Solution to Problems 12 and 13 in Michael Taylor's volume 3 in PDE

Alon Brook-Ray

December 22, 2020

Abstract

Solutions to problems 12 and 13 in chapter 16 of volume 3 of PDE textbook by Michael Taylor.

Contents

1 Problem 12

Definition 1. We define Schwartz class as $\mathcal{S}(\mathrm{R}^n) := \{\varphi \in C^\infty : q_N(\varphi) < \infty, \text{ for } N = 0, 1, 2, \ldots\}$, where $q_N(\varphi) := \sup_{x \in \mathrm{R}^n, |\alpha| \leq N} (1 + |x|^2)^N |D^\alpha \varphi(x)|$.

We have:

$$d/dt(u_\epsilon, u_\epsilon) = 2(\partial_t u_\epsilon, u_\epsilon) = 2\Big(J_\epsilon L J_\epsilon u_\epsilon, u_\epsilon\Big) + 2\Big(J_\epsilon g(J_\epsilon u_\epsilon), u_\epsilon\Big) \quad (1)$$

Since J_ϵ is self-adjoint, we get: $2\Big(J_\epsilon L J_\epsilon u_\epsilon, u_\epsilon\Big) = 2\Big(L J_\epsilon u_\epsilon, J_\epsilon u_\epsilon\Big)$.
Now, we shall use [1, eq. (1.11), page 415] , plug $\alpha = 0$ into [1, eq. (1.11), page 415] to get:

$$2\Big(L J_\epsilon u_\epsilon, J_\epsilon u_\epsilon\Big) \leq C \|J_\epsilon u_\epsilon\|_{L^2}^2 \quad (2)$$

Now, we shall use Young's inequality for convolution on the RHS of (2), i.e:

$$\|J_\epsilon u_\epsilon\|_{L^2} = \|j_\epsilon * u_\epsilon\|_{L^2} \leq \|j_\epsilon\|_{L^1} \|u_\epsilon\|_{L^2} \leq C \|u_\epsilon\|_{L^2} \quad (3)$$

Now we shall estimate the second term in (1), we are using lemmas 1.6 and 1.5 from the previous file:

Combine (??), (2) and (3), to get: $d/dt\|u_\epsilon\|_{L^2}^2 \leq C\|u_\epsilon\|_{L^2}^2$.
For $d/dt\|\nabla u_\epsilon\|_{L^2}^2 \leq C\|\nabla u_\epsilon\|_{L^2}^2$ We have:

$$d/dt(\nabla u_\epsilon, \nabla u_\epsilon) = 2(\nabla \partial_t u_\epsilon, \nabla u_\epsilon) = 2\Big(\nabla J_\epsilon L J_\epsilon u_\epsilon, \nabla u_\epsilon\Big) + 2\Big(\nabla J_\epsilon g(J_\epsilon u_\epsilon), \nabla u_\epsilon\Big) \tag{4}$$

Notice that:

$$\begin{aligned}
2\Big(\nabla J_\epsilon g(J_\epsilon u_\epsilon), \nabla u_\epsilon\Big) &\underset{J_\epsilon \text{ commutes with } \nabla}{=} 2\Big(J_\epsilon \nabla g(J_\epsilon u_\epsilon), \nabla u_\epsilon\Big) \\
&\underset{J_\epsilon \text{ is self-adjoint}}{=} 2\Big(\nabla g(J_\epsilon u_\epsilon), J_\epsilon \nabla u_\epsilon\Big) \\
&= 2\Big(g'(J_\epsilon u_\epsilon) J_\epsilon \nabla u_\epsilon, J_\epsilon \nabla u_\epsilon\Big) \\
&\underset{\text{Cauchy-Schwartz inequality}}{\leq} C\|J_\epsilon \nabla u_\epsilon\|_{L^2} \|g'(J_\epsilon u_\epsilon) J_\epsilon \nabla u_\epsilon\|_{L^2} \\
&\leq C\|J_\epsilon \nabla u_\epsilon\|_{L^2}^2 \sup_v |g'(v)| \\
&\underset{\text{we used } |g'| \leq C \text{ , and (3)}}{\leq} C\|\nabla u_\epsilon\|_{L^2}^2
\end{aligned} \tag{4}$$

In eq. (4), the first term becomes: $2\Big(\nabla(J_\epsilon L J_\epsilon u_\epsilon), \nabla u_\epsilon\Big) = 2\Big(\nabla(L J_\epsilon u_\epsilon), J_\epsilon \nabla u_\epsilon\Big) = 2\Big(L J_\epsilon \nabla u_\epsilon, J_\epsilon \nabla u_\epsilon\Big) + 2\Big([\nabla, L] J_\epsilon u_\epsilon, J_\epsilon \nabla u_\epsilon\Big)$.
The first term is bounded by $C\|\nabla u\|_{L^2}^2$, as can be inferred by the next reference [1, eq. (1.11), page 415].
The second term can be seen to be bounded by the same bound, by the next equation:

$$\begin{aligned}
([\nabla, L] J_\epsilon u_\epsilon, J_\epsilon \nabla u_\epsilon) &= \sum_j (\nabla A_j \partial_j (J_\epsilon u_\epsilon), J_\epsilon \nabla u_\epsilon) \\
&= \int \sum_j \sum_{i,k} \sum_m (J_\epsilon \partial_m (u_\epsilon)_i)(\partial_m a_{ik}^j) \partial_j (J_\epsilon (u_\epsilon)_k)
\end{aligned} \tag{4}$$

So we get by Cauchy-Schwartz that this is less or equals to: $C\|\nabla J_\epsilon u_\epsilon\|_{L^2}^2$ where the constant C depends on bounds on derivatives of entries of the

matrix A_j which are smooth functions. Now we know from the fact that J_ϵ commutes with ∇ we have: $\|\nabla J_\epsilon u_\epsilon\|_{L^2}^2 = \|J_\epsilon \nabla u_\epsilon\|_{L^2}^2$, and from (3) it follows that this is less than: $C\|\nabla u_\epsilon\|_{L^2}^2$.
From the two inequalities: $d/dt\|u_\epsilon\|_{L^2}^2 \leq C\|u_\epsilon\|_{L^2}^2$ and $d/dt\|\nabla u_\epsilon\|_{L^2}^2 \leq C\|\nabla u_\epsilon\|_{L^2}^2$, now add both inequalities to get: $d/dt\|u_\epsilon\|_{H^1}^2 \leq C\|u_\epsilon\|_{H^1}^2$.
Thus, $\|u_\epsilon\|_{H^1}^2 \leq A\exp(Ct)$ for a positive constant A.
Since $\|u_\epsilon\|_{H^1}^2 \leq A\exp(Ct)$, the bound exists for all time t, thus also our solution $u_\epsilon \in H^1$ exists for each time, t.This follows from the ODE continuation theorem, which says that a solution to an ODE exists as long as the norm of the solution is finite. So we need to show that $\|F(u)\|_{L^2} \leq h(\|u\|_{L^2})$ for some continuous function h.

$$\begin{aligned} \|F(u_\epsilon)\|_{L^2} &= \|J_\epsilon L J_\epsilon u_\epsilon + J_\epsilon g(J_\epsilon u_\epsilon)\|_{L^2} \\ &\leq \|J_\epsilon L J_\epsilon u_\epsilon\|_{L^2} + \|J_\epsilon g(J_\epsilon u_\epsilon)\|_{L^2} \end{aligned} \tag{4}$$

In (4), we know that $\|J_\epsilon g(J_\epsilon u_\epsilon)\|_{L^2} \leq \|j_\epsilon\|_{L^1}^2 \sup_{v\in \mathrm{R}^n} |g'(v)| \|u\|_{L^2} \leq C\|u\|_{L^2}$.
As for the first term in the RHS after the inequality sign in (4): $\|J_\epsilon L J_\epsilon u_\epsilon\|_{L^2} \leq \|j_\epsilon\|_{L^1}\|LJ_\epsilon u_\epsilon\|_{L^2}$. Now, we only need to estimate the second factor:

$$\begin{aligned} \|LJ_\epsilon u_\epsilon\|_{L^2} &= \left\| \sum_k A_k \partial_{x_k} \Big(\int j(\epsilon^{-1}(\cdot - s))\epsilon^{-n} u(t,s)ds \Big) \right\|_{L^2} \\ &= \left\| \sum_k A_k \Big(\int j_{x_k}(\epsilon^{-1}(\cdot - s))\epsilon^{-n-1} u(t,s)ds \Big) \right\|_{L^2} \\ &\underset{\text{young's inequality for convolution}}{\leq} \sum_k \|A_k\|_{L^\infty} \epsilon^{-1} \|\epsilon^{-n} j_{x_k}(\epsilon^{-1}(\cdot))\|_{L^1} \|u\|_{L^2} \end{aligned} \tag{4}$$

Note that $\int_{\mathrm{R}^n} \frac{dx}{(1+|x|^2)^N} = \omega_n \int_0^\infty \frac{r^{n-1}}{(1+r^2)^N} dr < \infty$, whenever $N > n/2$ (where ω_n is a constant that depends on n). Then, if $N > n/2$ and $j \in \mathcal{S}(\mathrm{R}^n)$, then we get:

$$\begin{aligned} \|j_{x_k}\|_{L^1} &\leq \int_{\mathrm{R}^n} q_N(j)(1+|x|^2)^{-N} dx \\ &= C q_N(j) < \infty \end{aligned} \tag{4}$$

Thus, $\epsilon^{-1}\|\epsilon^{-n} j_{x_k}(\epsilon^{-1}(\cdot))\|_{L^1} \leq C/\epsilon$.
Now, inserting this into (4), we get: $\|LJ_\epsilon u_\epsilon\|_{L^2} \leq \sum_k \|A_k\|_{L^\infty} C/\epsilon \cdot \|u_\epsilon\|_{L^2}$.
So by combining everything together we get:

$$\|F(u_\epsilon)\|_{L^2} \leq \sum_k \|A_k\|_{L^\infty} C/\epsilon \cdot \|u_\epsilon\|_{L^2} + C\|u_\epsilon\|_{L^2} = h(\|u_\epsilon\|_{L^2}) \tag{4}$$

Now, we shall show Lipschitz criterion is satisfied. Take two points $t, s \in I = [t_1, t_2]$, and estimate:

$$\begin{aligned}
\|u(t,\cdot) - u(s,\cdot)\|_{L^2} &= \| \int_s^t \partial_{t'} u(t',x) dt' \|_{L^2} \\
&= \Big\| \int_s^t (J_\epsilon L J_\epsilon u_\epsilon(t') + J_\epsilon g(J_\epsilon u_\epsilon(t'))dt' \Big\|_{L^2} \\
&\leq \big|t-s\big| \sup_{t' \in I} \Big\|(J_\epsilon L J_\epsilon u_\epsilon(t') + J_\epsilon g(J_\epsilon u_\epsilon(t'))\Big\|_{L^2} \\
&= \big|t-s\big| \sup_{t' \in I} \left(\Big\|(J_\epsilon L J_\epsilon u_\epsilon(t')\Big\|_{L^2} + \Big\|J_\epsilon g(J_\epsilon u_\epsilon(t'))\Big\|_{L^2} \right)
\end{aligned} \tag{4}$$

The first term inside the sup in (4), is less or equal $C\|\nabla u_\epsilon(t')\|_{L^2}$, since A_j is a bounded matrix and J_ϵ as well is a bounded operator on L^2 and ∇ includes all the spatial derivatives of L; and also from above we know that:$C\|\nabla u_\epsilon\|_{L^2} \leq C_0 \exp(Ct')$ for positive constants C, C_0, and this is smaller than $C_0 \exp(Ct_2)$. The second term is estimated as follows: from what we've seen above it's less than $C\|g(J_\epsilon u_\epsilon)\|_{L^2}^2$, which is again smaller than $C\|u_\epsilon\|_{L^2} \sup|g'| \leq C_1 \exp(Ct')$, for C_1, C positive constants, which is less than $C_1 \exp(Ct_2)$. From all of the above we'll conclude that: $\|u(t,\cdot) - u(s,\cdot)\|_{L^2} \leq |t-s|c(I)$, where $c(I)$ is a constant that depends on the interval, I.

2 Problem 13

Definition 2. The space $L^\infty(C, B)$, where C is a subset of R and B is a Banach space, is defined as the set of all functions $f : C \to B$ which their supremum norm is finite, $\|f\|_{L^\infty(C,B)} := \sup_{x\in C} \|f(x)\|_B < \infty$;
$Lip(C, B)$ is the space of functions $f : C \to B$ which their Lipschitz's norm is finite, $\|f\|_{Lip(C,B)} := \sup_{x,y\in C, x\neq y} \frac{\|f(x)-f(y)\|_B}{|x-y|} < \infty$.

When $s \in \mathsf{Z}_+$, M is a manifold and N is another manifold, we define the space $C^s(M; N)$ as the space of functions $f : M \to N$ such that $f, f', \ldots, f^{(s)}$ are continuous functions; and $C^\infty(M; N)$ as the space of functions which are differentiable in all orders inside M.

Theorem 2.1. Let A_j be a $K \times K$ matrix, smooth in its arguments and symmetric, $A_j = A_j^*$. Suppose g is smooth in its arguments, with values in R^K s.t $g(0) = 0$, $|g'(u)| \leq C$. Then there exists a unique solution $u \in L^\infty_{loc}(\mathsf{R}, H^1(M)) \cap Lip_{loc}(\mathsf{R}, L^2(M))$,(where $M = \mathsf{T}^n$) to the PDE: $u_t = Lu+$

$g(u)$,and initial condition $u(0) = f$, where $f \in H^1(M)$, and the operator L is defined by: $L(t, x, u, D_x)u = \sum_j A_j(t,x)\frac{\partial}{\partial x_j}u$.

Proof. Suppose u_1, u_2 solve the PDE above, i.e $u_t = Lu + g(u)$, $u(0) = f$. Take $w = u_1 - u_2$, then w satisfies: $w_t = Lw + h(w, u_2)$, where $h(w(x,t), u_2(x,t)) = g(w(t,x) + u_2(t,x)) - g(u_2(t,x))$, $w(0) = 0$.
Since $w(0) = 0$ we must have $\|w(0)\|^2_{L^2} = 0$. Notice that

$$
\begin{aligned}
(h(w(t), u_2), w(t)) &\underset{\text{Cauchy-Schwartz inequality}}{\leq} \|w(t)\|_{L^2}\|h(w(t), u_2(t))\|_{L^2} \\
&\underset{h(0,u_2)=g(u_2)-g(u_2)=0}{=} \|w(t)\|_{L^2}\|h(w(t), u_2(t)) - h(0, u_2(t))\|_{L^2} \\
&= \|w(t)\|_{L^2}\|\int_0^1 w(t)h_w(rw(t), u_2(t))dr\|_{L^2} \\
&\leq \|w(t)\|_{L^2}\|w(t)\|_{L^2}\|\int_0^1 h_w(rw(t), u_2(t)dr\|_{L^\infty} \\
&\leq \|w(t)\|_{L^2}\|w(t)\|_{L^2} \sup_{v\in\mathbb{R}^n, x\in M} |h_w(v, u_2(x,t))| \\
&= \|w(t)\|_{L^2}\|w(t)\|_{L^2} \sup_{v\in\mathbb{R}^n, x\in M} |g'(v + u_2(x,t))| \\
&\leq C\|w(t)\|_{L^2}\|w(t)\|_{L^2} \qquad (12)
\end{aligned}
$$

Notice the following: $\partial_t(w,w) = 2(w_t, w) = 2(\sum_j A_j\partial_{x_j}w, w) + 2(h(w,u), w)$. we get:$2(\sum_j A_j\frac{\partial}{\partial x_j}w, w) = -\sum_j \int w^* \cdot \frac{\partial A_j}{\partial x_j} \cdot w dx$ by the following calculation:

$$
(A_j\frac{\partial}{\partial x_j}w, w) = \int w^* \cdot A_j \cdot \partial_{x_j}w dx \underset{\text{integration by parts}}{=} -\int w^*_{x_j} \cdot A_j \cdot w dx - \int w^* \cdot \frac{\partial}{\partial x_j}A_j \cdot w dx
$$

by the fact that the transpose of a number equals the number, we get that: $w^*_{x_j} \cdot A_j \cdot w = (w^*_{x_j} \cdot A_j \cdot w)^* \underset{A_j^* = A_j}{=} (w^* \cdot A_j \cdot w_{x_j})$. Now, use the Cauchy-Schwarz inequality: $2(A_j\partial_{x_j}w, w) \leq 2\sum_j \|A_j(t,\cdot)\|_{C^1} \leq C(t)\|w(t)\|^2_{L^2}$, where we used the fact that $A_j(x,t)$ is C^∞-smooth in its arguments x, t, the variables x are defined on T^n which is compact; thus $A_j(x,t)$ and its derivatives are bounded by a function of t only. Gathering everything together we get: $\partial_t\|w(t)\|^2_{L^2} \leq C_1(t)\|w(t)\|^2_{L^2}$ by integration and using Gronwall's inequality lemma we get that $\|w(t)\|^2_{L^2} \leq \|w(0)\|^2_{L^2} \exp\left(\int_0^t C_1(s)ds\right) \underset{\text{since } w(0)=0}{=} 0$; thus $w(t) = 0$ and we have uniqueness. Now, for the existence part.
Arzela-Ascoli theorem states the following:

Theorem 2.2. Let $\mathcal{F}$ be an equicontinuous family of functions from a separable space X to a metric space Y. Let $\{f_n\}$ be a sequence in $\mathcal{F}$ such that for each $x \in X$ the closure of the set $\{f_n(x) : 0 \leq n < \infty\}$ is compact. Then there is a subsequence $\{f_{n_k}\}$ that converges pointwise to a continuous function f, and the convergence is uniform on each compact subset of X. [3, page 169]

u_ϵ is bounded in $L^\infty(I, H^1(M)) \cap Lip(I, L^2(M))$ (this follows from Problem 12), it has a weak limit point by Alaoglu theorem:

Theorem 2.3. (Alaoglu Theorem) For a real Banach space X, the closed unit ball: $\mathcal{D}(X^*) = \{f \in X^* : \|f\| \leq 1\}$, where X^* is the dual to X, is compact in the weak-* topology. [4]

(where X in this theorem is $H^1(M)$ which is a Banach space, we are looking at this space since the function $u_\epsilon : I \to H^1(M)$; and the dual to $H^1(M)$ is the space of bounded linear functionals $F : H^1(M) \to \mathsf{R}$). So there exists $u \in L^\infty_{loc}(I, H^1(M)) \cap Lip_{loc}(I, L^2(M))$ such that $u_\epsilon \rightharpoonup v$ Furthermore, by Arzela-Ascoli theorem, there's a subsequence: $u_{\epsilon_k} \to u$ in $C(I, L^2(M))$,where in the theorem of Arzela-Ascoli we pick $f_n = u_\epsilon$, where $\epsilon = \epsilon(n)$, i.e ϵ depends on n, $X = I$ and $Y = H^1(M)$. Since $u_{\epsilon_k} \rightharpoonup v$ as well, we must have that $v = u$ in $L^2(M)$.(The proof of the last claim is a simple observation that if we take $w \in L^2(M)$ then $< u - v, w >= \int_M (u - u_{\epsilon_k})w + \int_M (u_{\epsilon_k} - v)w$, the second integral converges to zero since $u_{\epsilon_k} \rightharpoonup v$, and the first integral converges to zero as well since $u_{\epsilon_k} \to u$, we have $|\int_M (u - u_{\epsilon_k})w| \leq \sup_{x \in M} |u - u_{\epsilon_k}| \cdot C \cdot \|w\|_{L^2(M)} \to 0$.)

Definition 3. A sequence of functions f_n in L^2 is said to converge weakly to a function f in L^2 provided: $\lim_{n\to\infty} \int f_n g = \int fg \ \forall g \in L^2$

While $\partial_t u_{\epsilon_k} \rightharpoonup \partial_t u$ weakly, since

in our case here the sequence $u_\epsilon \in H^1$ so both $u_\epsilon, \nabla u_\epsilon \in L^2$, the claim that justifies that $\partial_t u_{\epsilon_k} \rightharpoonup \partial_t u$ weakly is since $u_{\epsilon_k} \in L^\infty(I, H^1(M)) \cap Lip(I, L^2(M))$, we have $\partial_t u_{\epsilon_k}$ is bounded in $L^\infty(I, L^2(M)) \cap Lip(I, L^2(M))$,($\partial_t u_{\epsilon_k}$ is bounded since the weak derivative of a Lipschitz continuous function (which is u_{ϵ_k}) is bounded, the bound on the weak derivative is the Lipschitz constant).(This last fact follows from Theorem 4 in [6, pages 294-295] which we will adapt here for our case).

Theorem 2.4. (Characterization of $W^{1,\infty}$) Assume U is bounded and ∂U is Lipschitz. Assume that $f : U \to \mathsf{R}$, then:

$$f \text{ is loacally Lipschitz continuous in } U$$

if and only if:

$$f \in W^{1,\infty}_{loc}(U)$$

Proof. First suppose that f is locally Lipschitz continuous. Fix $i \in \{1, \ldots, n\}$, then for each $V \subset\subset W \subset\subset U$, pick $0 < h < dist(V, \partial W)$, and define $g_i^h(x) := \frac{f(x+he_i)-f(x)}{h}$ $(x \in V)$. Now, $\sup_{h>0} |g_i^h| \leq Lip(f|_W) < \infty$. Then according to weak compactness in L^p where $1 < p < \infty$ we have: a sequence $h_j \to 0$ and a function $g_i \in L^\infty_{loc}(U)$ such that:

$$g_i^{h_j} \rightharpoonup g_i \text{ weakly in } L^p_{loc}(U)$$

for all $1 < p < \infty$. But if $\phi \in C^1_c(V)$, we have:

$$\int_U f(x)\frac{\phi(x+he_i)-\phi(x)}{h}dx = -\int_U g_i^h(x)\phi(x+he_i)dx.$$

We set $h_j = h$ and let $j \to \infty$ to get:

$$\int_U f\phi_{x_i}dx = -\int_U g_i\phi dx$$

Hence g_i is the weak partial derivative of f with respect to x_i for $i = 1, \ldots n$ and thus $f \in W^{1,\infty}_{loc}(U)$.

Conversely, suppose $f \in W^{1,\infty}_{loc}(U)$. Let $B \subset\subset U$ be any closed ball contained in U. Then by properties of mollifiers we know that:

$$\sup_{0<\epsilon\ <\epsilon_0} \|Df^\epsilon\|_{L^\infty(B)} < \infty$$

for $\epsilon_0 > 0$ sufficiently small where $f^\epsilon = \eta_\epsilon * f$ is the usual mollification. Since $f^\epsilon \in C^\infty$ we have $f^\epsilon(x) - f^\epsilon(y) = \int_0^1 Df^\epsilon(y + t(x-y))dt \cdot (x-y)$ for $x, y \in B$; whence, $|f^\epsilon(x) - f^\epsilon(y)| \leq C|x-y|$. The constant C is independent of ϵ now as $\epsilon \to 0$ we get that $|f(x)-f(y)| \leq C|x-y|$. Hence $f|_B$ is Lipschitz continuous for each ball $B \subset\subset U$, and so f is locally Lipschiz continuous in U. ☐

so by Alaoglu theorem $\partial_t u_{\epsilon_k} \rightharpoonup w$ weakly in $L^\infty(I, L^2(M)) \cap Lip(I, L^2(M))$ for some w

and then by uniqueness of the limit $\partial_t u_{\epsilon_k} \rightharpoonup w$ in $L^\infty(I, L^2(M))$ (there is uniqueness since $L^\infty(I, L^2(M)$ is a Hausdorff space)

we get: $w = \partial_t u$, since $u_{\epsilon_k} \to u$ in $C(I, L^2(M))$). For the last assertion we need to state the Dominated Convergence Theorem and prove another claim which will prove our assertion that $w = \partial_t u$.

Theorem 2.5. (Dominated Convergence Theorem) Let $\{f_n\}$ be a sequence of real-valued measurable functions on a measure space (S, Σ, μ). Suppose the sequence converges pointwise to a function f and is dominated by some integrable function g in the sense $|f_n(x)| \leq g(x)$ for all n and for all $x \in S$, then f is integrable and $\lim_{n\to\infty} \int_S f_n(x)dx = \int f(x)dx$. [5, page 26]

Theorem 2.6. If $\{u_{\epsilon_k}(t)\} \subset L^2(M)$ where M is a compact manifold, and assume that the sequence converges uniformly in $C(I, L^2(M))$ to u where $I \subset \mathsf{R}$ is compact, assume also that $\partial_t u_{\epsilon_k}(t) \rightharpoonup w$, then $w = \partial_t u$.

Proof. We shall prove the claim (2.6). Take some $v \in L^2(M)$, write down:

$$< w - \partial_t u(t), v >= \int_M (w(x) - \partial_t u_{\epsilon_k}(t))v(x)dx + \int_M (\partial_t u_{\epsilon_k}(t) - \partial_t u(t))v(x)dx. \tag{4}$$

The first integral above in the RHS of (2) tends to zero as $k \to \infty$ since $\partial_t u_{\epsilon_k} \rightharpoonup w$; as for the second integral we shall use the Dominated Convergence Theorem. Since $u_{\epsilon_k}(t) \to u(t)$ in $C(I, L^2(M))$ we must have: $\int_M \partial_t (u_{\epsilon_k}(t) - u(t))vdx = \partial_t \int_M (u_{\epsilon_k}(t) - u(t))vdx$; now since $u_{\epsilon_k}(t)$ is bounded above by a constant that depends on t, this constant function is an integrable function since our domain of integration is a compact manifold, namely M, we get by the Dominated Convergence theorem that $\int_M u_{\epsilon_k} vdx \to \int_M uvdx$ as $k \to \infty$, where we have taken the measure to be vdx. In this case we get by the next chain of equalities that the second integral in (2) tends to zero as well:

$$\begin{aligned}\lim_{k\to\infty} \int_M \partial_t (u_{\epsilon_k}(t) - u(t))vdx &= \lim_{k\to\infty} \partial_t \int_M (u_{\epsilon_k}(t) - u(t))vdx \\ &= \partial_t \lim_{k\to\infty} \int_M (u_{\epsilon_k}(t) - u(t))vdx \\ &= \partial_t 0 = 0\end{aligned}$$

This ends the proof of the claim, since we get that $< w - \partial_t u, v >= 0 \ \forall v \in L^2(M)$, thus $w = \partial_t u$. ☐

$J_{\epsilon_k} u_{\epsilon_k}$ converges in L^2 norm to u, since we have: $\|J_{\epsilon_k} u - u\|_{L^2} \to 0$ and also $\|u_{\epsilon_k} - u\|_{L^2} \to 0$, by the triangle inequality we must have: $\|J_{\epsilon_k} u_{\epsilon_k} - u\|_{L^2} \leq \|J_{\epsilon_k} u_{\epsilon_k} - J_{\epsilon_k} u\|_{L^2} + \|J_{\epsilon_k} u - u\|_{L^2} \leq \|j_{\epsilon_k}\|_{L^1} \|u_{\epsilon_k} - u\|_{L^2} + \|J_{\epsilon_k} u - u\|_{L^2} \to 0$ (since $\|j_{\epsilon_k}\|_{L^1}$ is bounded, and from the above we know that: $\|u_{\epsilon_k} - u\|_{L^2} \to 0$). To show this we need to show that $\|J_{\epsilon_k} u - u\|_{L^2} \to 0$ is fulfilled, for this we have the next claim to prove.

Theorem 2.7. Let $\varphi \geq 0$ with $\int_{\mathbb{R}^n} \varphi(y)dy = 1$, $\varphi_\epsilon(x) = 1/\epsilon^n \varphi(x/\epsilon)$. Suppose $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$. Then:

$$\lim_{\epsilon \to 0} \|f * \varphi_\epsilon - f\|_{L^p} = 0$$

Proof. $|f * \varphi_\epsilon - f| = |\int_{\mathbb{R}^n}(f(x-y) - f(x))\varphi_\epsilon(y)dy|$. By Minkowski integral inequality, which says the following:
Suppose $(S_1, \mu_1), (S_2, \mu_2)$ are two measure spaces, and $F : S_1 \times S_2 \to \mathbb{R}$ is measurable, then: $\left[\int_{S_2} |\int_{S_1} F(x,y)d\mu_1(x)|^p d\mu_2(y)\right]^{1/p} \leq \int_{S_1} \left(\int_{S_2} |F(x,y)|^p d\mu_2(y)\right)^{1/p} d\mu_1(x))$
.

$$\begin{aligned}\|f * \varphi_\epsilon - f\|_{L^p} &\leq \| \int_{\mathbb{R}^n} |f(x-y) - f(x)|\varphi_\epsilon(y)dy\|_{L^p} \\ &\leq \int_{\mathbb{R}^n} \|f(x-y) - f(x)\|_{L^p(dx)}\varphi_\epsilon(y)dy\end{aligned}$$

Set: $I = \int_{|y| \leq \delta} \|f(x-y) - f(x)\|_{L^p(dx)}\varphi_\epsilon(y)dy$, and $II = \int_{|y|>\delta} \|f(x-y) - f(x)\|_{L^p(dx)}\varphi_\epsilon(y)dy$. The translation operator $y \to f(x-y)$ is continuous from $\mathbb{R}^n$ to $L^p(\mathbb{R}^n)$ for $1 \leq p < \infty$. So given $\eta > 0$ there exists $\delta > 0$ s.t:

$$\|f(x-y) - f(x)\|_{L^p(dx)} < \eta \ \forall |y| \leq \delta.$$

Thus with such a δ, $I < \eta \int_{|y| \leq \delta} \varphi_\epsilon(y)dy \leq \eta \int_{\mathbb{R}^n} \varphi_\epsilon(y)dy = \eta$. From the fact that: $\|f(x-y) - f(x)\|_{L^p(dx)} \leq 2\|f\|_{L^p}$, it follows that: $II \leq 2\|f\|_{L^p} \int_{|y|>\delta} \varphi_\epsilon(y)dy = 2\|f\|_{L^p} \frac{1}{\epsilon^n} \int_{|y|>\delta} \varphi(y/\epsilon)dy = 2\|f\|_{L^p} \int_{|y|>\delta/\epsilon} \phi(y)dy \to 0$ as $\epsilon \to 0$. Thus, $\|f * \varphi_\epsilon - f\|_{L^p} \to 0$. □

Thus, we apply the theorem on $p = 2$ we must have $\|J_{\epsilon_k} u - u\|_{L^2} \to 0$, and from the above argumentation indeed $\|J_{\epsilon_k} u_{\epsilon_k} - u\|_{L^2} \to 0$. Since the derivative of g, is bounded by C, we have a Lipschitz constant C ,s.t $|g(J_{\epsilon_k} u_{\epsilon_k}) - g(u)| \leq C|J_{\epsilon_k} u_{\epsilon_k} - u|$, we get that: $\|g(J_{\epsilon_k} u_{\epsilon_k}) - g(u)\|_{L^2} \leq C\|J_{\epsilon_k} u_{\epsilon_k} - u\|_{L^2} \to 0$; thus we have: $g(J_{\epsilon_k} u_{\epsilon_k}) \to g(u)$ in $C(\mathbb{R}, L^2(M))$ norm.And also we have:

$$\begin{aligned}\|J_{\epsilon_k} g(J_{\epsilon_k} u_{\epsilon_k}) - g(u)\|_{L^2} &\leq \|J_{\epsilon_k} g(J_{\epsilon_k} u_{\epsilon_k}) - J_{\epsilon_k} g(u)\|_{L^2} + \|J_{\epsilon_k} g(u) - g(u)\|_{L^2} \\ &\leq \|j_{\epsilon_k}\|_{L^1} \|g(J_{\epsilon_k} u_{\epsilon_k}) - g(u)\|_{L^2} + \|J_{\epsilon_k} g(u) - g(u)\|_{L^2} \underset{k \to \infty}{\longrightarrow} 0\end{aligned}$$

Where in the above last chain of inequalities the first term converges to zero as we have seen above it since $\|j_{\epsilon_k}\|_{L^1} < \infty$ and $\|g(J_{\epsilon_k} u_{\epsilon_k}) - g(u)\|_{L^2} \to 0$ as shown above, and $\|J_{\epsilon_k} g(u) - g(u)\|_{L^2} \to 0$ follows from theorem (2.7).

Definition 4. A continuous operator, $T : A \to A$, at a point x_0; where A is a Banach space, is an operator that is continuous in some topology. There is the strong continuity by the norm of A, i.e $\lim_{x\to x_0} \|T(x)-T(x_0)\|_A = 0$, and there's also weak-topology continuity, by the inner product, i.e: $< T(x) - T(x_0), v >_A \to 0 \ \forall v \in A$ as $x \to x_0$.

L is a weak-topology continuous operator from the space $H^1(M) \to L^2(M)$

by the fact that $L = \sum_j A_j(t,x)\partial_j$, we want to show weak convergence of L operator, where $u \rightharpoonup u_0$. Take $v \in L^2$ then: $| < L(u) - L(u_0), v > | = | \int \sum_j A_j \partial_j (u - u_0) v| \le C_2(t) \sum_j | < \partial_j(u - u_0), v > | \to 0$ as $u \rightharpoonup u_0$ in $H^1(M)$. Where we used the fact that $A_j(x,t)$ is smooth in its arguments in a compact manifold T^n and thus A_j is bounded by a constant that depends on t (just as in the uniqueness part of this problem); so by the weak convergence of $u \rightharpoonup u_0$ in $H^1(M)$ we have: $| < \partial_j(u - u_0), v > | \to 0$.

Then by the weak continuity of L $J_{\epsilon_k} L J_{\epsilon_k} u_{\epsilon_k} \rightharpoonup Lu$ weakly (since $LJ_{\epsilon_k} u_{\epsilon_k} \rightharpoonup Lu = v$ weakly, and if we denote by: $v_{\epsilon_k} = LJ_{\epsilon_k} u_{\epsilon_k}$ we also have $J_{\epsilon_k} v_{\epsilon_k} \rightharpoonup Lu = v$ from what was proven above),

so by the fact that $\frac{\partial u_\epsilon}{\partial t} = J_\epsilon L J_\epsilon u_\epsilon + J_\epsilon g(J_\epsilon u_\epsilon), u_\epsilon(0) = f$ and u_{ϵ_k} is a subsequence of u_ϵ that satisfy the same PDE and gathering all the limits we get that: $\partial_t u_{\epsilon_k} \rightharpoonup \partial_t u$ weakly , $J_{\epsilon_k} L J_{\epsilon_k} u_{\epsilon_k} \rightharpoonup Lu$ weakly , $J_{\epsilon_k} g(J_{\epsilon_k} u_{\epsilon_k}) \to g(u)$ in L^2 norm,and thus by the fact that strong convergence implies weak convergence, we also have here weak convergence:$J_{\epsilon_k} g(J_{\epsilon_k} u_{\epsilon_k}) \rightharpoonup g(u)$. By the uniqueness of the limit, which means that since $\partial_t u_{\epsilon_k} = J_{\epsilon_k} L J_{\epsilon_k} u_{\epsilon_k} + J_{\epsilon_k} g(J_{\epsilon_k} u_{\epsilon_k})$ and $\partial_t u_{\epsilon_k} \rightharpoonup \partial_t u$ weakly; and also $J_{\epsilon_k} L J_{\epsilon_k} u_{\epsilon_k} + J_{\epsilon_k} g(J_{\epsilon_k} u_{\epsilon_k}) \rightharpoonup Lu + g(u)$ weakly, thus we must have equality between the limits, i.e, $\partial_t u = Lu + g(u)$.

And since $u_{\epsilon_k}(0) = f$ in the weak limit we have: $f = u_{\epsilon_k}(0) \rightharpoonup u(0) \Rightarrow u(0) = f$. □

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