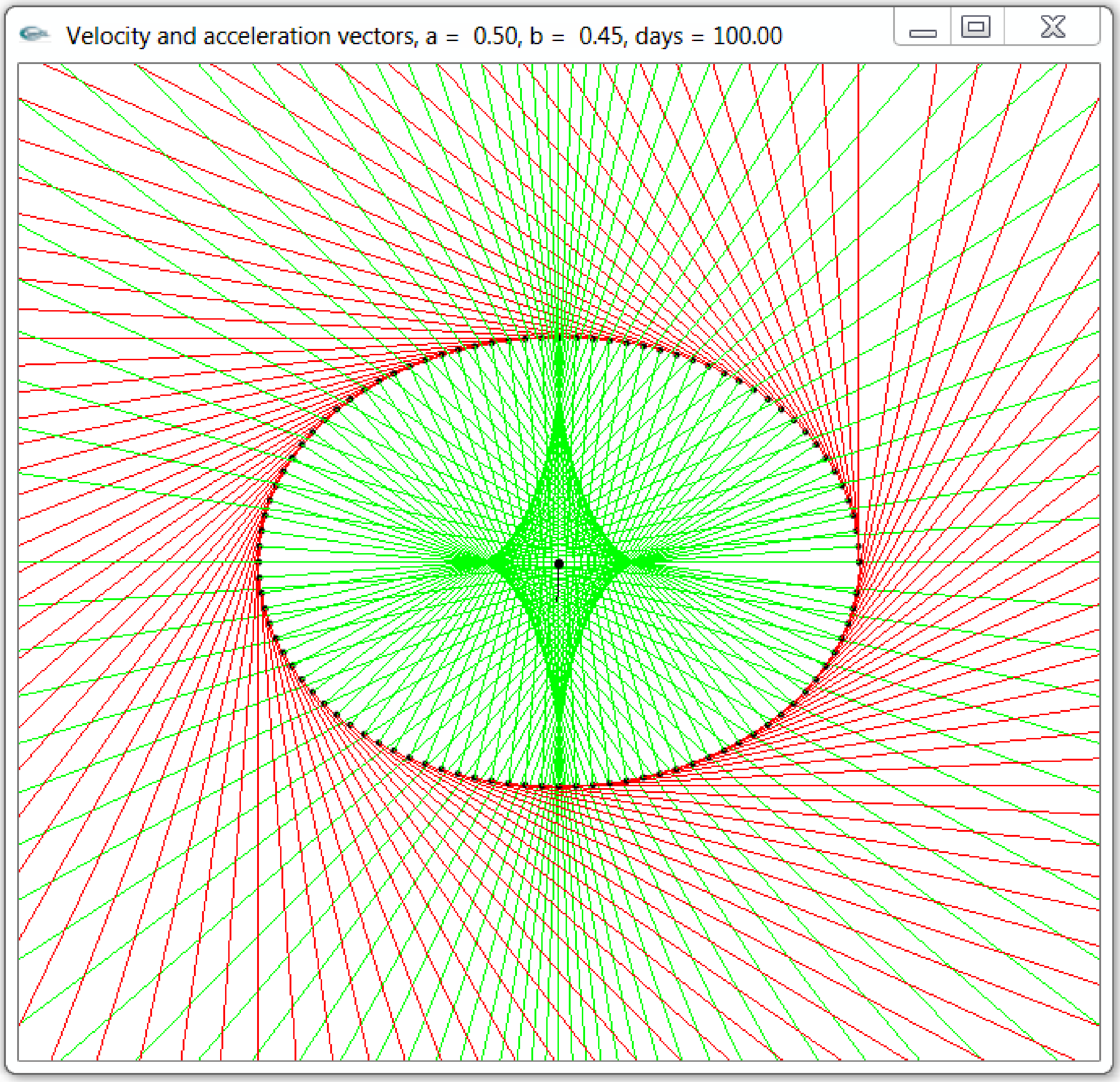


Figure 20

Conclusions

From all of the above, it follows that the focal point lies only on the lines of the three acceleration vectors: perihelion, aphelion and resultant vector, in the case of the center of coordinates at the focus of the ellipse. If the center of the ellipse is aligned with the center of coordinates, five lines of acceleration vectors pass through the center.

References

1. V. Strohm, programs and data files,
https://drive.google.com/file/d/1u4_ZTM3YKxr75K6bC9370jZSmCMt07Yy/view?usp=sharing