General Formula for a Polynomial Generated Sequence

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Abstract

In this paper, we will introduce a simple formula that relates polynomial and polynomial sequence. The derivation is based on a method that used for calculating the coefficient of a polynomial generated sequence which we will show in later chapters.

1 Introduction

When someone gives you a sequence and would like you to tell him or her the next term or even the 100th term, we would often study the sequence by identifying what kind of sequence is it.In general, if the sequence turns out to be a polynomial generated sequence, we can proceed to subtract the excessive terms,identify the degree of polynomial and carry out the usual way to find the general formula by calculating the coefficient of the polynomial.

However, we believe that if the sequence given to us is having a large degree,it would take up a lot time to carry out the process. So, we used this method and generalise it to a general formula.

For more information on the suggested method, one can read the article labeled [1] in the reference section.

2 Definitions

2.1 Mirrored Worpitzky Number Triangle(MWNT)

Even this method is strongly based on MWNT but we will not look into this types of numbers. But it is still important to note down the formula we used in the calculation.

$$MWNT(n,k) = \frac{1}{k}\sum_{i=0}^{k}(-1)^{k-i}\left(\begin{array}{c} k \\ i \end{array}\right) i^n for\, n \geq 1, k \geq 1$$

2.2 Polynomial Difference Theorem

The theorem states that a P(x) is a polynomial iff for a set of x values,P(x) generates an arithmetic sequence such that when we subtract the excessive terms,

we obtain numbers of row of differences and corresponding values in the last row is constant.
In other word, the degree of P(x) is the number of row of difference it has.

2.3 Polynomial Sequence's residue

We give the "number of difference of row" a fancy name, known as polynomial sequence's residue(PSR). For all polynomial sequence, there exist two kind of PSR: constant residue variable residue.
Constant Residue refers to the row which corresponding values are equal but not equal to 0.
Variable residue refers to the other row other than constant residue.

Other fact about PSR is that every PSR is a sequence of degree that is less than the pervious one.In another word, if a polynomial sequence having degree d, then the first PSR is a sequence of degree (d-1).

3 Simple Derivation

We let our polynomial in its general form be:

$$p(x) = \sum_{n=1}^{d} C_n x^n$$

where $C_1 C_2, ..., C_d$ are constant coefficient and d is the degree of polynomial.
With set of x values, we formed a polynomial sequence: p(1),p(2).... .Next, we subtract excessive terms to obtain PSR and the first value appear in every PSR which we denote these values as $a_0, a1...$ where a_0 comes from the constant residue.

There is a relationship between the coefficient, MWNT and a_n which can be expressed as a series of linear equations.

$$C_d MWNT(d+1, d+1) = a_0$$

$$\sum_{n=0}^{1} C_(d+n-1) MWNT(d+n, d) = a_1$$

$$\sum_{n=0}^{2} C_(d+n-2) MWNT(d+n-1, d-1) = a_2$$

$$\sum_{n=0}^{3} C_(d+n-3) MWNT(d+n-2, d-2) = a_3$$

$$\vdots$$

$$\sum_{n=0}^{d} C_n MWNT n+1, 1 = P(1)$$

It is impossible to solve the equation(solving for coefficient) without knowing what is d. So,to continue the next step, we have to assume d is a very large and finite value. After obtaining the coefficient, we substitute the coefficient into P(x) and rearrange the following terms by distributing the variable and regroup terms with a_0, a_1, a_2..... Then, we get:

$$p(x) = p(1) + \frac{a_{d-1}(x-1)}{1!} + \frac{a_{d-2}(x-1)(x-2)}{2!} + ... \frac{a_0(x-1)(x-2)(x-3)....(x-d)}{d!}$$

$$p(x) = p(1) + \sum_{i=1}^{d} a_{d-i} \prod_{n=1}^{i} \frac{x-n}{n}$$

where d is the degree of polynomial.

4 Reference

[1] John K. Sikora, On Calculating the Coefficients of a Polynomial Generated Sequence Using the Worpitzky Number Triangle, ArVix.