On the existence of identical periodic solutions between differential equations

J. Akande[1], K. K. D. Adjaï[1], A.V. R. Yehossou[1], M. D. Monsia[1*]

1- Department of Physics, University of Abomey-Calavi, Abomey-Calavi, 01.BP.526, Cotonou, BENIN

Abstract

We present for the first time a singular quadratic Lienard type equation having the linear harmonic oscillator solution. We show also the existence of a singular quadratic Lienard type equation having the solution of the Ermakov-Pinney equation. Consequently, these equations may be used as truly nonlinear oscillators.

Keywords: Linear harmonic oscillator, singular quadratic Lienard type equation, truly nonlinear oscillator, exact periodic solution.

Introduction

In theory of second-order autonomous differential equations, the linear harmonic oscillator

$$\ddot{x} + \omega^2 x = 0 \tag{1}$$

where the overdot denotes differentiation with respect to time, and ω is a constant, is taken as the prototype. It is known that the general solution of the equation (1) is

$$x(t) = A_0 \sin(\omega t + \alpha) \tag{2}$$

where A_0 and α are arbitrary parameters.

As the analytical properties of the sine function are well known and very convenient for the engineering practice, the problem of finding sinusoidal solution to differential equations has becomes an attractive research field for the physics and engineering. However, to day, there are no known nonlinear differential equations having the formula (2) as exact and explicit general

[*] Corresponding author : E-mail: monsiadelphin@yahoo.fr

solution. Another celebrated differential equation like the linear harmonic oscillator is the Ermakov-Pinney equation [1]

$$\ddot{x} + a\,x + \frac{b}{x^3} = 0 \tag{3}$$

where a and b are arbitrary parameters. Recently in [2] it has been shown the existence of a new periodic solution to this equation in the form

$$x(t) = \pm\left(\frac{b}{a}\right)^{1/4}\sqrt{\sin\left[\pm 2\sqrt{a}\,(t+K)\right]} \tag{4}$$

where K is an arbitrary parameter. The solution (4) is such that

$$x^2(t) = \sqrt{\frac{b}{a}}\sin\left(\pm 2\sqrt{a}\,(t+K)\right) \tag{5}$$

which is identical to (2) when $A_0^2 = \dfrac{b}{a}$, $\omega = \pm 2\sqrt{a}$ and $\alpha = \pm 2\sqrt{a}K$. It is reasonable to ask whether one can map the Ermakov-Pinney equation onto a differential equation having the formula (5) as general periodic solution. It is also logic to ask whether there exists a differential equations having the Ermakov-Pinney equation solution. In this situation the objective in this paper is to show that the formula (5) is the solution of a singular quadratic Lienard type equation and that the formula (4) is also the solution of a singular quadratic Lienard type equation. In section (2) we show the first prediction, and the second prediction is shown in the section (3). Finally a conclusion is carried out for the work.

2. Equation having (5) as solution

Let us consider the point transformation

$$u = x^p \tag{6}$$

Using the equation (6) one may get the first derivative

$$\frac{dx}{dt} = \frac{1}{p}u^{\frac{1-p}{p}}\frac{du}{dt} \tag{7}$$

and the second derivative

$$\frac{d^2x}{dt^2} = \frac{1}{p}\left\{\ddot{u}\,u^{\frac{1-p}{p}} + \frac{1-p}{p}\dot{u}^2 u^{\frac{1-2p}{p}}\right\} \tag{8}$$

Substituting (6) and (8) into (3) yields

$$\ddot{u}\,u^{\frac{1-p}{p}}+\frac{1-p}{p}\dot{u}^2u^{\frac{1-2p}{p}}+ap\,u^{\frac{1}{p}}+bp\,u^{\frac{-3}{p}}=0 \tag{9}$$

from which one may secure the quadratic Lienard type equation

$$\ddot{u}+\frac{1-p}{p}\frac{\dot{u}^2}{u}+ap\,u+bp\,u^{\frac{p-4}{p}}=0 \tag{10}$$

Putting $p=2$, reduces (10) to

$$\ddot{u}-\frac{1}{2}\frac{\dot{u}^2}{u}+2au+\frac{2b}{u}=0 \tag{11}$$

which admits the solution (5), that is

$$u(t)=\sqrt{\frac{b}{a}}\sin\left(\pm 2\sqrt{a}\,(t+K)\right) \tag{12}$$

Now we may show the existence of a singular quadratic Lienard type equation having the Ermakov-Pinney equation solution.

3. Equation having the formula (4) as solution

In this section we present the singular quadratic Lienard type equation that exhibits the formula (4) as solution. To do so we consider the general class of quadratic Lienard type equations introduced recently by Akande et al. [3].

Let

$$y''(\tau)+\beta^2 y(\tau)=0 \tag{13}$$

be the equation of the linear harmonic oscillator in this theory, where prime designes the derivative with respect to τ and β a constant. Let us also consider the generalized Sundman transformation [3]

$$y(\tau)=F(t,x), \qquad d\tau=G(t,x)dt, \qquad \frac{\partial F(t,x)}{\partial x}\neq 0 \tag{14}$$

where $\quad F(t,x)=\int g^{\ell}(x)dx$, $\qquad G(t,x)=e^{\gamma\varphi(x)}$

$g(x)\neq 0$, and $\varphi(x)$ are arbitrary functions of x, and ℓ and γ are arbitrary parameters. The application of (14) transforms (13) into [3]

$$\ddot{x}+\left(\ell\frac{g'(x)}{g(x)}-\gamma\,\varphi'(x)\right)\dot{x}^2+\frac{\beta^2\,e^{2\gamma\varphi(x)}\int g(x)^\ell\,dx}{g(x)^\ell}=0 \qquad (15)$$

where prime means differentiation with respect to the argument. Subtituting $\varphi(x)=\ln\big(f(x)\big)$, yields as equation

$$\ddot{x}+\left(\ell\frac{g'(x)}{g(x)}-\gamma\frac{f'(x)}{f(x)}\right)\dot{x}^2+\frac{\beta^2 f(x)^{2\gamma}\int g(x)^\ell\,dx}{g(x)^\ell}=0 \qquad (16)$$

Choosing $f(x)=x^2$, and $g(x)=x$, leads to obtain the class of singular quadratic Lienard type equations.

$$\ddot{x}+(\ell-2\gamma)\frac{\dot{x}^2}{x}+\frac{\beta^2}{\ell+1}x^{4\gamma+1}=0 \qquad (17)$$

In this context the nonlocal transformation (14) becomes

$$y(\tau)=\frac{1}{\ell+1}x^{\ell+1}\ ,\quad d\tau=x^{2\gamma}dt\,,\qquad \ell\neq-1 \qquad (18)$$

By making $\ell=1$, and $\gamma=0$, one may get the singular quadratic Lienard type equation of interest in the form

$$\ddot{x}+\frac{\dot{x}^2}{x}+\frac{\beta^2}{2}x=0 \qquad (19)$$

Now the objective is to show that the exact and explicit general periodic solution of (19) is identical to the solution (4) of the Ermakov-Pinney equation (3). To that end, the system of equations (18) become

$$y(\tau)=\frac{1}{2}x^2\ ,\quad d\tau=dt \qquad (20)$$

If we take the solution of the equation (13) as

$$y(\tau)=A\sin\big(\beta\tau+\lambda\big) \qquad (21)$$

where A and λ are arbitrary parameters, then the system of equations (20) reduces to

$$x(t)=\pm\big[2A\sin\big(\beta\tau+\lambda\big)\big]^{1/2}\ ,\quad d\tau=dt \qquad (22)$$

From $d\tau=dt$, one may obtain

$$\tau = t + K_1 \tag{23}$$

where K_1 is an arbitrary constant. In this situation the solution $x(t)$ is given by

$$x(t) = \pm\left[2A\sin\left(\beta\left(t + K_1\right) + \lambda\right)\right]^{1/2} \tag{24}$$

which takes the definitive form

$$x(t) = \pm\left[2A\right]^{1/2}\sqrt{\sin\left(\beta\left(t + K_2\right)\right)} \tag{25}$$

where $\beta K_2 = \beta K_1 + \lambda$. The solution (25) is identical to the solution (4) where $\frac{b}{a} = 4A^2$, $\beta = \pm 2\sqrt{a}$, and $K = K_2$. As the Ermakov-Pinney equation one may see that the singular quadratic Lienard type equation (19) is a truly nonlinear oscillator. Now a conclusion may be formulated for the work.

Conclusion

We investigate in this paper the possibility to two differential equations to have identical solutions. In this context we have shown the existence for the first time of a singular quadratic Lienard type equation that has the linear harmonic oscillator equation solution. Secondly the existence for the first time of a singular quadratic Lienard type equation having the Ermakov-Pinney equation solution has also been highlighted. As a result these equations can be used as truly nonlinear oscillators.

References

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