Stirling Numbers Via Combinatorial Sums

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Abstract. In this paper, we have derived a formula to find combinatorial sums of the type $\sum_{r=0}^{n} r^k \binom{n}{r}$ for $k \in \mathbb{N}$. The formula is conveniently expressed as a linear combination of terms involving the falling factorial. The co-efficients in this linear expression satisfy a recurrence relation, which is identical to that of the Stirling numbers of the first and second kind.

Keywords: Combinatorics · Stirling Numbers

1 Introduction to Stirling Numbers

The Stirling Numbers [12] arise in combinatorics in describing a set of n distinct objects by specific groupings. Stirling Numbers of the first kind describe the number of ways to permute n objects such that it forms k disjoint cycles, denoted by $\left[{n \atop k}\right]$. On the other hand, $\left\{{n \atop k}\right\}$ describes the number of ways to partition n objects into k non-intersecting subsets. They are described by the following recurrence relations -

$$\begin{bmatrix} n+1 \\ k \end{bmatrix} = n \begin{bmatrix} n \\ k \end{bmatrix} + \begin{bmatrix} n \\ k-1 \end{bmatrix} \tag{1}$$

$$\begin{Bmatrix} n+1 \\ k \end{Bmatrix} = k \begin{Bmatrix} n \\ k \end{Bmatrix} + \begin{Bmatrix} n \\ k-1 \end{Bmatrix} \tag{2}$$

One subtlety to note is that eq (1) is the recurrence for the *unsigned* Stirling numbers of the first kind. The *signed* Stirling numbers of the first kind follow an alternative recurrence -

$$s(n,k) = s(n,k-1) - ns(n,k) \tag{3}$$

The Stirling Numbers find their applications in Computer Science as has been exclusively pointed by Donald Knuth [8].

2 Related Works

We summarise here some of the related works in the literature involving power sums of binomial coefficients and Stirling Numbers. [2] has analysed power sums involving Stirling Numbers in the complex domain. In [5], sums involving product of binomial co-efficients and Harmonic numbers H_n have been analysed and similar work has been done by [1] pursuing a connection between Stirling and Harmonic numbers. [6] has pursued results related power sums of binomial co-efficients with other binomial coefficients.

There's a connection between these power sums and Wilson's congruence which has been demonstrated in [7]. They have analysed power sums of the type in eq (4) which closely resemble the sums studied in this work —

$$\sum_{r=0}^{n}(-1)^{n-r}r^k\binom{n}{r} \tag{4}$$

A similar analysis has also been performed in [11], [3] and is well known classical result in combinatorics. [4] has analysed sums of the type in eq (5) —

$$\sum_{r=0}^{n}\binom{n}{r}^a \tag{5}$$

Theorems due to de Bruijn [9] suggest that no closed forms for it exist for all $a \in \mathbb{N}$. However, some of its divisibility properties have been explored in [4].

3 Basis of the Conjecture

Using the binomial expansion —

$$(1+x)^n = \sum_{r=0}^{n} x^r\binom{n}{r} \tag{6}$$

On differentiating —

$$n(1+x)^{n-1} = \binom{n}{1} + 2x\binom{n}{2} + 3x^2\binom{n}{3} + \ldots + nx^{n-1}\binom{n}{n} \tag{7}$$

Setting x = 1 in equation (6), we get —

$$\sum_{r=0}^{n} r\binom{n}{r} = n2^{n-1} \tag{8}$$

which is our required answer. As a logical extension, one can ask what the sum of $\sum_{r=0}^{n} r^2\binom{n}{r}$ is. We first multiply eq (7) with x and differentiate to obtain —

$$n(1+x)^{n-1} + nx(n-1)(1+x)^{n-2} = \binom{n}{1} + 2^2x\binom{n}{2} + \ldots + n^2x^{n-1}\binom{n}{n} \tag{9}$$

On setting $x = 1$ in eq (9), we get —

$$\sum_{r=0}^{n} r^2 \binom{n}{r} = n2^{n-1} + n(n-1)2^{n-2} \tag{10}$$

If we use the notation, $S_{nk} = \sum_{r=0}^{n} r^k \binom{n}{r}$, we also find that —

$$S_{n3} = n2^{n-1} + 3n(n-1)2^{n-2} + n(n-1)(n-2)2^{n-3} \tag{11}$$

As it can be seen, the $n(n-1)2^{n-2}$ term is padded with a co-efficient of 3. If we denote $\prod_{i=0}^{k-1}(n-i) = (n)_k$ (A.K.A the falling factorial for $k \geq 1$), and $T_{ni} = (n)_i 2^{n-i}$, then we conjecture that -

$$S_{nk} = \sum_{i=1}^{k} a_{ki}(n)_i 2^{n-i} = \sum_{i=1}^{k} a_{ki} T_{ni} \qquad 1 \leq k \leq n \tag{12}$$

The a_{ki}'s are general co-efficients which are padded to T_{ni} terms. Note that $(n)_n = n!$ and for any $k > n$, we have $(n)_k = 0 \implies T_{nk} = 0$. We can also assume $(n)_0 = 1$. Hence our conjecture (eq 12) is valid in general with the summation limit upto n —

$$S_{nk} = \sum_{i=1}^{n} a_{ki}(n)_i 2^{n-i} = \sum_{i=1}^{n} a_{ki} T_{ni} \qquad k \in \mathbb{N} \tag{13}$$

Since S_{nk} is obtained by setting $x = 1$ in a polynomial, we denote this polynomial as $S_{nk}^{(x)}$. Our process of multiplication by x followed by differentiation can be encapsulated as —

$$S_{n(k+1)}^{(x)} = \frac{d}{dx}\left[x S_{nk}^{(x)}\right]$$
$$\implies S_{n(k+1)}^{(x)} = S_{nk}^{(x)} + x\frac{d}{dx} S_{nk}^{(x)} \tag{14}$$

where

$$S_{nk}^{(x)} = \sum_{i=1}^{n} a_{ki}(n)_i x^{i-1}(1+x)^{n-i} \tag{15}$$

It is easy to see that every polynomial $S_{nk}^{(x)}$ is a polynomial in x of degree $(n-1)$ for $k \in \mathbb{N}$. This is because each successive polynomial is obtained by multiplication by x followed by a differentiation — a step that retains the degree of a polynomial. The recurrence relation among the polynomials (eq 14. Consider the i^{th} term of $S_{nk+1}^{(x)}$ and its co-efficient $a_{(k+1)i}$. This term is built from terms in $S_{nk}^{(x)}$ as follows —

$$\begin{aligned} a_{(k+1)i}(n)_i x^{i-1}(1+x)^{n-i} &= a_{ki}(n)_i x^{i-1}(1+x)^{n-i} \\ &+ a_{ki}(n)_i(1+x)^{n-i} x \frac{d}{dx} x^{i-1} \\ &+ a_{k(i-1)}(n)_{i-1} x^{i-2} x \frac{d}{dx}(1+x)^{n-i+1} \end{aligned}$$

$$\begin{aligned} \implies a_{(k+1)i}(n)_i x^{i-1}(1+x)^{n-i} &= i a_{ki}(n)_i x^{i-1}(1+x)^{n-i} \\ &+ a_{k(i-1)}(n)_i x^{i-1}(1+x)^{n-i} \qquad (16) \end{aligned}$$

We can extract the recurrence relation by equating the co-efficients in (16) —

$$a_{(k+1)i} = i a_{ki} + a_{k(i-1)} \qquad (17)$$

The co-efficients a_{ki} satisfy the same recurrence as that of the eq (2) — Stirling Numbers of the Second Kind.

4 Triangle of Stirling Numbers of The Second Kind

It is well known that $\sum_{r=0}^{n} \binom{n}{r} = 2^n$. This represents S_{n0}. Since $(n)_0 = 1$ and $T_{n0} = 2^n$, we get $S_{n0} = a_{00} T_{n0}$ and $a_{00} = 1$. We also define $a_{k0} = 0$ for $k > 0$ and $a_{ki} = 0$ for $i > k$. These definitions are consistent with our conjecture (eq 13) and also define the base case of the Stirling Numbers of the Second Kind. Hence, our derived coefficients a_{ki} *must* be the Stirling Numbers of the Second Kind. We can display these numbers in a triangular fashion in table (4) [14] —

i k	0	1	2	3	4	5	6	7
0	1	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0
2	0	1	1	0	0	0	0	0
3	0	1	3	1	0	0	0	0
4	0	1	7	6	1	0	0	0
5	0	1	15	25	10	1	0	0
6	0	1	31	90	65	15	1	0
7	0	1	63	301	350	140	21	1

Table 1. Stirling Numbers of the Second Kind

4.1 Verifying The Formula for S_{nk}

We confirm the validity of the method for two examples - one with $k \leq n$ and another with $k > n$ [13]

Case 1 - $n = 5, k = 4$

$$\begin{aligned}\sum_{r=0}^{5} r^4 \binom{5}{r} &= 0^4\binom{5}{0} + 1^4\binom{5}{1} + 2^4\binom{5}{2} + 3^4\binom{5}{3} + 4^4\binom{5}{4} + 5^4\binom{5}{5} \\ &= 0 + (1\times 5) + (16\times 10) + (81\times 10) + (256\times 5) + (625\times 1) \\ &= 2880\end{aligned}$$

From (13), we have -

$$\begin{aligned}S_{54} &= \sum_{i=0}^{5} a_{4i}T_{5i} \\ &= a_{40}T_{50} + a_{41}T_{51} + a_{42}T_{52} + a_{43}T_{53} + a_{44}T_{54} + a_{45}T_{55} \\ &= 0 + (1\times 80) + (7\times 160) + (6\times 240) + (1\times 240) + 0 \\ &= 2880\end{aligned}$$

Case 2 - $n = 3, k = 6$

$$\begin{aligned}\sum_{r=0}^{3} r^6 \binom{3}{r} &= 0^6\binom{3}{0} + 1^6\binom{3}{1} + 2^6\binom{3}{2} + 3^6\binom{3}{3} \\ &= 0 + (1\times 3) + (64\times 3) + (729\times 1) \\ &= 924\end{aligned}$$

From (13), we have -

$$\begin{aligned}S_{36} &= \sum_{i=0}^{3} a_{6i}T_{3i} \\ &= a_{60}T_{30} + a_{61}T_{31} + a_{62}T_{32} + a_{63}T_{33} \\ &= 0 + (1\times 12) + (31\times 12) + (90\times 6) \\ &= 924\end{aligned}$$

5 An Alternate Approach

We shall derive the inverse relation i.e T_{nk} in as a linear sum of S_{nk}'s. From this point on, we shall asume strictly $k \le n$

$$(1+x)^n = \sum_{r=0}^{n} x^r \binom{n}{r}$$

Differentiating k times -

$$\left[\prod_{i=0}^{k-1}(n-i)\right](1+x)^{n-k} = \sum_{r=k}^{n}\left[\prod_{i=0}^{k-1}(r-i)\right] x^{r-k}\binom{n}{r} \tag{18}$$

The product on the LHS is just the falling factorial. One can expand the product on the RHS with generalised coefficients b_{ki} —

$$\prod_{i=0}^{k-1}(r-i) = \sum_{i=1}^{k} b_{ki} r^i \tag{19}$$

Plugging in (19) in (18) and multiplying both sides by x^k —

$$\begin{aligned}
(n)_k x^k (1+x)^{n-k} &= \sum_{r=k}^{n}\sum_{i=1}^{k} b_{ki} r^i x^r \binom{n}{r} \\
&= \sum_{r=0}^{n}\sum_{i=1}^{k} b_{ki} r^i x^r \binom{n}{r} - \sum_{r=0}^{k-1}\sum_{i=1}^{k} b_{ki} r^i x^r \binom{n}{r} \\
&= \sum_{i=1}^{k} b_{ki} \sum_{r=0}^{n} r^i x^r \binom{n}{r} - \sum_{r=0}^{k-1} x^r \binom{n}{r} \left[\sum_{i=1}^{k} b_{ki} r^i\right] \\
&= \sum_{i=1}^{k} b_{ki} \sum_{r=0}^{n} r^i x^r \binom{n}{r} - \sum_{r=0}^{k-1} x^r \binom{n}{r} \left[\prod_{i=0}^{k-1} (r-i)\right]
\end{aligned} \tag{20}$$

Plugging in $x = 1$ in (20) —

$$\begin{aligned}
(n)_k 2^{n-k} &= \sum_{i=1}^{k} b_{ki} \left[\sum_{r=0}^{n} r^i \binom{n}{r}\right] - \sum_{r=0}^{k-1} \binom{n}{r} \left[\prod_{i=0}^{k-1}(r-i)\right] \\
\implies T_{nk} &= \sum_{i=1}^{k} b_{ki} S_{ni} - \sum_{r=0}^{k-1} \binom{n}{r} \left[\prod_{i=0}^{k-1}(r-i)\right]
\end{aligned} \tag{21}$$

In the summation indexed by r on the RHS of (21), r can only take values from $\{0, 1, \cdots, k-1\}$. The product vanishes for every value of r as i indexes from 0 to $k-1$. Hence the second summation term is identically zero. Ultimately we obtain the inverse relation to eq (13) as —

$$T_{nk} = \sum_{i=1}^{k} b_{ki} S_{ni} \tag{22}$$

6 Obtaining The Recurrence Relation For b_{ki}

From the definition of b_{ki}, it can be seen that the co-efficient of the lowest power is $b_{k1} = \prod_{i=1}^{k-1}(-1)^i i = (-1)^{k-1}(k-1)!$. Moreover, the co-efficient of the highest power is $b_{kk} = 1$.

We have established the base cases and can continue to establish the recurrence relation. From (19), we have -

$$\prod_{i=0}^{k}(r-i) = \sum_{i=1}^{k+1} b_{(k+1)i} r^i$$

$$\implies \sum_{i=1}^{k+1} b_{(k+1)i} r^i = (r-k)\left[\prod_{i=0}^{k-1}(r-i)\right]$$

$$= (r-k)\left[\sum_{i=1}^{k} b_{ki} r^i\right]$$

$$= \sum_{i=1}^{k} b_{ki} r^{i+1} - \sum_{i=1}^{k} k b_{ki} r^i$$

We equate the coefficients in the relation —

$$\sum_{i=1}^{k+1} b_{(k+1)i} r^i = \sum_{i=2}^{k-1} b_{k(i-1)} r^i - \sum_{i=1}^{k} k b_{ki} r^i \tag{23}$$

A recurrence from (23) could be extracted as follows —

$$b_{(k+1)i} = b_{k(i-1)} - k b_{ki} \tag{24}$$

The coefficients b_{ki} satisfy the same recurrence as that of the Signed Stirling Numbers of the First Kind (eq 3).

7 Triangle of Signed Stirling Numbers of the First Kind

i / k	0	1	2	3	4	5	6	7
0	1	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0
2	0	-1	1	0	0	0	0	0
3	0	2	-3	1	0	0	0	0
4	0	-6	11	-6	1	0	0	0
5	0	24	-50	35	-10	1	0	0
6	0	-120	274	-225	85	-15	1	0
7	0	720	-1764	1624	-735	175	-21	1

Table 2. Signed Stirling Numbers of the First Kind

As argued previously, we set $b_{00} = 1$ to satisfy our conjecture (eq 13) and $b_{k0} = 0$ for $k > 0$ and $b_{ki} = 0$ for $i > k$. Since they satisfy the same base case as the Signed Stirling Numbers of the First Kind, the coefficients b_{ki} *must* be the Signed Stirling Numbers of the First Kind. We can again display these numbers in a triangular fashion in table (7) [13].

7.1 Verifying The Formula For T_{nk}

It has already been stated that $k \leq n$. Hence, we shall verify the formula (22) for the two cases below [14]

Case 1 - $n = 5, k = 3$

$$\begin{aligned} T_{53} &= 240 \\ S_{51} &= 80 \\ S_{52} &= 240 \\ S_{53} &= 800 \end{aligned}$$

From eq (22), we have —

$$\begin{aligned} b_{31}S_{51} + b_{32}S_{52} + b_{33}S_{53} &= (2 \times 80) - (3 \times 240) + (1 \times 800) \\ &= 240 \end{aligned}$$

Case 2 - $n = 6, k = 4$

$$\begin{aligned} T_{64} &= 1440 \\ S_{61} &= 192 \\ S_{62} &= 672 \\ S_{63} &= 2592 \\ S_{64} &= 10752 \end{aligned}$$

From eq (22), we have —

$$\begin{aligned} b_{41}S_{61} + b_{42}S_{62} + b_{43}S_{63} + b_{44}S_{64} &= (-6 \times 192) + (11 \times 672) - (6 \times 2592) + (1 \times 10752) \\ &= 1440 \end{aligned}$$

8 Proving The Inverse Nature of The Sequences

We shall concern ourselves only with a square sub-section of the table of Stirling numbers (i.e $1 \leq k \leq n$). By using (25) in our conjecture (eq 12), we get —

$$\begin{aligned} T_{nk} &= \sum_{i=1}^{k} b_{ki} S_{ni} \\ &= \sum_{i=1}^{k} b_{ki} \left(\sum_{l=1}^{i} a_{il} T_{nl} \right) \\ &= \sum_{i=1}^{k} \sum_{l=1}^{i} b_{ki} a_{il} T_{nl} \end{aligned}$$

The inner sum runs from $l = 1$ to i. We note that $i \leq k$ due to the outer sum and it does not make a difference to change the upper limit of the inner sum to k. Since the same represented is the same, we must have —

$$\begin{aligned} \sum_{i=1}^{k} \sum_{l=1}^{k} b_{ki} a_{il} T_{nl} &= \sum_{l=1}^{k} \delta_{kl} T_{nl} \\ &= T_{nk} \end{aligned}$$

With the Einstein summation convention [10], we can say $b_{ki} a_{ij} = \delta_{kj}$. The square matrices represented by a square-section of the two tables (4 and 7) are inverse to each other. Hence, we have also established the inverse relationship between the Signed Stirling Numbers of the First Kind and Stirling Numbers of the Second Kind.

9 Conclusion

We have developed a novel way to evaluate a power sum of binomial co-efficients with the derivation of a new series of co-efficients and proven them equivalent to the Stirling Numbers of the First and Second Kind.

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