Existence of non-periodic real-valued and complex-valued solutions of Mathews-Lakshmanan oscillator equation

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Abstract

We show in this paper the existence of non-periodic real-valued and complex-valued solutions of the Mathews-Lakshmanan oscillator equation. The theory allows also to find the sinusoidal periodic solution given by the authors. As an oscillator can only have periodic solutions for the same model parameters, we conclude that the Mathews-Lakshmanan equation is a pseudo-oscillator.

Keywords: Mathews-Lakshmanan oscillator, differential equation, periodic solution, complex-valued solution.

Introduction

In 1974, Mathews and Lakshmanan published a paper entitled [1]: ‘‘ On a unique nonlinear oscillator’’ in which they presented a remarkable quadratic Lienard type differential equation using the Lagrangian approach. The authors [1] show that the periodic solution of such an equation has the harmonic form but with amplitude-dependent frequency. Later a vast literature has been carried out on the usefullness of this oscillator equation. So, a rich variety of studies from classical as well as quantum mechanical point of view has been developed in the literature [2- 8]. Recently the authors of the present work have shown that there exists a lot of such quadratic Lienard type equations that can have periodic solutions exhibiting harmonic oscillations with amplitude-dependent frequency [9, 10]. In [11] the authors have shown that the equation recorded as equation **6.111** in Kamke book [12] is equivalent to an oscillator equation since it may be obtained from the nonlocal transformation of the harmonic oscillator equation such that its solution is periodic and has harmonic form but with amplitude-

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dependent frequency [11, 12] when the independent variable is the time. More recently, in [13, 14] the authors presented also quadratic Lienard type equations that have periodic solutions exhibiting harmonic behavior but with amplitude-dependent frequency. In [15] the authors have shown the existence for the first time of a quadratic Lienard type equation having the Jacobi elliptic function Cn, which is a generalization of the cosine function, as general solution. The above shows that the problem of finding sinusoidal periodic solutions of nonlinear differential equations constitutes an attractive research field of mathematics and physics. In the present work one may ask whether the Mathews-Lakshmanan oscillator equation may exhibit non-periodic solutions. This is reasonable, since, as can be seen recently in the literature, several nonlinear differential equations assumed to be oscillator equations, have non-periodic solutions for the same model parameters. In this way the objective in this work is to show that the famous Mathews-Lakshmanan oscillator equation has non-periodic solutions for the same model or design parameters. To do this, we show the complex-valued solution in section (2) and exhibit real non-periodic solution in section (3). We show also in section (3) that the developed theory enables to calculate in a direct fashion the periodic solution given in [1]. Finally a conclusion is given for the work.

2. Complex- valued solution

Let us consider the second-order differential equation [16]

$$\ddot{x}+\frac{g'(x)}{g(x)}\frac{\dot{x}^2}{2}+\frac{a}{2}\frac{f'(x)}{g(x)}=0 \tag{1}$$

where the overdot means differentiation with respect to time and prime stands for the derivative with respect to the argument. The function $f(x)$, and $g(x)$ are arbitrary functions of x, and a is an arbitrary parameter. Choosing $g(x)=\frac{1}{1+\mu x^2}$, yields the first derivative $g'(x)=-\frac{2\mu x}{\left(1+\mu x^2\right)^2}$ such that $\frac{g'(x)}{g(x)}=-\frac{2\mu x}{1+\mu x^2}$. So the general equation (1) becomes

$$\ddot{x}-\frac{\mu x}{1+\mu x^2}\dot{x}^2+\frac{a}{2}\left(1+\mu x^2\right)f'(x)=0 \tag{2}$$

The choice $f(x) = -\frac{1}{\mu\left(1+\mu x^2\right)}$, that is $f'(x) = \frac{2x}{\left(1+\mu x^2\right)^2}$, reduces the equation (2) to the Mathews-Lakshmanan equation [1]

$$\ddot{x} - \frac{\mu x}{1+\mu x^2}\dot{x}^2 + \frac{a x}{1+\mu x^2} = 0 \tag{3}$$

where μ is an arbitrary parameter. The corresponding first-order differential equation [16]

$$g(x)\dot{x}^2 + a f(x) = b \tag{4}$$

may read

$$\frac{\dot{x}^2}{\left(1+\mu x^2\right)} - \frac{a}{\mu\left(1+\mu x^2\right)} = b \tag{5}$$

so that one may write

$$\dot{x}^2 = b\left(1+\mu x^2\right) + \frac{a}{\mu} \tag{6}$$

where b is an arbitrary parameter. Using the equation (6) one may obtain the quadrature defined by

$$\pm\left(t+K\right) = \int \frac{dx}{\sqrt{b\left(1+\mu x^2\right) + \frac{a}{\mu}}} \tag{7}$$

where K is an arbitrary parameter and $\mu \neq 0$. The equation (7) may be also written as

$$\pm\left(t+K\right) = \int \frac{dx}{\sqrt{\left(b+\frac{a}{\mu}\right) + b\mu x^2}} \tag{8}$$

As b is an arbitrary parameter, it is always possible to set $b + \frac{a}{\mu} = 0$, that is $b = -\frac{a}{\mu}$ where a and μ are always arbitrary parameters. Therefore the equation (8) becomes

$$\int \frac{dx}{x} = \pm\sqrt{-a}\,(t+K) \tag{9}$$

to yield

$$\ell n(x) = \pm\sqrt{-a}\,(t+K) \tag{10}$$

from which one may secure the general solution to the Mathews-Lakshmanan oscillator equation (3) in the form

$$x(t) = e^{\pm\sqrt{-a}\,(t+K)} \tag{11.a}$$

or

$$x(t) = e^{\pm i\sqrt{a}\,(t+K)} \tag{11.b}$$

when $a > 0$ and $i^2 = -1$.

Now we may show the existence of real non-periodic solution to the equation (3).

3. Periodic and non-periodic solutions

3.1 Non-periodic solution

The equation (8) may be rewritten in this form

$$\int \frac{dx}{\sqrt{1+\frac{b\mu^2}{a+b\mu}x^2}} = \pm\sqrt{\frac{b\mu+a}{\mu}}\,(t+K) \tag{12}$$

Letting $\frac{b\mu^2}{a+b\mu} > 0$, as b is an arbitrary parameter, the equation (12) may lead to

$$sh^{-1}\left[\frac{\mu\sqrt{b(a+b\mu)}}{a+b\mu}x\right] = \pm\sqrt{b\mu}\,(t+K) \tag{13}$$

from which one may secure the general solution to the Mathews-Lakshmanan oscillator equation (3) as

$$x(t) = \frac{\sqrt{b(a+b\mu)}}{\mu b}sh\left[\pm\sqrt{b\mu}\,(t+K)\right] \tag{14}$$

where $b \neq 0$. If $a > 0$, and $\mu > 0$, then it suffices to set $b > 0$, to secure $\frac{b\mu^2}{a + b\mu} > 0$.

3.2 Periodic solution

The application of $\frac{b\mu^2}{a + b\mu} < 0$, that is $b < 0$, and $a + b\mu > 0$, as $a > 0$, and $\mu > 0$ allows one, from (12), to get

$$\sin^{-1}\left[-\frac{\mu\sqrt{-b(a + b\mu)}}{a + b\mu}x\right] = \pm\sqrt{-b\mu}\,(t + K) \tag{15}$$

From (15) one may obtain the general solution of the Mathews-Laskhmanan equation (3) in the form

$$x(t) = -\frac{\sqrt{-b(a + b\mu)}}{b\mu}\sin\left[\pm\sqrt{-b\mu}\,(t + K)\right] \tag{16}$$

The amplitude is given by $A = -\frac{\sqrt{-b(a + b\mu)}}{b\mu}$, such that $A^2 = \frac{1}{\mu}\left(-1 + \frac{a}{(-b)\mu}\right)$ and the positive angular frequency $\omega = \sqrt{-b\mu}$, that is $\omega^2 = -b\mu$. Using the notation adopted in [1], ω^2 may be written as $\omega^2 = \varepsilon = -b\mu$, such that $A^2 = \frac{1}{\lambda}\left(-1 + \frac{\alpha}{\varepsilon}\right)$, for $\lambda = \mu$, and $\alpha = a$. The equation (16) shows the existence of solution with negative angular frequency $\omega = -\sqrt{-b\mu}$, which has not been highlighted previously in the literature. That being so a conclusion may be addressed for the work.

Conclusion

In this work the celebrated Mathews-Lakshmanan oscillator equation has been investigated. It has been possible to show that such an equation may exhibit real non-periodic solution and complex-valued solution. The periodic solution given in [1] has been in a direct fashion calculated in this paper. The work developed in this paper shows that the Mathews-Lakshmanan equation is, in fact, a pseudo-oscillator.

References

[1] P. M. Mathews and M. Lakshmanan, On a unique nonlinear oscillator, Quarterly of Applied Mathematics, 32 (1974) 215-218, Doi: 10.1090/qam/430422.

[2]O. Mustafa, Position-dependent mass Lagrangians: Nonlocal transformations, Euler-Lagrange invariance and exact solvability. J. Phys. A: Math. Theor., 48: 225206-225206. DOI: 10.1088/1751-8113/48/22/225206

[3] A. Bhuvaneswa, V. K. Chandrasekar, M. Senthilvelan and M. Lakshmanan, On the complete integrability of a nonlinear oscillator from group theoretical/perspective, 2012, arXiv: 1207. 4945V1.

[4] A. Schulze-Halberg and J. Wang, Two-parameter double-oscillator model of Mathews-Lakshmanan type: Series solutions and supersymmetric partners , J. Math. Phys. 56 (2015) 072106.

[5] A. Schulze-Halberg, Closed-form solutions and supersymmetric partners of the inverted Mathews-Lakshmanan oscillator. Eur. Phys. J. Plus A. (2015) 130:134, pages 1–8.

[6] J.F. Cariñena, M. F. Ranada and M. Santander, Two important examples of nonlinear oscillators, Proceedings of 10th International Conference in Modern Group Analysis, (MGA 05), Larnaca, Cyprus, (2005) pp: 39-46.

[7] B. Midya and B. Roy, A generalized quantum nonlinear oscillator, J. Phys. A: Math. Theor. 42 (2009) 285301, 1-16.

[8] M. S. Bruzon, M.L. Gandarias and M. Senthilvelan, On the nonlocal symmetries of certain nonlinear oscillators and their general solution, Physics Letters A 375 (2011) 2985-2987.

[9] M. D. Monsia, J. Akande, D. K. K. Adjaï, L. H. Koudahoun and Y. J. F. Kpomahou, A class of position-dependent mass Liénard differential equations via a general nonlocal transformation Math.Phys.,viXra.org/1608.0226v1.pdf (2017).

[10] J. Akande, D. K. K Adjaï and M. D. Monsia, Theory of exact trigonometric periodic solutions to quadratic Liénard type equations, Journal of Mathematics and Statistics 14 (1) (2018) 232-240, Doi:10.3844jmssp.2018.232.240.

[11] M. Nonti, A.V. R. Yehossou, J. Akande and M.D. Monsia, On a singular Kamke equation equivalent to linear harmonic oscillator, Math.Phys.,viXra.org/1808.0572v1.pdf (2018).

[12] E. Kamke, Differentialgleichungen lösungsmethoden und lösungen, Springer Fachmedien Weisbaden GMBH, 10th edition, 1977.

[13] M. D. Monsia, On a nonlinear differential equation of Lienard type, Math.Phys.,viXra.org/2011.0050v3.pdf (2020).

[14] M. D. Monsia, On polynomial differential equations of Duffing type, Math.Phys.,viXra.org/11787018.pdf (2020).

[15] M. Nonti, K.K. D. Adjaï, J. Akande and M.D. Monsia, On truly nonlinear oscillator equations of Ermakov-Pinney type, Math.Phys.,viXra.org/2012.v1.pdf (2020).

[16] M. D. Monsia, Analysis of a purely nonlinear generalized isotonic oscillator equation, Math.Phys.,viXra.org/2010.0195v1.pdf (2020).