Smooth Functions Vanishing at Zero Map H^s with $s > \frac{d}{2}$ into itself

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1 Statement of Results

Definition 1.1. 1. A vector of the form $\alpha = (\alpha_1, \ldots, \alpha_n)$, where each component α_i is a nonnegative integer, is called multi-index of order

$$|\alpha| = \alpha_1 + \ldots + \alpha_n$$

2. Given a multi-index α, define:

$$D^\alpha u(x) := \frac{\partial^{|\alpha|} u(x)}{\partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n}}$$

3. Given a multi-index α, define: $k^\alpha := \prod_j (k_j)^{\alpha_j}$, where $(k_j)^0$ equals one even when $k_j = 0$.

4. If $u \in H^k(U)$ for k integer, we define its norm to be:

$$\|u\|_{H^k} := \|u\|_{W^{k,2}} = (\sum_{|\alpha| \leq k} \int_U |D^\alpha u|^2 dx)^{1/2}$$

Lemma 1.2. *For $s > n/2$, then:*

$$\|\hat{u}\|_{L^1} \leq C \|u\|_{H^s}$$

Proof. Write down,

$$\|\hat{u}\|_{L^1} = \int |\hat{u}(\xi)| < \xi >^s < \xi >^{-s} d\xi \tag{1.1}$$

where $< \xi > = \sqrt{1 + |\xi|^2}$, now apply Cauchy- Schwartz inequality, where the factors are $|\hat{u}(\xi)| < \xi >^s$ and $< \xi >^{-s}$. on equation (1.1), and use the fact that $s > n/2$ to get integrability of $< \xi >^{-2s}$, and the desired inequality.

□

Lemma 1.3. *Let s be an integer greater than $\frac{n}{2}$, where n is the spatial dimension. Then for every positive integer m there exists a constant C depending only on s, d, and m such that*

$$\sum_{0\leq \sum_{j=1}^{m}|\alpha^{(j)}|\leq s} \left\| \prod_{j=1}^{m} D^{\alpha^{(j)}} u_j \right\|_{L^2} \leq C \prod_{j=1}^{m} \|u_j\|_{H^s}. \tag{1.2}$$

Proof. We will need to estimate $|k^{(1)}|^{|\alpha^{(1)}|} \cdots |k^{(m)}|^{|\alpha^{(m)}|}$, we use successively Young's inequality for numbers, to get:

$$|k^{(1)}|^{|\alpha^{(1)}|} \cdots |k^{(m)}|^{|\alpha^{(m)}|} \leq \sum_{j=1}^{m} C(|k^{(j)}|^{|\alpha^{(1)}|+\cdots|\alpha^{(m)}|}) \tag{1.3}$$

The proof of the above inequality goes as follows by induction on m. Suppose we proved for the case m and let's prove for $m+1$.

We have: $|k^{(1)}|^{|\alpha^{(1)}|} \cdots |k^{(m)}|^{|\alpha^{(m)}|} \leq C(|k^{(1)}|^{|\alpha^{(1)}|+\ldots+|\alpha^{(m)}|} + \ldots + |k^{(m)}|^{|\alpha^{(1)}|+\ldots+|\alpha^{(m)}|})$; so by Young's inequality for numbers:

$$\begin{aligned}
|k^{(1)}|^{|\alpha^{(1)}|} \cdots |k^{(m)}|^{|\alpha^{(m)}|} |k^{(m+1)}|^{|\alpha^{(m+1)}|} &\leq C(|k^{(1)}|^{|\alpha^{(1)}|+\ldots+|\alpha^{(m)}|} + \ldots + |k^{(m)}|^{|\alpha^{(1)}|+\ldots+|\alpha^{(m)}|}) \cdot \\
\cdot |k^{(m+1)}|^{|\alpha^{(m+1)}|} \underset{\text{Young's inequality for numbers}}{\leq} & \; C(|k^{(1)}|^{|\alpha^{(1)}|+\ldots+|\alpha^{(m)}|+|\alpha^{(m+1)}|} + \ldots \\
& + |k^{(m)}|^{|\alpha^{(1)}|+\ldots+|\alpha^{(m)}|+|\alpha^{(m+1)}|} + |k^{(m+1)}|^{|\alpha^{(1)}|+\ldots+|\alpha^{(m+1)}|})
\end{aligned}$$

The constant after the second $\leq$ isn't the same as the constant C after the first $\leq$. For the base case of the induction, obviously for $m=1$: $|k^{(1)}|^{|\alpha^{(1)}|} \leq C|k^{(1)}|^{|\alpha^{(1)}|}$.

Denote by $S := \{j : |\alpha^{(j)}| = 0\}$.

(1.4)

$$\begin{aligned}
&|\underset{j\in S}{*}\hat{u}_j * \underset{t\notin S}{*} D^{\alpha^{(t)}}u_t(y)| \\
&= \left|\int\dots\int \hat{u}_1(y-x_1)\hat{u}_2(x_1-x_2)\dots D^{\alpha^{(k+1)}}u_{k+1}(x_k-x_{k+1})\right. \\
&\qquad \left.\dots D^{\alpha^{(n)}}u_n(x_n)dx_1\dots dx_n\right| \\
&\le \int\dots\int |\hat{u}_1(y-x_1)\hat{u}_2(x_1-x_2)\dots D^{\alpha^{(k+1)}}u_{k+1}(x_k-x_{k+1})\dots \\
&\qquad D^{\alpha^{(n)}}u_n(x_n)|dx_1\dots dx_n \\
&= \int\dots\int |x_k-x_{k-1}|^{|\alpha^{(k+1)}|}\dots|x_n|^{|\alpha^{(n)}|}|\hat{u}_1(y-x_1)\dots\hat{u}_k(x_{k-1}-x_k) \\
&\qquad \hat{u}_{k+1}(x_k-x_{k+1})\dots\hat{u}_n(x_n)|dx_1\dots dx_n \\
&\underset{\text{by } (1.3)}{\le} C\int\dots\int[1+|x_k-x_{k+1}|^{|\alpha^{(k+1)}|+\dots+|\alpha^{(n)}|}]|\hat{u}_1(y-x_1)|\dots \\
&\qquad |\hat{u}_k(x_{k-1}-x_k)|\,|\hat{u}_{k+1}(x_k-x_{k+1})|\dots\hat{u}_n(x_n)|dx_1\dots dx_n \\
&+\dots C\int\dots\int[1+|x_n|^{|\alpha^{(k+1)}|+\dots+|\alpha^{(n)}|}]|\hat{u}_1(y-x_1)\dots\hat{u}_n(x_n)|dx_1\dots dx_n \\
&\le C|(1+|.|^{|\alpha^{(k+1)}|+\dots+|\alpha^{(n)}|})\hat{u}_{k+1}(\cdot)| * |\hat{u}_1 * \dots * \hat{u}_n| + \dots \\
&\quad + C|(1+|.|^{|\alpha^{(k+1)}|+\dots+|\alpha^{(n)}|})\hat{u}_n(\cdot)| * |\hat{u}_1 * \dots * \hat{u}_{n-1}|,
\end{aligned}$$

where we have reordered the factors without the derivative to appear first, and let k denote the number of factors not differentiated; This is allowed since convolution is commutative and associative.

Now, by Parseval identity we get:

$$\|\underset{j\in S}{*}\hat{u}_j * \underset{t\notin S}{*} D^{\alpha^{(t)}}u_t\|_{L^2} = \|\prod_{j\in S}u_j\prod_{t\notin S}D^{\alpha^{(t)}}u_t\|_{L^2}.$$

We now employ Young's convolution inequality, on the last RHS in eq. (1.4)(after taking L^2 norm from both sides) above:

$$\begin{aligned}
\|\prod_{j\in S}u_j\prod_{t\notin S}D^{\alpha^{(t)}}u_t\|_{L^2} \le & C\|u_{k+1}\|_{H^{|\alpha^{(k+1)}|+\dots+|\alpha^{(n)}|}}\|\hat{u}_1 * \dots * \hat{u}_n\|_{L^1} + \dots \\
& + C\|u_n\|_{H^{|\alpha^{(k+1)}|+\dots+|\alpha^{(n)}|}}\|\hat{u}_1 * \dots * \hat{u}_{n-1}\|_{L^1},
\end{aligned}$$

where we've used the inequality: $\|(1+|.|^m)\hat{u}\|_{L^2} \underset{\text{cf [1, Theorem 8, p.297]}}{\le} C\|u\|_{H^m}$ for m an integer.

We have by Young's inequality for convolution (after repeated use of this inequality, (each use for a convolution of two factors)):

$$\|\underset{j}{*}\hat{u}_j\|_{L^1} \le \prod_j \|\hat{u}_j\|_{L^1}$$

As we've seen in Lemma 1.2 we have: $\|\hat{u}\|_{L^1} \leq C\|u\|_{H^s}$. And:

$$\|u\|^2_{H^{|\alpha^{(k+1)}|+\ldots+|\alpha^{(n)}|}} \leq \int_{|y|\geq 1} (1+|y|^s)^2 \hat{u}^2 dy + \int_{|y|\leq 1} 4\hat{u}^2 dy.$$

So we get eventually that: $\|u\|_{H^{|\alpha^{(k+1)}|+\cdots+|\alpha^{(n)}|}} \leq C\|u\|_{H^s}$. □

Lemma 1.4.
(1.5)

$$\|F(u)\|_{H^s} \leq C(\|F(u)\|_{L^2} + \sum_{|b|=s} \sum_{j=1}^{s} \sum_{\alpha^{(1)}+\cdots+\alpha^{(j)}=b} \|F^{(j)}(u)\|_{L^\infty} \|D^{\alpha^{(1)}} u \cdots D^{\alpha^{(j)}} u\|_{L^2}).$$

Proof. We have:

$$\begin{aligned}
\|F(u)\|_{H^s} &\underset{\text{by [1]}}{\leq} C\|(1+|k|^s)F(u)(k)\|_{L^2} \quad (1.6)\\
&\underset{\text{from triangle inequality for } L^2\text{-norm}}{\leq} C(\|F(u)(k)\|_{L^2} + \||k|^s F(u)(k)\|_{L^2})\\
&\underset{\text{from Parseval identity}}{=} C(\|F(u)\|_{L^2} + \||k|^s F(u)(k)\|_{L^2})\\
&\leq C(\|F(u)\|_{L^2} + \sum_{|b|=s} \|D^b F(u)\|_{L^2}) \underset{\text{from Parseval identity}}{=}\\
&= C(\|F(u)\|_{L^2} + \sum_{|b|=s} \|D^b F(u)\|_{L^2}),
\end{aligned}$$

where we have used the facts that

$$|D^b F(u)(k)| = |\int \exp(-ix\cdot k) D^b F(u)(x) dx| = |\int \exp(-ix\cdot k) \prod_j \partial^{b_j}_{x_j} F(u(x)) dx|$$

$$\underset{\text{Integration by parts}}{=} |\int \prod_j k_j^{b_j} \exp(-ix\cdot k) F(u(x)) dx| = |\prod_j k_j^{b_j} F(u)(k)|$$

and

$$c|k|^{2s} = c(\sum_j k_j^2)^s \leq \sum_{|b|=s} \prod_j k_j^{2b_j} \quad (1.7)$$

for some positive constant $c > 0$. The last inequality above is obtained since both sides of the inequality are homogenous of order $2s$ in k, and the fact that its RHS is positive and continuous everywhere on the unit sphere. Since $\sum_{|b|=s} \prod_j k_j^{2b_j}$ is a polynomial in $k_1, \ldots, k_n$, this expression is continuous on the unit sphere. This expression is positive since every term in the sum is either positive or zero because of the even exponent in $k_j^{2b_j}$. And there's at least one positive term in the sum, because the sum is over $|b| = s$; we have the next list of vectors: $(s, 0, \ldots, 0), \ldots, (0, 0, \ldots, 0, s)$ such that whatever k is chosen such that $|k| = 1$, we must have that $\exists k_j \neq 0$, and $b_j = s$, $b_i = 0$ for $i \neq j$, such that the term $k_j^{2s} \cdot 1 \cdot \ldots \cdot 1$ is positive.Hence by the theorem that says that every real-valued continuous function on a compact set achieves its minimum and maximum value on this set,and when the function is positive, as is the case we are dealing with then the minimum value it takes on is positive, which provides the positive c such that the function is greater than c , there exists a positive c such that $\sum_{|b|=s} \prod_j k_j^{2b_j} \geq c$ for $|k| = 1$, which means that the inequality in (1.7) holds.

Thus,

$$\sum_{|b|=s} \|D^b F(u)(k)\|_{L^2} \geq c\||k|^{|b|} F(u)(k)\|_{L^2}$$

Now the multivariate Faa di Bruno's Formula is (`http://mathworld.wolfram.com/FaadiBrunosFormula.html`):

$$D^b F(u) = \sum_{j=1}^{|b|} \sum_{|\gamma^{(k)}| \geq 1, \gamma^{(1)} + \cdots + \gamma^{(j)} = b} F^{(j)}(u) c_{\gamma^{(1)}, \ldots, \gamma^{(j)}} \prod_{k=1}^{j} D^{\gamma^{(k)}} u$$

To see how this formula is obtained, note the following heuristic explanation. We apply $\sum_{|b|=s} D^b$ on $F(u)$, i.e $\sum_{b_1 + \ldots b_n = s} \frac{\partial^{|b|}}{\partial_{x_1}^{b_1} \ldots \partial_{x_n}^{b_n}} F(u)$. Let's use color designation on the partial derivatives. Denote by C_j^i the color designators of the partial derivative $\partial_{x_i}^{b_i}$, where the index j goes from 1 to b_i. So we turned D^b into:$D^b = \prod_{i=1}^n \prod_{j=1}^{b_i} \partial_{x_i}^{C_j^i}$. The first derivative that is applied to $F(u(x))$ yields $F'(u(x)) \frac{\partial u}{\partial x_m^{C_1^m}}$ for some m. After that, successive derivatives may be applied either to whatever derivative of F currently appears or to one of the derivatives of u that are already present. Each time a derivative is applied to F the chain rule yields a factor of $\frac{\partial u}{\partial x_m^{C_k^m}}$, for some k and some m. Hence if F is differentiated j times and the remaining derivatives are applied directly to the factor of u then a term $F^{(j)}(u(x)) \prod_{k=1}^j D^{\gamma^{(k)}} u$ is obtained. (where the $\gamma^{(k)}$s have color designators added).

If for example we apply $\partial_{x_i}^{C_1^i} \partial_{x_i}^{C_2^i}$ to $u_1 u_2$ then we get: $[\partial_{x_i}^{C_1^i} \partial_{x_i}^{C_2^i} u_1] u_2 + u_1 [\partial_{x_i}^{C_1^i} \partial_{x_i}^{C_2^i}] u_2 + \partial_{x_i}^{C_1^i} u_1 \partial_{x_i}^{C_2^i} u_2 + \partial_{x_i}^{C_2^i} u_1 \partial_{x_i}^{C_1^i} u_2$

each term has coefficient one because every different way of applying the derivative operators yields a different result.

Once we neglect the color designators, then the last two terms become the same, and hence the resulted combined term has coefficient two. I.e, coefficients higher than one arise because neglecting the color designators makes some terms that used to be different, become the same terms, and combining them yields a coefficient equal to the number of terms so combined. If one of the ∂_{x_i} that appears in D^b has a power b_i greater than one and those derivatives are distributed among at least two of the $\gamma^{(k)}$, then the coefficient $c_{\gamma^{(1)}, \ldots, \gamma^{(j)}}$ multiplying $F^{(j)}(u(x)) \prod_{k=1}^j D^{\gamma^{(k)}} u$ will be greater than one. For example, $u_{1_{x_1}} u_{2_{x_1 x_2}}$

will have a coefficient greater than one, while $u_{1_{x_1x_1}}u_{2_{x_2x_2}}$ will have coefficient one.

The reason that some terms have a factor greater than one is that there's more than one way to choose which factor of ∂_{x_i} is applied to which factor of $u_1 \dots u_j$.

Now, plug the formula for D^bF back into (1.6), into $C(\|F(u)\|_{L^2}+\sum_{|b|=s}\|D^bF(u)\|_{L^2})$ and we get the desired result:

$$\begin{aligned}
&C\Big(\|F(u)\|_{L^2}+\sum_{|b|=s}\|D^bF(u)\|_{L^2}\Big)\\
&=C\Big(\|F(u)\|_{L^2}+\sum_{|b|=s}\|\sum_{j=1}^{|b|}\sum_{|\gamma^{(k)}|\geq 1,\gamma^{(1)}+\dots+\gamma^{(j)}=b}\\
&\qquad F^{(j)}(u)c_{\gamma^{(1)},\dots,\gamma^{(j)}}\prod_{k=1}^{j}D^{\gamma^{(k)}}u\|_{L^2}\Big)\\
&\leq C\Big(\|F(u)\|_{L^2}+\sum_{|b|=s}\sum_{j=1}^{|b|}\sum_{|\gamma^{(k)}|\geq 1,\gamma^{(1)}+\dots+\gamma^{(j)}=b}\\
&\qquad \|F^{(j)}(u)\|_{L^\infty}\|\prod_{k=1}^{j}D^{\gamma^{(k)}}u\|_{L^2}\Big)
\end{aligned} \tag{1.8}$$

Above the constant C takes different values from both sides of the inequality. □

Lemma 1.5. *Suppose $u \in H^s$ and $v \in L^\infty$, then there exists a constant $C > 0$ s.t:*

$$\|uv\|_{L^2} \leq C\|u\|_{H^s}\|v\|_{L^\infty} \tag{1.9}$$

Proof.

$$\|uv\|_{L^2} = (\int u^2v^2)^{1/2} \leq \sup|v|(\int u^2)^{1/2} = \|v\|_{L^\infty}\|\hat{u}\|_{L^2} \leq \|(1+|y|^s)\hat{u}\|_{L^2}\|v\|_{L^\infty} \leq C\|u\|_{H^s}\|v\|_{L^\infty}$$

□

Lemma 1.6. *For $F \in C^s, u \in H^s$ such that $F(0) = 0$*

$$\|F(u)\|_{L^2} \leq C\|u\|_{H^s} \sup_{|v|\leq C\|u\|_{H^s}} |F'(v)| \tag{1.10}$$

Proof. We have $F(u) = F(u)-0 = F(u)-F(0) = \int_{r=0}^1 \frac{d}{dr}F(ru)dr = u\int_{r=0}^1 F'(ru)dr$, so:

$$\|F(u)\|_{L^2} = \|u\int_{r=0}^1 F'(ru)dr\|_{L^2} \leq C\|u\|_{H^s}\|\int_{r=0}^1 F'(ru)dr\|_{L^\infty} \tag{1.11}$$

The last inequality is proved in the last lemma above, 1.5.

$$(1.11) \leq \|u\|_{H^s} \sup_{x \in \mathsf{R}^n} \sup_{r \in [0,1]} |F'(ru(x))|$$

Now, we have:

$$\sup_{x \in \mathsf{R}^n} \sup_{r \in [0,1]} |F'(ru(x))| \leq \sup_{x \in \mathsf{R}^n \, \sup_{v \in L^\infty : |v| \leq c\|u\|_{H^s}}} |F'(v(x))|$$

$$\leq \sup_{v \in \mathsf{R} : |v| \leq c\|u\|_{H^s}} |F'(v)|$$

The first inequality follows since the set: $\{ru : r \in [0,1] \; u \in H^s\}$ is a subset of $\{v \in L^\infty : |v| \leq c\|u\|_{H^s}\}$; while the second inequality follows from the fact that the inequality $|v| \leq c\|u\|_{H^s}$ is satisfied for every $x \in \mathsf{R}^n$, we see that to find the supremum of $|F'(v(x))|$ it's enough to look at the supremum of $|F'(v)|$ in $\{v \in \mathsf{R} : |v| \leq c\|u\|_{H^s}\}$, since the supremum doesn't depend on x, only on the bounds of v. □

Lemma 1.7. *Under the assumptions of lemmas: 1.3,1.4,1.6 we have:*

(1.12)

$$\|F(u)\|_{H^s} \leq C\Big(\|u\|_{H^s} \sup_{|v| \leq c\|u\|_{H^s}} |F'(v)| + \sum_{|b|=s} \sum_{j=1}^{s} \sup_{\{v \in \mathsf{R}^n : |v| \leq c\|u\|_{H^s}\}} |F^{(j)}(v)| \|u\|^j_{H^s} \Big)$$

Proof. Now, from lemma 1.2 we have: $\sum_{\alpha^{(1)}+\ldots+\alpha^{(j)}=b} \|D^{\alpha^{(1)}} u \cdots D^{\alpha^{(j)}} u\|_{L^2} \leq C\|u\|^j_{H^s}$.

So, by the lemmas: 1.3,1.4,1.6

$$\begin{aligned} &\|F(u)\|_{H^s} \\ &\leq C\Big(\|u\|_{H^s} \sup_{|v| \leq c\|u\|_{H^s}} |F'(v)| + \sum_{|b|=s} \sum_{j=1}^{|b|} \sum_{|\gamma^{(k)}| \geq 1, \gamma^{(1)}+\cdots+\gamma^{(j)}=b} \\ &\qquad \|F^{(j)}(u)\|_{L^\infty} \| \prod_{k=1}^{j} D^{\gamma^{(k)}} u\|_{L^2} \Big) \qquad (1.13) \\ &\leq C\Big(\|u\|_{H^s} \sup_{|v| \leq c\|u\|_{H^s}} |F'(v)| + \sum_{|b|=s} \sum_{j=1}^{|b|} \\ &\qquad \sup_{\{v \in \mathsf{R}^n : |v| \leq c\|u\|_{H^s}\}} |F^{(j)}(v)| \|u\|^j_{H^s} \Big) \end{aligned}$$

The estimate $\|F^{(j)}(u(x))\|_{L^\infty} \leq \sup_{\{v \in \mathsf{R}^n : |v| \leq c\|u\|_{H^s}\}} |F^{(j)}(v)|$ was proven in lemma 1.6, for the case $j = 1$, but that proof works just as well for $j > 1$.

□

References

[1] Lawrence C. Evans. *Partial Differential Equations*. American Mathematical Society, 2010.