Continuous Bernoulli distribution
--- simulator and test statistic

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Abstract

We discussed the simulator and test statistic of continuous Bernoulli distribution which is important to test the pervasive error of variational autoencoders in deep learning. We provided the sufficient statistic, the point estimator, the confidence interval, test statistic, goodness of fit, and one-way test for continuous Bernoulli distribution. Besides, continuous binomial distribution can be derived, so the the confidence interval and the test can be worked under two continuous Bernoulli populations. Continuous trinomial distribution can also be find.

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Chapter 1, The Continuous Bernoulli distribution

1.The probability density function of Continuous Bernoulli distribution

The Bernoulli distribution and parameter= p ,
$f_X(x;p)=p^x(1-p)^{1-x}, x=0,1, 0<p<1,$
X is discrete random variable,

Let X is continuous random variable and λ is the parameter which replaces p .

$$f_X(x;\lambda)=C(\lambda)\lambda^x(1-\lambda)^{1-x}, 0\le x\le 1, 0<\lambda<1,$$

$$f_X(x;\lambda)=C(\lambda)(1-\lambda)\int_0^1\left(\frac{\lambda}{1-\lambda}\right)^x dx---(1.1),$$

$$(i)\lambda\ne\frac{1}{2},(1.1)=C(\lambda)(1-\lambda)\frac{\left(\frac{\lambda}{1-\lambda}\right)^x}{ln\left(\frac{\lambda}{1-\lambda}\right)}\Bigg|_0^1=C(\lambda)\frac{2\lambda-1}{ln\left(\frac{\lambda}{1-\lambda}\right)}=1,$$

$$C(\lambda)=\frac{ln(1-\lambda)-ln(\lambda)}{1-2\lambda},$$

$$(ii)\lambda=\frac{1}{2},(1.1)=C(\lambda)\int_0^1\frac{1}{2}dx=2C(\lambda)=1, C(\lambda)=2,$$

Section 1, The Continuous Bernoulli distribution,
$X\sim CB(\lambda)$, this probability distribution for "machine learning".
(1)The probability density function,

$$f_X(x;\lambda)=C(\lambda)\lambda^x(1-\lambda)^{1-x}, 0\le x\le 1, 0<\lambda<1,$$

$$C(\lambda)=\begin{cases}\frac{2\,tanh^{-1}(1-2\lambda)}{1-2\lambda}, \lambda\ne\frac{1}{2}\\ 2, \lambda=\frac{1}{2}\end{cases}$$

$$tanh^{-1}(x)=\frac{1}{2}log_e\left(\frac{1+x}{1-x}\right)=\frac{1}{2}ln\left(\frac{1+x}{1-x}\right), -1<x<1,$$

(2)The distribution function,

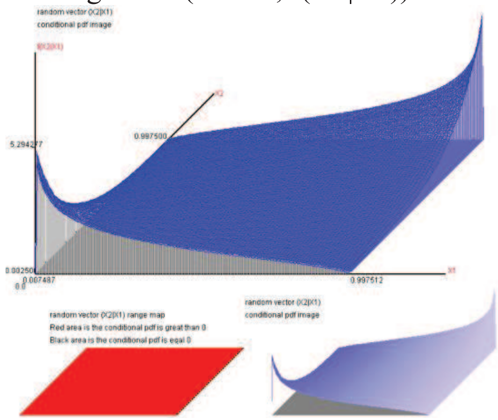
$$F_X(x;\lambda)=\begin{cases}\frac{\lambda^x(1-\lambda)^{1-x}+\lambda-1}{2\lambda-1}, \lambda\ne\frac{1}{2}\\ x, \lambda=\frac{1}{2}\end{cases}, 0<x<1$$

(3) The λ is the shape parameter,

Let X~Continous Bernoulli(λ), the λ is the shape parameter from the below diagram. The f(X|λ) is the conditional probability density in λ ,$0<\lambda<1$, but the E(X)=λ is the function of λ.

The following diagram, let X2=X, X1=λ, f(X2|X1)= f(X|λ),

the diagram is (X1=λ, f(X2|X1)).



The red area is the range of (X, λ).

Section 2, The simulator of Continuous Bernoulli distribution,

The inverse of $F_X(x;\lambda)$

$$x=\begin{cases}\dfrac{log_e\left(F_X(x;\lambda)\times(2\lambda-1)-(\lambda-1)\right)-log_e(1-\lambda)}{log_e\left(\dfrac{\lambda}{1-\lambda}\right)},\lambda\neq\dfrac{1}{2}\\ F_X(x;\lambda),\lambda=\dfrac{1}{2}\end{cases}$$

The random number= $RND=F_X(x;\lambda)\sim Uniform(0,1)$,

$$x\ \text{ simulated value }=\begin{cases}\dfrac{log_e\left(RND\times(2\lambda-1)-(\lambda-1)\right)-log_e(1-\lambda)}{log_e\left(\dfrac{\lambda}{1-\lambda}\right)},\lambda\neq\dfrac{1}{2}\\ RND,\lambda=\dfrac{1}{2}\end{cases}$$

(1)The simulated data generator,

do

{

getting RND,

converting x simulated value,

}

(2)The probability distribution simulator,

The probability distribution simulated database,

do 100,000,000 times,

{

getting RND,

converting x simulated value and saving the database,

}

This frequency distribution is likely to the probability density function, the sample mean of database is closed to the population mean and the relative error is below 1/10000.

Note: The computer program is C:\C_Bernoulli\C_Bernoulli_01.exe, which can compute the simulated data of Continuous Bernoulli distribution.

Section 3, The expectation and variance,

(1) $E(X)=C(\lambda)(1-\lambda)\int_0^1 x\left(\frac{\lambda}{1-\lambda}\right)^x dx---(1.2),$

$$(i)\lambda \neq \frac{1}{2},(1.2)=C(\lambda)(1-\lambda)\left(x\times\frac{\left(\frac{\lambda}{1-\lambda}\right)^x}{ln\left(\frac{\lambda}{1-\lambda}\right)}\Bigg|_0^1-\int_0^1\frac{\left(\frac{\lambda}{1-\lambda}\right)^x}{ln\left(\frac{\lambda}{1-\lambda}\right)}dx\right)$$

$$=C(\lambda)(1-\lambda)\left(\frac{\frac{\lambda}{1-\lambda}}{ln\left(\frac{\lambda}{1-\lambda}\right)}-\frac{\left(\frac{\lambda}{1-\lambda}\right)^x}{\left(ln\left(\frac{\lambda}{1-\lambda}\right)\right)^2}\Bigg|_0^1\right)$$

$$=C(\lambda)\left(\frac{\lambda}{ln(\lambda)-ln(1-\lambda)}+\frac{1-2\lambda}{(ln(\lambda)-ln(1-\lambda))^2}\right)$$

$(ii)\lambda=\frac{1}{2},(1.2)=\int_0^1 xdx=0.5,$

$$\mu=E(X)=\begin{cases}\frac{\lambda}{2\lambda-1}+\frac{1}{2\,tan^{-1}(1-2\lambda)} & if\ \lambda\neq\frac{1}{2}\\ \frac{1}{2} & if\ \lambda=\frac{1}{2}\end{cases}$$

(2) $E(X^2)=C(\lambda)(1-\lambda)\int_0^1 x^2\left(\frac{\lambda}{1-\lambda}\right)^x dx---(1.3),$

$$(i)\lambda \neq \frac{1}{2},(1.3)=C(\lambda)(1-\lambda)\left(x^2\times\frac{\left(\frac{\lambda}{1-\lambda}\right)^x}{ln\left(\frac{\lambda}{1-\lambda}\right)}\Bigg|_0^1-2\int_0^1\frac{x\left(\frac{\lambda}{1-\lambda}\right)^x}{ln\left(\frac{\lambda}{1-\lambda}\right)}dx\right)$$

$$=C(\lambda)\left(\frac{\lambda}{ln(\lambda)-ln(1-\lambda)}\right)-2E(X)$$

$(ii)\lambda=\frac{1}{2},(1.3)=\int_0^1 x^2dx=\frac{1}{3},$

$Var(X)=E(X^2)-E^2(X),$

$$Var(X)=\begin{cases}\frac{(1-\lambda)\lambda}{(1-2\lambda)^2}+\frac{1}{(2\,tan^{-1}(1-2\lambda))^2} & if\ \lambda\neq\frac{1}{2}\\ \frac{1}{12} & if\ \lambda=\frac{1}{2}\end{cases}$$

The estimated equation of $E(X)$, $Var(X)$,

$$\gamma_1(X)=E\left[\left(\frac{X-E(X)}{\sqrt{Var(X)}}\right)^3\right],\gamma_2(X)=E\left[\left(\frac{X-E(X)}{\sqrt{Var(X)}}\right)^4\right],$$

$\gamma_1(X)$ is skewed coefficient and $\gamma_2(X)$ is kurtosis coefficient.

Continuous Bernoulli distribution computed $E(X)$, $Var(X), \gamma_1(X)$ and $\gamma_2(X)$ is complexity, the estimated those moments using λ is easy way.
The Curvi-linear analysis(Taylor's expansion and regression combined) getting the mathematical model and computing the coefficients, the result could be accurately.

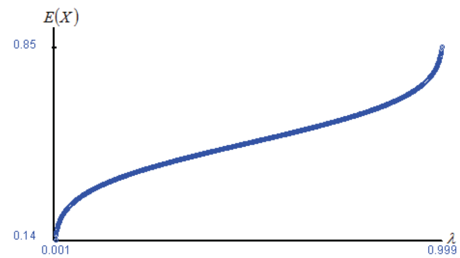
(1) $E(X)=G_1(\lambda)$, λ estimated $E(X)$,
The $E(X)$ estimated equation is $G_1(\lambda)$,
The $0.001 \le \lambda \le 0.999$, $0.143853919 \le \mu \le 0.856221427$,
The amount of paired data of (λ, $E(X)$) is 999, λ is setting value and $E(X)$ is computed by the simulator which has 100,000,000 data.

X= $0.279390+0.441311\times\lambda$,
The estimated equation------
$G_1(\lambda)$ =0.50005887293491469+
0.77359483065083623*(X-0.50004573071171143)^1+
-0.0151521129300817850000000000000*(X-0.50004573071171143)^2+
-27.27900934219360400*(X-0.50004573071171143)^3+
10.36370790004730200*(X-0.50004573071171143)^4+
15822.38842773437500000*(X-0.50004573071171143)^5+
-2817.42468261718750000*(X-0.50004573071171143)^6+
-3612752.6875*(X-0.50004573071171143)^7+
391281.72265625000000000*(X-0.50004573071171143)^8+
452401608.0000*(X-0.50004573071171143)^9+
-31440996.2500*(X-0.50004573071171143)^10+
-33874673664.0000*(X-0.50004573071171143)^11+
1540792624.0000*(X-0.50004573071171143)^12+
1582581137408.0000*(X-0.50004573071171143)^13+
-46642316288.0000*(X-0.50004573071171143)^14+
-46495537037312.0000*(X-0.50004573071171143)^15+
850124546048.0000*(X-0.50004573071171143)^16+
834533872107520.0000*(X-0.50004573071171143)^17+
-8542741594112.0000*(X-0.50004573071171143)^18+
-8357328558489600.0000*(X-0.50004573071171143)^19+
36339642531840.0000*(X-0.50004573071171143)^20+
35775834451083264.0000*(X-0.50004573071171143)^21
ANOVA

Source	df	SS	MS
Regression	21	16.7176990804	0.7960809086
Error	977	0.0001969542	0.0000002016
Total	998	16.7178960346	

H0:slope1=....=slope21=0, test statistic=3948994.157065,
sample size=999, R2=0.999988, R2(adj)=0.999988,MSE=0.000000,

(λ , $E(X)$) scatter diagram



(λ ,R= $G_1(\lambda)$,B= $E(X)$) scatter diagram

(2) $Var(X)=G_2(\lambda)$, λ estimated $Var(X)$,
The $Var(X)$ estimated equation is $G_2(\lambda)$,
The $0.001 \le \lambda \le 0.999$, $0.019960243 \le Var(X) \le 0.083352472$,
The amount of paired data of (λ , $Var(X)$) is 999, λ is setting value and $Var(X)$ is computed by the simulator which has 100,000,000 data.

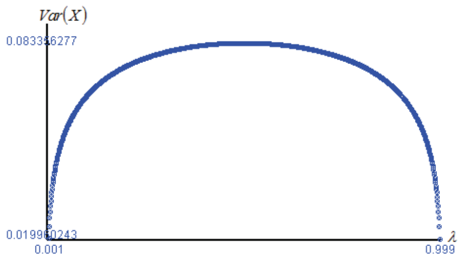
X=K(X1)=0.073806+-0.000019 $\times \lambda$,
The estimated equation ------
$G_2(\lambda)$=0.083298356117438743+
0.951844304800033570*(X-0.073795922003002973)^1+
-54413612.0*(X-0.073795922003002973)^2+
-200067416064.0*(X-0.073795922003002973)^3+
-508321342168110200000.0*(X-0.073795922003002973)^4+
7233666915898715700000000.0*(X-0.073795922003002973)^5+
775849316051104270000.0*(X-0.073795922003002973)^6+
-824069505565571400000000.0*(X-0.073795922003002973)^7+
-609322451431830740.0*(X-0.073795922003002973)^8+
4430717074039255700000.0*(X-0.073795922003002973)^9+
27276456959807344.0*(X-0.073795922003002973)^10+
-1314633807785993900000.0*(X-0.073795922003002973)^11+
-73922949398858467000000000000.0*(X-0.073795922003002973)^12+
228088785609802220.0*(X-0.073795922003002973)^13+
1233940925252432400000000000.0*(X-0.073795922003002973)^14+
-230539976819978550000000000000.0*(X-0.073795922003002973)^15+
-123962875241096120000000.0*(X-0.073795922003002973)^16+
1257626562718381800000000000.0*(X-0.073795922003002973)^17+
68709733665466692000000.0*(X-0.073795922003002973)^18+
-28621190224551843000000.0*(X-0.073795922003002973)^19+
-1614141452456421600.0*(X-0.073795922003002973)^20

ANOVA

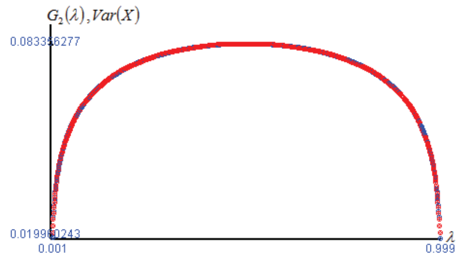
Source	df	SS	MS
Regression	20	0.1398193120	0.0069909656
Error	978	0.0000154000	0.0000000157
Total	998	0.1398347119	

H0:slope1=....=slope20=0, test statistic=443972.489429,
sample size=999, R2=0.999890, R2(adj)=0.999888,MSE=0.000000,

(λ , $Var(X)$) scatter diagram



(λ ,R= $G_2(\lambda)$,B= $Var(X)$) scatter diagram



(3) $\gamma_1(X)=G_3(\lambda)$, λ estimated $\gamma_1(X)$,

The $\gamma_1(X)$ estimated equation is $G_3(\lambda)$,

The $0.001 \le \lambda \le 0.999$, $-1.7961485553 \le \gamma_1(X) \le 1.795827056$,

The amount of paired data of (λ , $\gamma_1(X)$) is 999, λ is setting value and $\gamma_1(X)$ is computed by the simulator which has 100,000,000 data.

X=0.984739+-1.969753 $\times\lambda$,

The estimated equation ------

$G_3(\lambda)$=0.0001523718161990 9279+

0.7228857256474157 1000*(X--0.00013754206206167914)^1+
-0.0777136782344314 2700*(X--0.00013754206206167914)^2+
-1.4855569863102573 0000*(X--0.00013754206206167914)^3+
3.2366832731058821 0000*(X--0.00013754206206167914)^4+
44.196912858053111 00000*(X--0.00013754206206167914)^5+
-52.742141397669911 00000*(X--0.00013754206206167914)^6+
-514.35292186448351 000000*(X--0.00013754206206167914)^7+
441.66157603263855 000000*(X--0.00013754206206167914)^8+
3275.4831703230739 0000000*(X--0.00013754206206167914)^9+
-2160.6237526535988 0000000*(X--0.00013754206206167914)^10+
-12449.110818378627 00000000*(X--0.00013754206206167914)^11+
6596.0176293849945 0000000*(X--0.00013754206206167914)^12+
29480.764031872153 00000000*(X--0.00013754206206167914)^13+
-12939.831108570099 00000000*(X--0.00013754206206167914)^14+
-43855.796311795712 00000000*(X--0.00013754206206167914)^15+
16311.627405643463 00000000*(X--0.00013754206206167914)^16+
39823.573151856661 00000000*(X--0.00013754206206167914)^17+
-12768.250188350677 00000000*(X--0.00013754206206167914)^18+
-20163.347440525889 00000000*(X--0.00013754206206167914)^19+
5647.2611746788025 0000000*(X--0.00013754206206167914)^20+
4361.8745349179953 0000000*(X--0.00013754206206167914)^21+
-1078.2932203412056 0000000*(X--0.00013754206206167914)^22

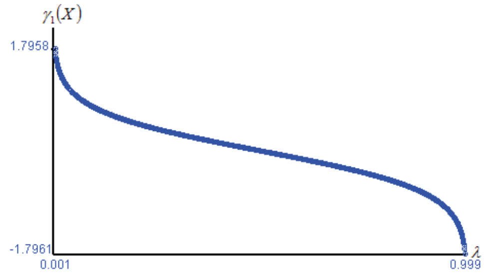
ANOVA

Source	df	SS	MS
Regression	22	340.2086189293	15.4640281332
Error	976	0.0059924144	0.0000061398
Total	998	340.2146113437	

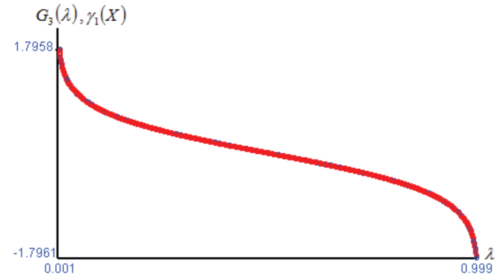
H0:slope1=....=slope22=0, test statistic=2518666.166276,

sample size=999, R2=0.999982, R2(adj)=0.999982,MSE=0.000006,

($\lambda, \gamma_1(X)$) scatter diagram



(λ,R=$G_3(\lambda)$,B=$\gamma_1(X)$) scatter diagram



(4) $\gamma_2(X)=G_4(\lambda)$, λ estimated $\gamma_2(X)$,
The $\gamma_2(X)$ estimated equation is $G_4(\lambda)$,
The $0.001 \le \lambda \le 0.999$, $1.799857270 \le \gamma_2(X) \le 7.0808074006$,
The amount of paired data of ($\lambda, \gamma_2(X)$) is 999, λ is setting value and $\gamma_2(X)$ is computed by the simulator which has 100,000,000 data.

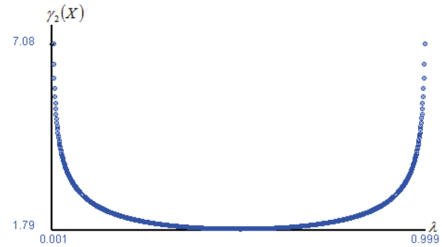
X=2.292589+0.000951$\times\lambda$,
The estimated equation ------
$G_4(\lambda)$=1.8082038890859193+
9.0944448420777917*(X-2.293064877314313400)^1+
-5649327.2372012138000000*(X-2.293064877314313400)^2+
-2840484322.50*(X-2.293064877314313400)^3+
1454772784505248.00*(X-2.293064877314313400)^4+
282173067709382660.00*(X-2.293064877314313400)^5+
-93623181371148578000000.00*(X-2.293064877314313400)^6+
-12843445897786422000000000.00*(X-2.293064877314313400)^7+
30545377164991993.00*(X-2.293064877314313400)^8+
3212971560766148400.00*(X-2.293064877314313400)^9+
-5682164267847958100000000.00*(X-2.293064877314313400)^10+
-482956905873362840000000000.00*(X-2.293064877314313400)^11+
6396856260882416 6.00*(X-2.293064877314313400)^12+
4544885501268294000.00*(X-2.293064877314313400)^13+
-443419149014227060000000.00*(X-2.293064877314313400)^14+
-26959294213922125000000000.00*(X-2.293064877314313400)^15+
18493181124335300.00*(X-2.293064877314313400)^16+
978467103510877170.00*(X-2.293064877314313400)^17+
-425413014879464930000000.00*(X-2.293064877314313400)^18+
-19833682514142766000000000.00*(X-2.293064877314313400)^19+
4146315834826265700000000000000.00*(X-2.293064877314313400)^20+
17195292699711689.00*(X-2.293064877314313400)^21

ANOVA

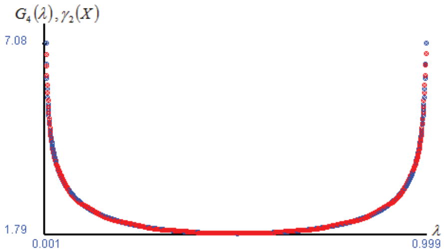
Source	df	SS	MS
Regression	21	553.4887357077	26.3566064623
Error	977	0.4692730413	0.0004803204
Total	998	553.9580087490	

H0:slope1=....=slope21=0, test statistic=54872.967861,
sample size=999, R2=0.999153, R2(adj)=0.999135,MSE=0.000480,

(λ, $\gamma_2(X)$) scatter diagram

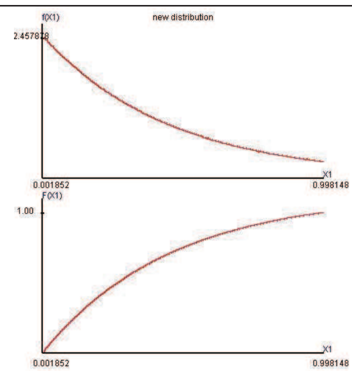


(λ, R=$G_4(\lambda)$, B=$\gamma_2(X)$) scatter diagram

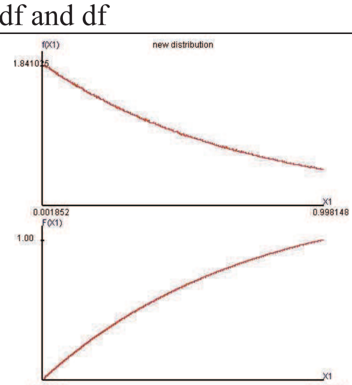


Note: The computer program is C:\C_Bernoulli\C_Bernoulli_02.exe, which can compute the $E(X)$, $Var(X), \gamma_1(X), \gamma_2(X)$ and frequency table when Continuous Bernoulli distribution(λ). The simulated data amount=100,000,000, the sample mean, sample variance, sample skewed coefficient and sample kurtosis coefficient is closed to $E(X)$, $Var(X), \gamma_1(X), \gamma_2(X)$ and the frequency distribution is similar to Continuous Bernoulli distribution (λ).

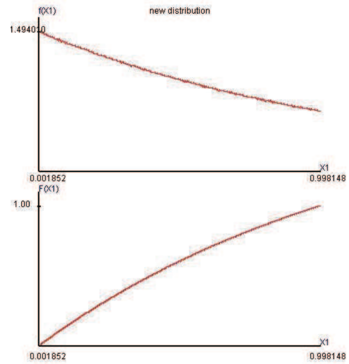
example 3-1, λ=0.1,

X1 pdf and df	Coefficient
	Mathematical Mean: 0.33015 Geometrical Mean : 0.20663 Harmonic Mean : 0.01882 Variance : 0.06652 S.D. : 0.25791 Skewed Coef. : 0.74382 Kurtosis Coef. : 2.58122 MAD : 0.21455 Range : 1.00000 Mid_range : 0.50000 Median : 0.26754 Q1 : 0.11441 Q2 : 0.26754 Q3 : 0.50003 IQR : 0.38562 C.V. : 0.78118

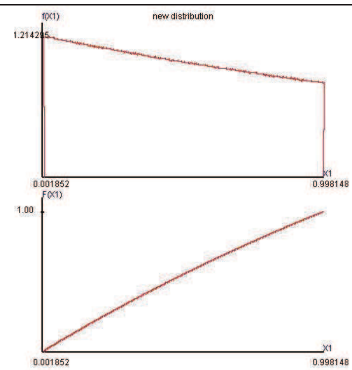
example 3-2, λ=0.2,

X1 pdf and df	Coefficient
	Mathematical Mean: 0.38814 Geometrical Mean : 0.25589 Harmonic Mean : 0.03197 Variance : 0.07595 S.D. : 0.27558 Skewed Coef. : 0.47578 Kurtosis Coef. : 2.11516 MAD : 0.23452 Range : 1.00000 Mid_range : 0.50000 Median : 0.33913 Q1 : 0.14981 Q2 : 0.33913 Q3 : 0.59652 IQR : 0.44671 C.V. : 0.71000

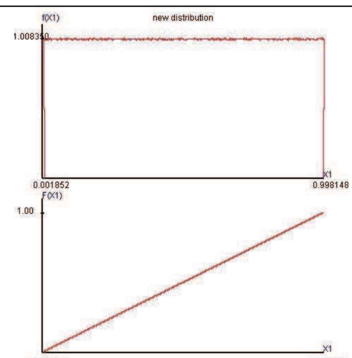
example 3-3, λ=0.3,

X1 pdf and df	Coefficient		
	Mathematical Mean:		0.43033
	Geometrical Mean	:	0.29538
	Harmonic Mean	:	0.03728
	Variance	:	0.08046
	S.D.	:	0.28365
	Skewed Coef.	:	0.29223
	Kurtosis Coef.	:	1.91812
	MAD	:	0.24399
	Range	:	1.00000
	Mid_range	:	0.50000
	Median	:	0.39722
	Q1	:	0.18196
	Q2	:	0.39722
	Q3	:	0.66073
	IQR	:	0.47877
	C.V.	:	0.65914

example 3-4, λ=0.4,

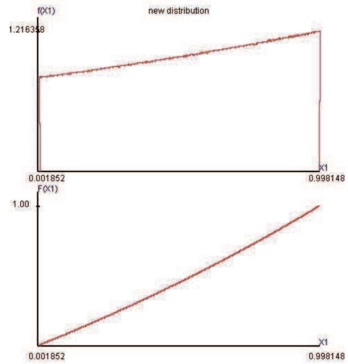
X1 pdf and df	Coefficient		
	Mathematical Mean:		0.46633
	Geometrical Mean	:	0.33176
	Harmonic Mean	:	0.03856
	Variance	:	0.08266
	S.D.	:	0.28751
	Skewed Coef.	:	0.14031
	Kurtosis Coef.	:	1.82714
	MAD	:	0.24860
	Range	:	1.00000
	Mid_range	:	0.50000
	Median	:	0.44968
	Q1	:	0.21460
	Q2	:	0.44968
	Q3	:	0.70952
	IQR	:	0.49492
	C.V.	:	0.61654

example 3-5, λ=0.5,此為 Uniform(0,1)。

X1 pdf and df	Coefficient		
	Mathematical Mean:		0.50002
	Geometrical Mean	:	0.36791
	Harmonic Mean	:	0.04653
	Variance	:	0.08334
	S.D.	:	0.28869
	Skewed Coef.	:	-0.00004
	Kurtosis Coef.	:	1.79990
	MAD	:	0.25002
	Range	:	1.00000
	Mid_range	:	0.50000
	Median	:	0.50002
	Q1	:	0.25001
	Q2	:	0.50002
	Q3	:	0.75001
	IQR	:	0.50000
	C.V.	:	0.57735

example 3-6, λ =0.6,

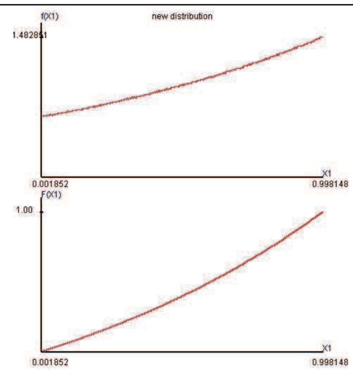
X1 pdf and df



Coefficient		
Mathematical Mean	:	0.53377
Geometrical Mean	:	0.40612
Harmonic Mean	:	0.06289
Variance	:	0.08267
S.D.	:	0.28752
Skewed Coef.	:	-0.14060
Kurtosis Coef.	:	1.82720
MAD	:	0.24861
Range	:	1.00000
Mid_range	:	0.50000
Median	:	0.55043
Q1	:	0.29050
Q2	:	0.55043
Q3	:	0.78554
IQR	:	0.49504
C.V.	:	0.53867

example 3-7, λ =0.7,

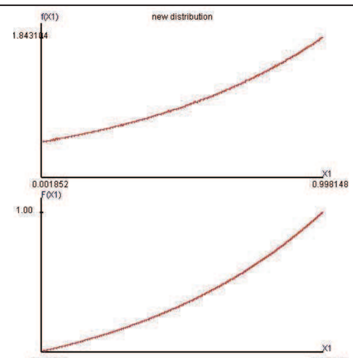
X1 pdf and df



Coefficient		
Mathematical Mean	:	0.56986
Geometrical Mean	:	0.44932
Harmonic Mean	:	0.08201
Variance	:	0.08044
S.D.	:	0.28362
Skewed Coef.	:	-0.29288
Kurtosis Coef.	:	1.91890
MAD	:	0.24395
Range	:	1.00000
Mid_range	:	0.50000
Median	:	0.60297
Q1	:	0.33959
Q2	:	0.60297
Q3	:	0.81822
IQR	:	0.47863
C.V.	:	0.49770

example 3-8, λ =0.8,

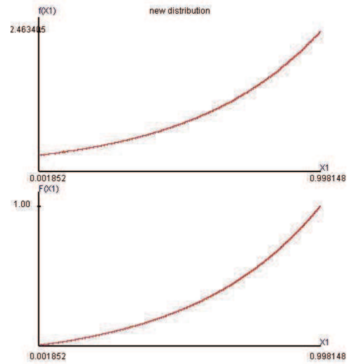
X1 pdf and df



Coefficient		
Mathematical Mean	:	0.61200
Geometrical Mean	:	0.50263
Harmonic Mean	:	0.09574
Variance	:	0.07590
S.D.	:	0.27551
Skewed Coef.	:	-0.47608
Kurtosis Coef.	:	2.11563
MAD	:	0.23446
Range	:	1.00000
Mid_range	:	0.50000
Median	:	0.66100
Q1	:	0.40365
Q2	:	0.66100
Q3	:	0.85024
IQR	:	0.44659
C.V.	:	0.45018

example 3-9, λ =0.9,

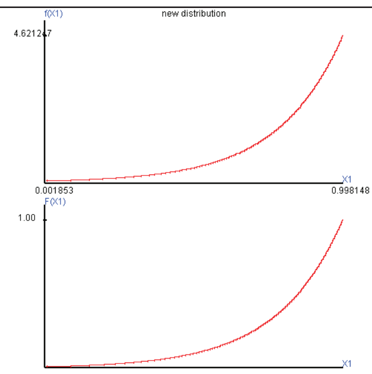
X1 pdf and df



Coefficient		
Mathematical Mean:		0.66987
Geometrical Mean :		0.58009
Harmonic Mean	:	0.14364
Variance	:	0.06651
S.D.	:	0.25790
Skewed Coef.	:	-0.74372
Kurtosis Coef.	:	2.58089
MAD	:	0.21455
Range	:	1.00000
Mid_range	:	0.50000
Median	:	0.73250
Q1	:	0.49996
Q2	:	0.73250
Q3	:	0.88561
IQR	:	0.38565
C.V.	:	0.38499

example 3-10, λ =0.99,

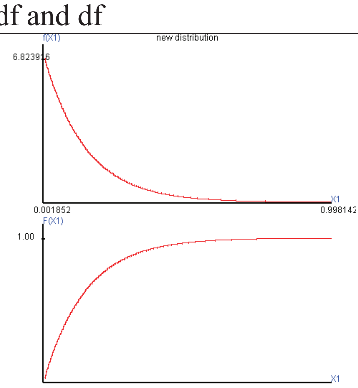
X1 pdf and df



Coefficient		
Mathematical Mean:		0.79258
Geometrical Mean :		0.75294
Harmonic Mean	:	0.51282
Variance	:	0.03707
S.D.	:	0.19253
Skewed Coef.	:	-1.41514
Kurtosis Coef.	:	4.82773
MAD	:	0.14894
Range	:	1.00000
Mid_range	:	0.50000
Median	:	0.85137
Q1	:	0.70480
Q2	:	0.85137
Q3	:	0.93816
IQR	:	0.23336
C.V.	:	0.24292

example 3-11, λ =0.001,

X1 pdf and df



Coefficient		
Mathematical Mean:		0.14384
Geometrical Mean :		0.08110
Harmonic Mean	:	0.00953
Variance	:	0.01999
S.D.	:	0.14138
Skewed Coef.	:	1.79668
Kurtosis Coef.	:	7.08231
MAD	:	0.10543
Range	:	0.99999
Mid_range	:	0.50000
Median	:	0.10020
Q1	:	0.04161
Q2	:	0.10020
Q3	:	0.20031
IQR	:	0.15870
C.V.	:	0.98292

Chapter 2, The sufficient statistic of Continuous Bernoulli distribution

The sufficient statistic of parameter is bassis on the parameter point estimator and the test statistic and confidence interval statistic.

$X_1, X_2, ..., X_n \overset{iid}{\sim} CB(\lambda)$, there are n independent random variables and same Continuous Bernoulli distribution (λ).

Section 1, The sufficient statistic of λ ,

(1)The likelihood function of λ ,

$X_1, X_2, ..., X_n \overset{iid}{\sim} CB(\lambda)$,

$$f_{X_1, X_2, ..., X_n}(x_1, x_2, ..., x_n; \lambda) = \prod_{i=1}^{n} f_{X_i}(x_i; \lambda) = (C(\lambda))^n \lambda^{\sum_{i=1}^{n} x_i} (1-\lambda)^{n-\sum_{i=1}^{n} x_i},$$

(2) The sufficient statistic of λ ,

$$f_{X_1, X_2, ..., X_n}(x_1, x_2, ..., x_n; \lambda) = ((1-\lambda)C(\lambda))^n \left(\frac{\lambda}{1-\lambda}\right)^{\sum_{i=1}^{n} x_i},$$

Let $T = \sum_{i=1}^{n} X_i$, $0 < x_n = t - \sum_{i=1}^{n-1} x_i < 1,$ $\sum_{i=1}^{n-1} x_i < t$ and $min(0, t-1) < \sum_{i=1}^{n-1} x_i$,

$$f_T(t; \lambda) = \int_0^1 \int_0^1 .... \int_0^1 (C(\lambda))^n \lambda^t (1-\lambda)^{n-t} dx_1 dx_2 ... dx_{n-1},$$

$$f_{X_1, X_2, ..., X_n | T=t}(x_1, x_2, ..., x_n | T = t) = \frac{((1-\lambda)C(\lambda))^n \left(\frac{\lambda}{1-\lambda}\right)^{\sum_{i=1}^{n} x_i}}{\int \int .... \int (C(\lambda))^n \lambda^t (1-\lambda)^{n-t} dx_1 dx_2 ... dx_{n-1}}$$

$= \frac{1}{\int \int .... \int 1 dx_1 dx_2 ... dx_{n-1}}$ is independent with λ ,

$\sum_{i=1}^{n} X_i$ is the sufficient statistic of λ ,(Fisher-Neymana factorization theorem).

Section 2, The sampling distribution of $\sum_{i=1}^{n} X_i$ is Continuous Binomial(n, λ),

$X_1, X_2, ..., X_n \overset{iid}{\sim}$ Continuous Bernoulli(λ),

1. The $X = X_1 + X_2 + ... + X_n$ pdf,

(1) n=2,

The probability density function,

$f_{X_1}(x_1; \lambda, n) = C(\lambda)\lambda^{x_1}(1-\lambda)^{1-x_1}, 0 \le x_1 \le 1, 0 < \lambda < 1,$

$f_{X_2}(x_2; \lambda, n) = C(\lambda)\lambda^{x_2}(1-\lambda)^{1-x_2}, 0 \le x_2 \le 1, 0 < \lambda < 1,$

$X_{,1}, X_2$ are independent random variables,

$f_{X_1,X_2}(x_1, x_2; \lambda, n) = f_{X_1}(x_1; \lambda, n) f_{X_2}(x_2; \lambda, n)$

$= (C(\lambda))^2 \lambda^{x_1+x_2}(1-\lambda)^{2-x_1-x_2}, 0 \le x_1 \le, 0 \le x_2 \le 1,$

$f_{X_1,X}(x_1, x; \lambda, n) = f_{X_1,X_2}(x_1, x_2 = x - x_1; \lambda, n),$

$= (C(\lambda))^2 \lambda^{x}(1-\lambda)^{2-x} \times \frac{\partial(x_1, x_2)}{\partial(x_1, x)}, \frac{\partial(x_1, x_2)}{\partial(x_1, x)} = 1,$

$X = X_{,1} + X_2, 0 < x_2 = x - x_1 < 1,$

$max(0, x-1) < x_1 < min(1, x), 0 \le x \le 2,$

$$\begin{cases} 0 < x_1 < x & if\ 0 \le x \le 1, \\ x-1 < x_1 < 1 & if\ 1 \le x \le 2, \end{cases}$$

$$f_X(x; \lambda, n) = \int_{max(0,x-1)}^{min(1,x)} (C(\lambda))^2 \lambda^x (1-\lambda)^{2-x} dx_1$$

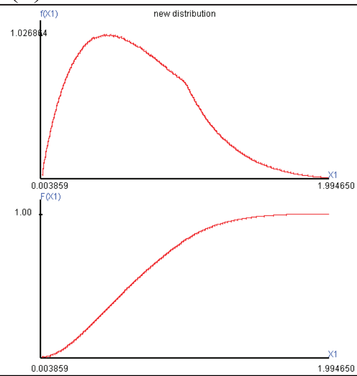
$$\begin{cases} f_X(x; \lambda, n) = (C(\lambda))^2 \lambda^x (1-\lambda)^{2-x} \int_0^x 1 dx_1 & if\ 0 \le x \le 1, \\ f_X(x; \lambda, n) = (C(\lambda))^2 \lambda^x (1-\lambda)^{2-x} \int_{x-1}^1 1 dx_1 & if\ 1 \le x \le 2, \end{cases}$$

$$f_X(x; \lambda, n) = \begin{cases} (C(\lambda))^2 x \lambda^x (1-\lambda)^{2-x} & if\ 0 \le x \le 1 \\ (C(\lambda))^2 (2-x) \lambda^x (1-\lambda)^{2-x} & if\ 1 \le x < 2 \end{cases}$$

for example, $\lambda = \frac{1}{2}$,

$$f_X(x; \lambda, n) = \begin{cases} x & if\ 0 \le x \le 1 \\ 2-x & if\ 1 \le x \le 2 \end{cases}$$

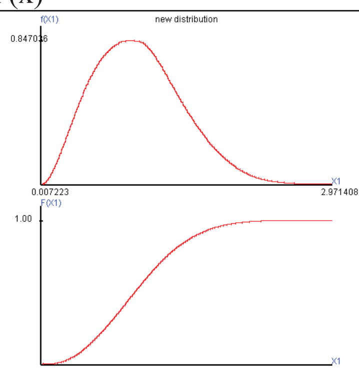
(1) λ =0.1,n=2, $X = X_1 + X_2 + ... + X_n$,

f(x), F(x)	Coefficient
	Mathematical Mean: 0.66038 Geometrical Mean : 0.54178 Harmonic Mean : 0.38075 Variance : 0.13309 S.D. : 0.36481 Skewed Coef. : 0.52557 Kurtosis Coef. : 2.78969 MAD : 0.29911 Range : 1.99819 Mid_range : 0.99925 Median : 0.62012 Q1 : 0.37209 Q2 : 0.62012 Q3 : 0.90821 IQR : 0.53612 C.V. : 0.55243

(2)n=3,

$$f_X(x;\lambda,n) = \begin{cases} (C(\lambda))^3 \dfrac{x^2}{2} \lambda^x (1-\lambda)^{3-x} & if\ 0 \le x \le 1 \\ (C(\lambda))^3 \dfrac{(-2x^2+6x-3)}{2} (2-x)\lambda^x (1-\lambda)^{3-x} & if\ 1 \le x < 2 \\ (C(\lambda))^3 \dfrac{(2-x)^2}{2} \lambda^x (1-\lambda)^{3-x} & if\ 2 \le x \le 3 \end{cases}$$

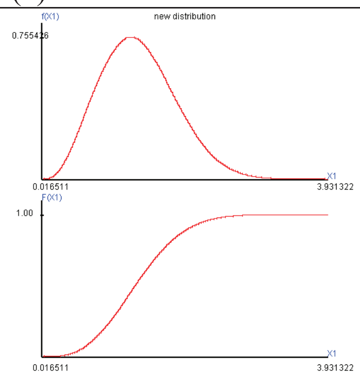
λ =0.1,n=3, $X = X_1 + X_2 + ... + X_n$

f(x), F(x)	Coefficient
	Mathematical Mean: 0.99053 Geometrical Mean : 0.87677 Harmonic Mean : 0.73594 Variance : 0.19966 S.D. : 0.44683 Skewed Coef. : 0.42949 Kurtosis Coef. : 2.86040 MAD : 0.36187 Range : 2.97520 Mid_range : 1.48932 Median : 0.95720 Q1 : 0.65421 Q2 : 0.95720 Q3 : 1.28357 IQR : 0.62936 C.V. : 0.45110

(3)n=4,

$$f_X(x;\lambda,n)=\begin{cases} (C(\lambda))^4\dfrac{x^3}{6}\lambda^x(1-\lambda)^{4-x} & if\ 0\le x\le 1 \\ (C(\lambda))^4\dfrac{(-3x^3+12x^2-12x+4)}{6}(2-x)\lambda^x(1-\lambda)^{4-x} & if\ 1\le x<2 \\ (C(\lambda))^4\dfrac{(3x^3-24x^2+60x-44)}{6}\lambda^x(1-\lambda)^{4-x} & if\ 2\le x\le 3 \\ (C(\lambda))^4\dfrac{(4-x)^3}{6}\lambda^x(1-\lambda)^{4-x} & if\ 3\le x\le 4 \end{cases}$$

f(x), F(x)

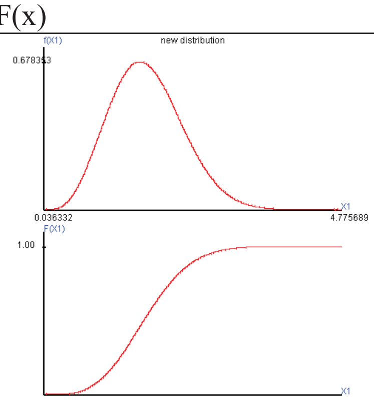


Coefficient	
Mathematical Mean:	1.32053
Geometrical Mean :	1.20985
Harmonic Mean :	1.08000
Variance :	0.26608
S.D. :	0.51583
Skewed Coef. :	0.37208
Kurtosis Coef. :	2.89474
MAD :	0.41595
Range :	3.92936
Mid_range :	1.97392
Median :	1.28631
Q1 :	0.94296
Q2 :	1.28631
Q3 :	1.65965
IQR :	0.71668
C.V. :	0.39062

(4)n=5,

$$f_X(x;\lambda,n)=\begin{cases} (C(\lambda))^5\dfrac{x^4}{24}\lambda^x(1-\lambda)^{5-x} & if\ 0\le x\le 1 \\ (C(\lambda))^5\dfrac{(-4x^4+20x^3-30x^2+20x-5)}{24}(2-x)\lambda^x(1-\lambda)^{5-x} & if\ 1\le x<2 \\ (C(\lambda))^5\dfrac{(6x^4-60x^3+210x^2-330x+155)}{24}\lambda^x(1-\lambda)^{5-x} & if\ 2\le x\le 3 \\ (C(\lambda))^5\dfrac{(-4x^4+60x^3-330x^2+780x-655)}{24}\lambda^x(1-\lambda)^{5-x} & if\ 3\le x\le 4 \\ (C(\lambda))^5\dfrac{(5-x)^4}{24}\lambda^x(1-\lambda)^{5-x} & if\ 4\le x\le 5 \end{cases}$$

f(x), F(x)



Coefficient	
Mathematical Mean:	1.65072
Geometrical Mean :	1.54198
Harmonic Mean :	1.41864
Variance :	0.33267
S.D. :	0.57677
Skewed Coef. :	0.33307
Kurtosis Coef. :	2.91623
MAD :	0.46410
Range :	4.75698
Mid_range :	2.40601
Median :	1.61668
Q1 :	1.23424
Q2 :	1.61668
Q3 :	2.03011
IQR :	0.79587
C.V. :	0.34941

$X \sim$ Continuous Binomial distribution(λ),

$X_1, X_2, ..., X_n \overset{iid}{\sim}$ Uniform($\alpha = 0, \beta = 1$),

$X = X_1 + X_2 + ... + X_n, h(x)$ is irwin-hall distribution and parameter n.
The pdf of Continuous Binomial distribution(λ) is

$$f_X(x;\lambda,n) = h(x)(C(\lambda))^n \lambda^x (1-\lambda)^{n-x}, 0 \le x \le n, 0 < \lambda < 1.$$

and $X = \sum_{i=1}^{n} X_i \xrightarrow[n\to\infty]{} Normal\left(E\left(X = \sum_{i=1}^{n} X_i\right), Var\left(X = \sum_{i=1}^{n} X_i\right)\right)$.

Section 3, The simulator of $\sum_{i=1}^{n} X_i$,

The Continuous Bernoulli simulated data $x(RND,\lambda)$ when random number= RND and parameter is λ ,

$$x(RND,\lambda)=\begin{cases}\dfrac{log_e\left(RND\times(2\lambda-1)-(\lambda-1)\right)-log_e(1-\lambda)}{log_e\left(\dfrac{\lambda}{1-\lambda}\right)}, \lambda\neq\dfrac{1}{2}\\ RND, \lambda=\dfrac{1}{2}\end{cases},$$

(1)The simulation process,

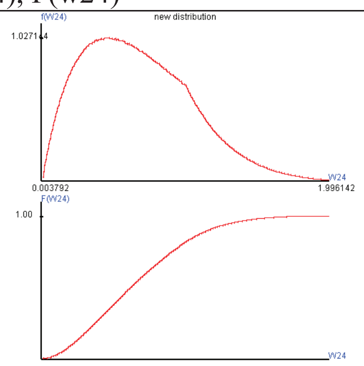
(i) Getting random number, $RND_1, RND_2, ..., RND_n$ are independently,

(ii) $x_1(RND_1,\lambda), x_2(RND_2,\lambda), ..., x_n(RND_n,\lambda)$

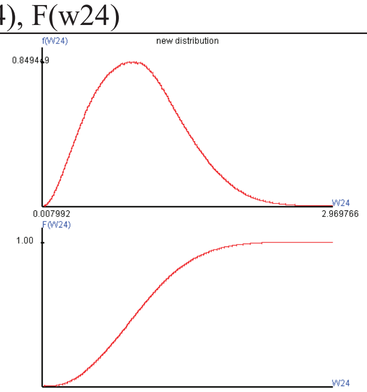
(iii) $x_j=\sum_{i=1}^{n} x_i(RND_i,\lambda)$, j=1,2,…..,100000000,

Repeat (i)~(iii) 100000000 times, the database of simulated data will be gotten.
This database can convert frequency distribution and $E(X)$, $Var(X), \gamma_1(X), \gamma_2(X)$,
This database is approached to Continuous Binomial distribution(λ).

(1)n=2, λ =0.1,W24= $X_1+X_2+....+X_n$,

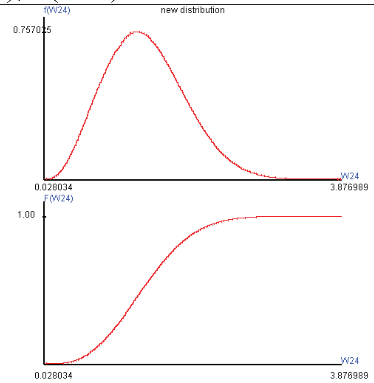
f(w24), F(w24)	Coefficient	
	Mathematical Mean:	0.66035
	Geometrical Mean :	0.54183
	Harmonic Mean :	0.38106
	Variance :	0.13306
	S.D. :	0.36478
	Skewed Coef. :	0.52586
	Kurtosis Coef. :	2.79028
	MAD :	0.29908
	Range :	1.99976
	Mid_range :	0.99997
	Median :	0.62011
	Q1 :	0.37209
	Q2 :	0.62011
	Q3 :	0.90807
	IQR :	0.53598
	C.V. :	0.55240

(2)n=3, λ =0.1,W24= $X_1+X_2+....+X_n$,

f(w24), F(w24)	Coefficient	
	Mathematical Mean:	0.99087
	Geometrical Mean :	0.87710
	Harmonic Mean :	0.73638
	Variance :	0.19978
	S.D. :	0.44696
	Skewed Coef. :	0.42939
	Kurtosis Coef. :	2.85937
	MAD :	0.36198
	Range :	2.97278
	Mid_range :	1.48888
	Median :	0.95754
	Q1 :	0.65442
	Q2 :	0.95754
	Q3 :	1.28397
	IQR :	0.62955
	C.V. :	0.45108

(3)n=4, λ =0.1,W24= $X_1 + X_2 + ... + X_n$,

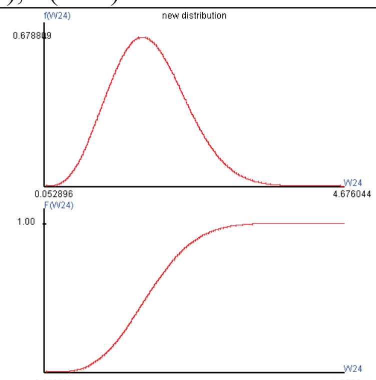
f(w24), F(w24)



Coefficient		
Mathematical Mean	:	1.32058
Geometrical Mean	:	1.20983
Harmonic Mean	:	1.07988
Variance	:	0.26622
S.D.	:	0.51597
Skewed Coef.	:	0.37193
Kurtosis Coef.	:	2.89496
MAD	:	0.41608
Range	:	3.86326
Mid_range	:	1.95251
Median	:	1.28629
Q1	:	0.94276
Q2	:	1.28629
Q3	:	1.65998
IQR	:	0.71722
C.V.	:	0.39071

(4)n=5, λ =0.1,W24= $X_1 + X_2 + ... + X_n$,

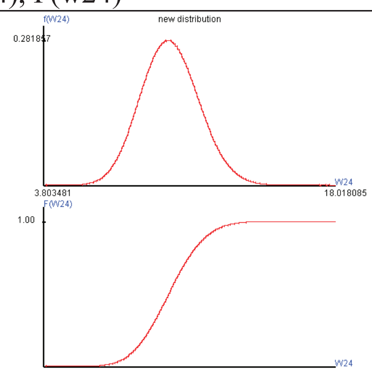
f(w24), F(w24)



Coefficient		
Mathematical Mean	:	1.65079
Geometrical Mean	:	1.54207
Harmonic Mean	:	1.41874
Variance	:	0.33261
S.D.	:	0.57672
Skewed Coef.	:	0.33301
Kurtosis Coef.	:	2.91701
MAD	:	0.46402
Range	:	4.64033
Mid_range	:	2.36447
Median	:	1.61687
Q1	:	1.23445
Q2	:	1.61687
Q3	:	2.03022
IQR	:	0.79577
C.V.	:	0.34936

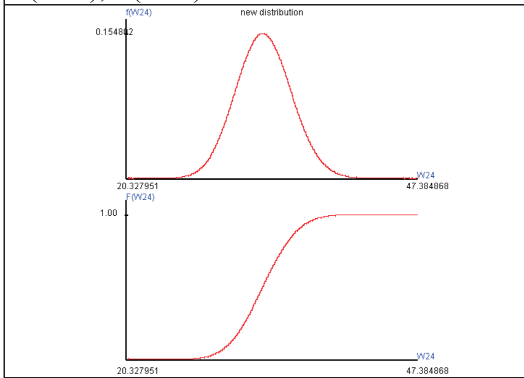
(5)n=30, λ =0.1,W24= $X_1 + X_2 + ... + X_n$,

f(w24), F(w24)

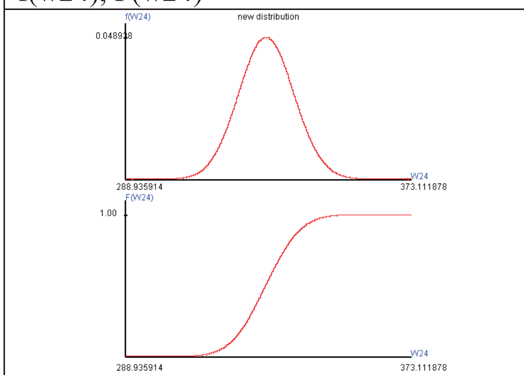


Coefficient		
Mathematical Mean	:	9.90661
Geometrical Mean	:	9.80458
Harmonic Mean	:	9.70068
Variance	:	1.99611
S.D.	:	1.41284
Skewed Coef.	:	0.13588
Kurtosis Coef.	:	2.98513
MAD	:	1.12887
Range	:	14.26745
Mid_range	:	10.91078
Median	:	9.87436
Q1	:	8.93257
Q2	:	9.87436
Q3	:	10.84534
IQR	:	1.91278
C.V.	:	0.14262

(6)n=100, λ =0.1,W24= $X_1 + X_2 + ... + X_n$,

f(w24), F(w24)	Coefficient		
	Mathematical Mean:		33.02027
	Geometrical Mean :		32.91910
	Harmonic Mean	:	32.81740
	Variance	:	6.65598
	S.D.	:	2.57992
	Skewed Coef.	:	0.07459
	Kurtosis Coef.	:	2.99595
	MAD	:	2.05937
	Range	:	27.15750
	Mid_range	:	33.85641
	Median	:	32.98780
	Q1	:	31.25982
	Q2	:	32.98780
	Q3	:	34.74515
	IQR	:	3.48533
	C.V.	:	0.07813

(7)n=1,000, λ =0.1,W24= $X_1 + X_2 + ... + X_n$,

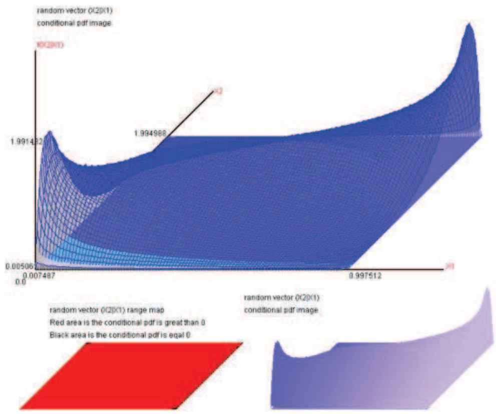
f(w24), F(w24)	Coefficient		
	Mathematical Mean:		330.20226
	Geometrical Mean :		330.10147
	Harmonic Mean	:	330.00063
	Variance	:	66.53806
	S.D.	:	8.15709
	Skewed Coef.	:	0.02381
	Kurtosis Coef.	:	2.99953
	MAD	:	6.50920
	Range	:	84.48889
	Mid_range	:	331.02390
	Median	:	330.16686
	Q1	:	324.67916
	Q2	:	330.16686
	Q3	:	335.68862
	IQR	:	11.00946
	C.V.	:	0.02470

Note: The computer program is C:\C_Bernoulli\C_Bernoulli_03.exe, which can compute the sample mean ($\overline{X} = \dfrac{X_1 + X_2 + ... + X_n}{n}$)sampling distribution of Continuous Bernoulli distribution.

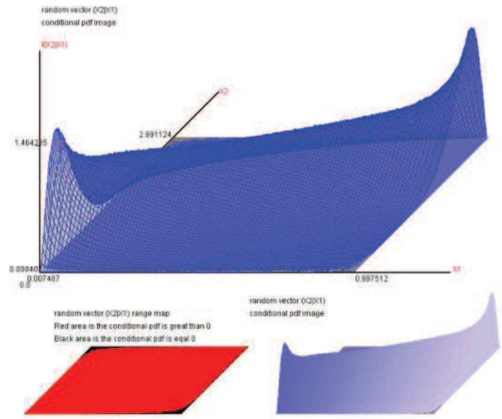
Section 4, $\sum_{i=1}^{n} X_i \xrightarrow[n\to\infty]{} Normal\left(E\left(\sum_{i=1}^{n} X_i\right), Var\left(\sum_{i=1}^{n} X_i\right)\right)$,

$X_1, X_2, ..., X_n \overset{iid}{\sim} CB(\lambda)$, $X2=\sum_{i=1}^{n} X_i$, the simulator and transformation can get f(X2|X1= λ), $0<\lambda<1$, the simulated data number=1,000,000,000.
The diagram is (X1= λ , f(X2|X1)).

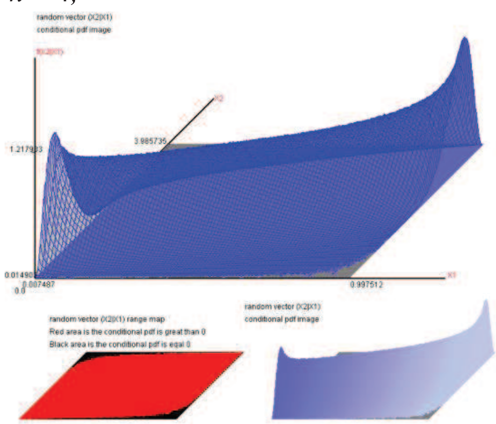
$n=2$,



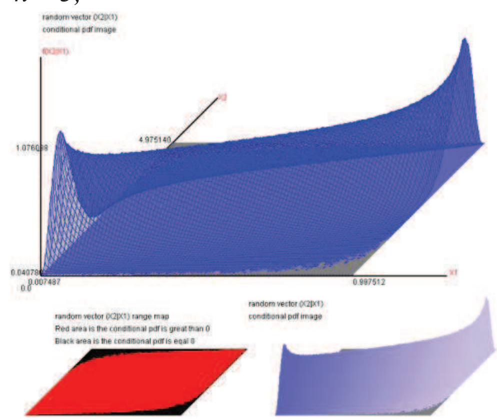
$n=3$,



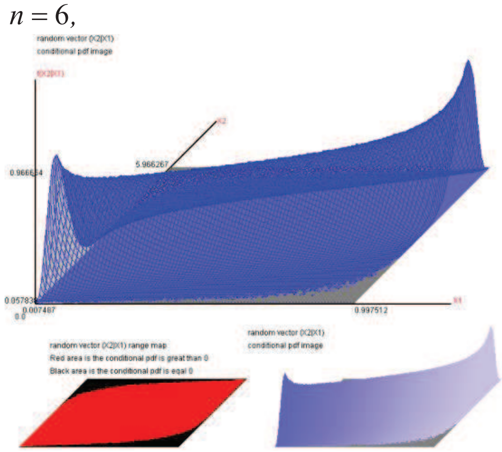
$n=4$,



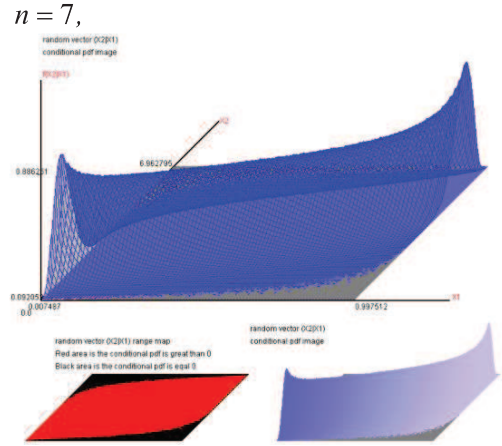
$n=5$,



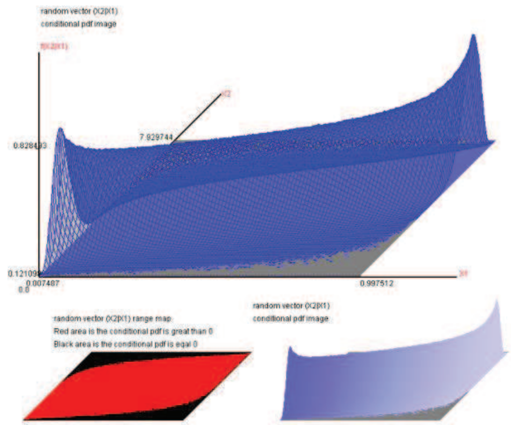
$n=6$,



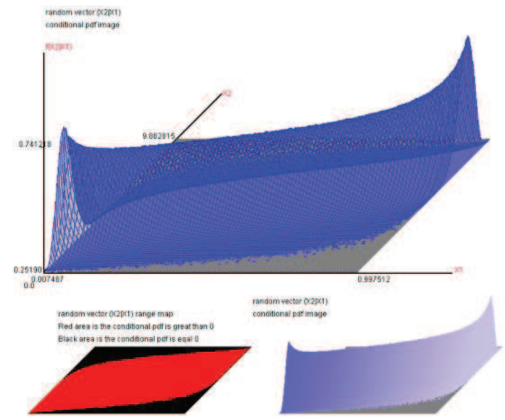
$n=7$,



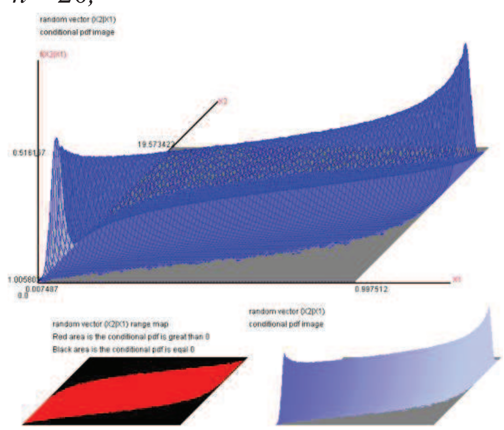
$n = 8,$



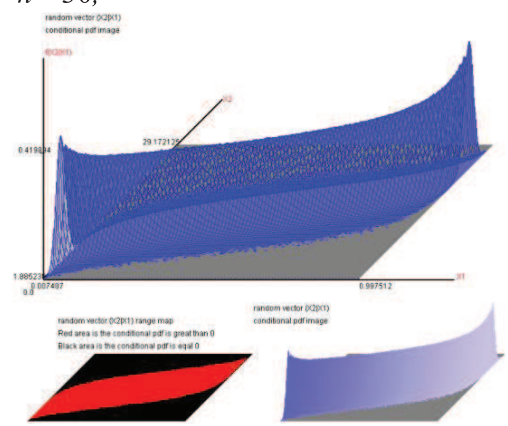
$n = 10,$



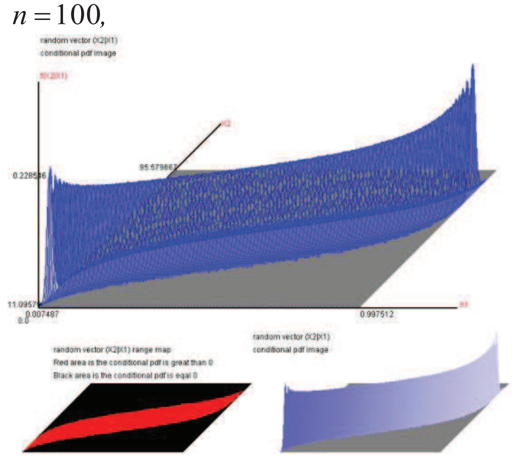
$n = 20,$



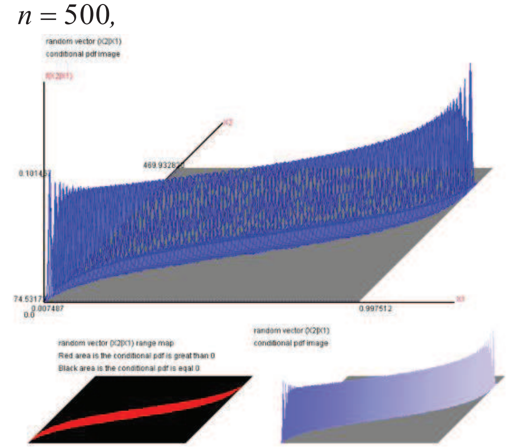
$n = 30,$



$n = 100,$



$n = 500,$



The red area is the range of $(\sum_{i=1}^{n} X_i , \lambda)$.

The λ in $\sum_{i=1}^{n} X_i$ which is changed to the shape parameter to the location parameter when n is very large. When $X_1, X_2,..., X_n \overset{iid}{\sim} CB(\lambda)$ and n is very large ($n \geq 500$),

$$\sum_{i=1}^{n} X_i \xrightarrow[n\to\infty]{} Normal\left(E\left(\sum_{i=1}^{n} X_i\right), Var\left(\sum_{i=1}^{n} X_i\right)\right).$$

Chapter 3, The λ point estimator of Continuous Bernoulli distribution

$X_1, X_2, ..., X_n \overset{iid}{\sim} CB(\lambda)$, n random samples come from the Continuous Bernoulli distribution (λ).

Section 1, UMVU(Uniformly minimum variance unbiased),

$$E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i) = n\mu, E\left(\overline{X} = \frac{\sum_{i=1}^{n} X_i}{n}\right) = \mu,$$

$$\mu = E(X) = \begin{cases} \frac{\lambda}{2\lambda - 1} + \frac{1}{2\tan^{-1}(1-2\lambda)} & if\ \lambda \neq \frac{1}{2} \\ \frac{1}{2} & if\ \lambda = \frac{1}{2} \end{cases}.$$

Let $U(\overline{X})$ is the function of λ and $E(U(\overline{X})) = \lambda$, but $U(\overline{X})$ cannot be found. The λ of UMVUE is not existed.

Section 2, Maximum likelihood estimator,

$$L(\lambda | x_1, x_2, ..., x_n) = (C(\lambda))^n \lambda^{\sum_{i=1}^{n} x_i} (1-\lambda)^{n - \sum_{i=1}^{n} x_i},$$

$$ln\, L(\lambda | x_1, x_2, ..., x_n) = n\, ln(C(\lambda)) + \sum_{i=1}^{n} x_i \times ln(\lambda) + \left(n - \sum_{i=1}^{n} x_i\right) \times ln(1-\lambda),$$

$$\frac{d\, ln\, L(\lambda | x_1, x_2, ..., x_n)}{d\lambda} = \frac{nC'(\lambda)}{C(\lambda)} + \frac{\sum_{i=1}^{n} x_i}{\lambda} - \frac{n - \sum_{i=1}^{n} x_i}{1-\lambda} = 0,$$

$$\frac{nC'(\lambda)}{C(\lambda)} + \frac{\sum_{i=1}^{n} x_i - n\lambda}{\lambda \times (1-\lambda)} = 0, \frac{\sum_{i=1}^{n} x_i}{n} = \overline{x} = \frac{\lambda}{\lambda \times (1-\lambda)} - \frac{C'(\lambda)}{C(\lambda)},$$

$\hat{\lambda} = \phi(\overline{x})$, $\phi(\ )$ cannot be derived from the above equation,

Section 3, The λ point estimator using sufficient statistic and estimated equation,

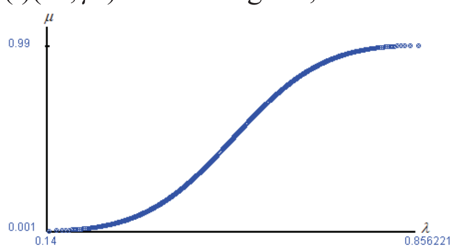
From chapter 2 and section 3, the μ and λ is one to one, $E(X)$ can be computed using Monte Carlo method and the relative error below 1/10000. This is a way to find the MLE of λ but using the software program and numerical analysis. It is the remedy method to construct the function of λ using μ, the analytics process is below,

(1) $\lambda = \phi^*(\mu), E(X) = \mu,$ the λ estimated equation of μ,

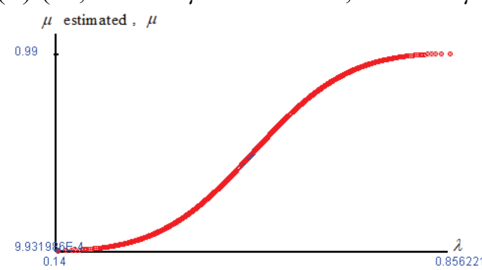
The λ estimated μ using curvi-linear model (Taylor's expansion and regression analysis) and μ is computed by the simulator in λ specific range (appendix 2).

The $0.001 \le \lambda \le 0.999$, $0.143853919 \le \mu \le 0.856221427$,

(i)(λ , μ) scatter diagram,



(ii) (λ , Red is μ estimated, Blue is μ),



(2) $\lambda = \phi^*(\mu)$, $\phi^*(\mu)$ estimated equation,

The estimated equation,

X= -0.596698+2.193196$\times\mu$,

λ=0.49997386580423608+1.36802409685464270*(X-0.5056)^1+
-0.00092474767006933689 0*(X-0.5056)^2+-2.73607823707760640*(X-0.5056)^3+
0.09510904364287853 2*(X-0.5056)^4+5.7483773675921839*(X-0.5056)^5+
-1.8419988453388214*(X-0.5056)^6+-12.357242575206328*(X-0.5056)^7+
16.361405849456787*(X-0.5056)^8+26.41792850010097*(X-0.5056)^9+
-80.02126121520996*(X-0.5056)^10+-48.621550429612398*(X-0.5056)^11+
228.76872253417969*(X-0.5056)^12+64.702439151704311*(X-0.5056)^13+
-380.75874328613281*(X-0.5056)^14+-51.895506033673882*(X-0.5056)^15+
341.66360473632812*(X-0.5056)^16+18.360968290828168*(X-0.5056)^17+
-127.70810317993164*(X-0.5056)^18,

ANOVA

Source	df	SS	MS
Regression	18	83.0834922851	4.6157495714
Error	980	0.0000077149	0.0000000079
Total	998	83.0835000000	

H0:slope1=....=slope18=0, test statistic=586328245.808614, p value=0.000000, sample size=999, R2=1.000000, R2(adj)=1.000000,MSE=0.000000,

The R2$\rightarrow$1 and MSE=0, $\phi^*(\ )$ is not error when $\phi^*(\mu)$ converting λ .

$\phi(\ )$ estimated equation is $\phi^*(\ )$, the MLE of λ which estimated equation is $\hat{\lambda} = \phi(\bar{x}) = \phi^*(\bar{x})$.

(3) $\hat{\lambda} = \phi(\overline{X})$, $\phi(\overline{X})$ is λ MLE estimated equation,

$\overline{X} = \mu + \varepsilon, E(\varepsilon) = 0, E(\varepsilon^2) = \dfrac{Var(X)}{n} \xrightarrow[n\to\infty]{}, \varepsilon \xrightarrow[n\to\infty]{} 0$.

The $\lambda = \phi^*(\mu), \phi^*(\overline{X} - \varepsilon) \xrightarrow[n\leftarrow\infty]{} \phi^*(\overline{X})$, λ MLE= $\phi(\overline{X}) = \phi^*(\overline{X})$.

$\phi(\overline{X})$ hqw asymptotic unbiased, $E(\phi(\overline{X})) \neq \lambda$, but $E(\phi(\overline{X})) \xrightarrow[n\to\infty]{} \lambda$, the estimated error is very small can be seen as 0.

But λ=0.5, $E(\overline{X})$= λ=0.5, the λ MLE= $\overline{X}$ is unbiased estimator if λ=0.5.

(4)The limitation of estimated equation, $\phi(\overline{X})$,

$0.143853919 \le \overline{X} \le 0.856221427$, the $\hat{\lambda} = \phi(\overline{X})$ could be reasonable number which is $0.001 \le \hat{\lambda} \le 0.999$.

Section 4, The simulator of $\hat{\lambda} = \phi(\overline{X})$ sampling distribution,

(1)The simulation process,

(i) Getting random number, $RND_1, RND_2, ..., RND_n$ are independently,

(ii) $x_1(RND_1, \lambda), x_2(RND_2, \lambda), ..., x_n(RND_n, \lambda)$

(iii) $\hat{\lambda}_j = \phi\left(\frac{\sum_{i=1}^{n} x_i(RND_i, \lambda)}{n}\right)$, j=1,2,…..,100000000, $\phi(\ )$ is estimated function.

Repeat (i)~(iii) 100000000 times, the database of simulated data will be gotten.
This database can convert frequency distribution and $E(\hat{\lambda})$, $Var(\hat{\lambda}), \gamma_1(\hat{\lambda}), \gamma_2(\hat{\lambda})$,
This database is approached to Continuous Binomial distribution(λ).

Note: The computer program is C:\C_Bernoulli\C_Bernoulli_04.exe, which can compute the $\hat{\lambda} = \phi(\overline{X})$ sampling distribution of Continuous Bernoulli distribution.

Section 5, $\hat{\lambda}$ being the consistent point estimator,

The simulator data to verified $E(\phi(\overline{X}))\xrightarrow[n\to\infty]{}\lambda$ and $Var(\phi(\overline{X}))$ closing to 0.

(5-1) The sampling distribution $\hat{\lambda}=\phi(\overline{X})$,

$E(\hat{\lambda})\xrightarrow[n\to\infty]{}\lambda$ and $Var(\hat{\lambda})\xrightarrow[n\to\infty]{}0$ and $Var(\hat{\lambda})\xrightarrow[n\to\infty]{}E((\hat{\lambda}-\lambda)^2)$.

The simulated data number of each time is 100,000,000.

(5-1),

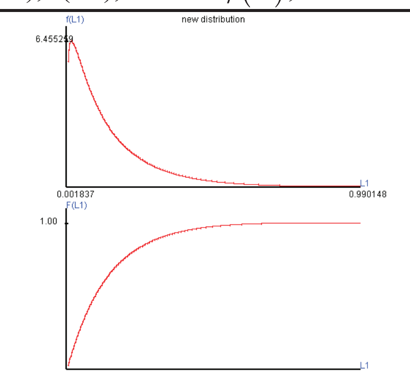
$X_1,X_2,...,X_n \overset{iid}{\sim} CB(\lambda)$, λ=0.1, E(X)= 0.33015, sigma(X)= 0.25791, Var(X)= 0.06652,

sample size	$E(\hat{\lambda})$	$Var(\hat{\lambda})$	$E((\hat{\lambda}-\lambda)^2)$	$Var(\overline{X})$
n=10	0.1370130162	0.0172280181	0.0185979815	0.006552
n=20	0.1195725332	0.0077028605	0.0080859445	0.003326
n=40	0.1100317818	0.0034962175	0.0035968541	0.001663
n=70	0.1057841817	0.0018941683	0.0019276250	0.000950
n=100	0.1040493104	0.0012945459	0.0013109428	0.0006652
n=150	0.1027003427	0.0008468626	0.0008541544	0.0004435
n=500	0.1007918487	0.0002466378	0.0002472648	0.00013304
n=5000	0.100050906	0.0000244040	0.0000244066	0.000013304

λ=0.1, the sampling distribution of $\hat{\lambda}=\phi(\overline{X})$,

(5-1-1)n=10, λ=0.1,

f(L1),F(L1),L1= $\hat{\lambda}=\phi(\overline{X})$,

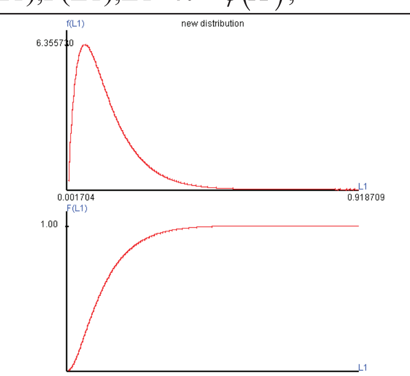


Coefficinet

Mathematical Mean:		0.13701
Geometrical Mean	:	none
Harmonic Mean	:	none
Variance	:	0.01723
S.D.	:	0.13126
Skewed Coef.	:	1.65015
Kurtosis Coef.	:	6.10894
MAD	:	0.09930
Range	:	0.99199
Mid_range	:	0.49599
Median	:	0.09554
Q1	:	0.04110
Q2	:	0.09554
Q3	:	0.19140
IQR	:	0.15030
C.V.	:	0.95798

(5-1-2) n=20, λ=0.1,

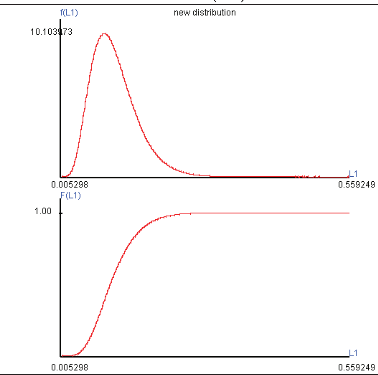
f(L1),F(L1),L1= $\hat{\lambda}=\phi(\overline{X})$,



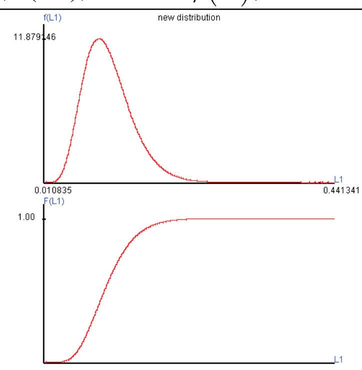
Coefficinet

Mathematical Mean:		0.11957
Geometrical Mean	:	none
Harmonic Mean	:	none
Variance	:	0.00770
S.D.	:	0.08777
Skewed Coef.	:	1.44830
Kurtosis Coef.	:	5.79970
MAD	:	0.06706
Range	:	0.92041
Mid_range	:	0.46021
Median	:	0.09779
Q1	:	0.05523
Q2	:	0.09779
Q3	:	0.16103
IQR	:	0.10580
C.V.	:	0.73400

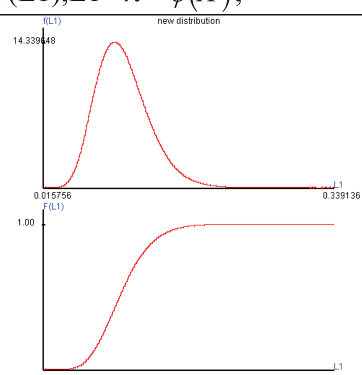
(5-1-3) n=70, λ=0.1,

f(L1),F(L1),L1= $\hat{\lambda} = \phi(\overline{X})$,	Coefficinet
	Mathematical Mean: 0.10578 Geometrical Mean : 0.09715 Harmonic Mean : 0.08849 Variance : 0.00189 S.D. : 0.04352 Skewed Coef. : 0.89480 Kurtosis Coef. : 4.20560 MAD : 0.03411 Range : 0.55601 Mid_range : 0.28227 Median : 0.09936 Q1 : 0.07414 Q2 : 0.09936 Q3 : 0.13050 IQR : 0.05635 C.V. : 0.41142

(5-1-4) n=100, λ=0.1,

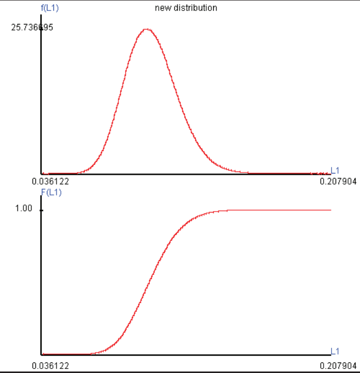
f(L1),F(L1),L1= $\hat{\lambda} = \phi(\overline{X})$,	Coefficinet
	Mathematical Mean: 0.10405 Geometrical Mean : 0.09799 Harmonic Mean : 0.09193 Variance : 0.00129 S.D. : 0.03598 Skewed Coef. : 0.75937 Kurtosis Coef. : 3.87646 MAD : 0.02834 Range : 0.43211 Mid_range : 0.22609 Median : 0.09953 Q1 : 0.07804 Q2 : 0.09953 Q3 : 0.12517 IQR : 0.04712 C.V. : 0.34580

(5-1-5) n=150, λ=0.1,

f(L1),F(L1),L1= $\hat{\lambda} = \phi(\overline{X})$,	Coefficinet
	Mathematical Mean: 0.10270 Geometrical Mean : 0.09865 Harmonic Mean : 0.09460 Variance : 0.00085 S.D. : 0.02910 Skewed Coef. : 0.62631 Kurtosis Coef. : 3.59876 MAD : 0.02301 Range : 0.32458 Mid_range : 0.17745 Median : 0.09968 Q1 : 0.08182 Q2 : 0.09968 Q3 : 0.12030 IQR : 0.03847 C.V. : 0.28336

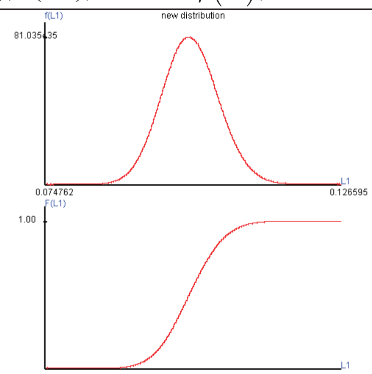
(5-1-6) n=500, λ=0.1,

f(L1),F(L1),L1= $\hat{\lambda} = \phi\left(\overline{X}\right)$,	Coefficinet	
	Mathematical Mean:	0.10079
	Geometrical Mean :	0.09958
	Harmonic Mean :	0.09836
	Variance :	0.00025
	S.D. :	0.01570
	Skewed Coef. :	0.34669
	Kurtosis Coef. :	3.18418
	MAD :	0.01250
	Range :	0.17242
	Mid_range :	0.12201
	Median :	0.09989
	Q1 :	0.08976
	Q2 :	0.09989
	Q3 :	0.11083
	IQR :	0.02107
	C.V. :	0.15581



(5-1-7) n=5000, λ=0.1,

f(L1),F(L1),L1= $\hat{\lambda} = \phi\left(\overline{X}\right)$,	Coefficinet	
	Mathematical Mean:	0.10005
	Geometrical Mean :	0.09993
	Harmonic Mean :	0.09981
	Variance :	0.00002
	S.D. :	0.00494
	Skewed Coef. :	0.10760
	Kurtosis Coef. :	3.01920
	MAD :	0.00394
	Range :	0.05202
	Mid_range :	0.10068
	Median :	0.09996
	Q1 :	0.09667
	Q2 :	0.09996
	Q3 :	0.10333
	IQR :	0.00666
	C.V. :	0.04938



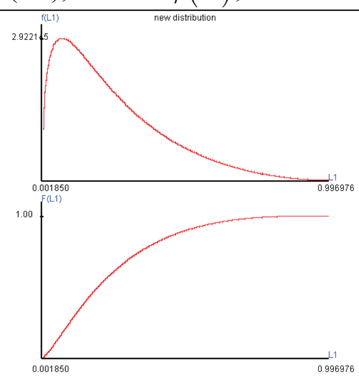
(5-2),

$X_1, X_2, \ldots, X_n \overset{iid}{\sim} CB(\lambda)$, λ =0.2, E(X)= 0.38814, sigma(X)= 0.27558, Var(X)= 0.07595,

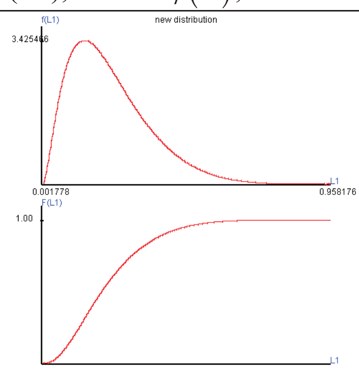
sample size	$E(\hat{\lambda})$	$Var(\hat{\lambda})$	$E((\hat{\lambda}-\lambda)^2)$	$Var(\overline{X})$
n=10	0.2394982408	0.0329158192	0.0344759302	0.007595
n=20	0.2219193607	0.0170552531	0.0175357114	0.0037975
n=40	0.2116025519	0.0085570515	0.0086916707	0.00189875
n=70	0.2068029237	0.0048734547	0.0049197345	0.001085
n=100	0.2048065669	0.0034028042	0.0034259073	0.0007595
n=150	0.2032259079	0.0022625926	0.0022729991	0.0005063
n=500	0.2009754695	0.0006751068	0.0006760583	0.0001519

λ =0.2, the sampling distribution of $\hat{\lambda} = \phi(\overline{X})$,

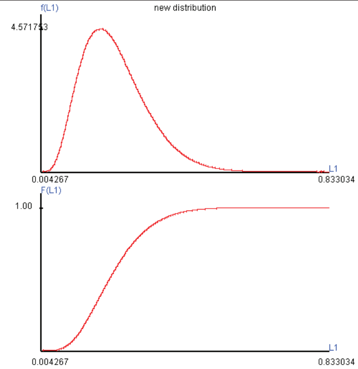
(5-2-1) n=10, λ =0.2,

f(L1),F(L1),L1= $\hat{\lambda} = \phi(\overline{X})$,	Coefficinet
	Mathematical Mean: 0.23950 Geometrical Mean : none Harmonic Mean : none Variance : 0.03292 S.D. : 0.18143 Skewed Coef. : 1.00124 Kurtosis Coef. : 3.53251 MAD : 0.14571 Range : 0.99883 Mid_range : 0.49941 Median : 0.19530 Q1 : 0.09619 Q2 : 0.19530 Q3 : 0.34298 IQR : 0.24678 C.V. : 0.75753

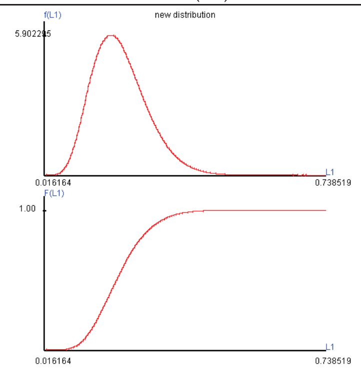
(5-2-2) n=20, λ =0.2,

f(L1),F(L1),L1= $\hat{\lambda} = \phi(\overline{X})$,	Coefficinet
	Mathematical Mean: 0.22192 Geometrical Mean : none Harmonic Mean : none Variance : 0.01706 S.D. : 0.13060 Skewed Coef. : 0.91215 Kurtosis Coef. : 3.69857 MAD : 0.10393 Range : 0.95995 Mid_range : 0.47998 Median : 0.19764 Q1 : 0.12244 Q2 : 0.19764 Q3 : 0.29732 IQR : 0.17488 C.V. : 0.58848

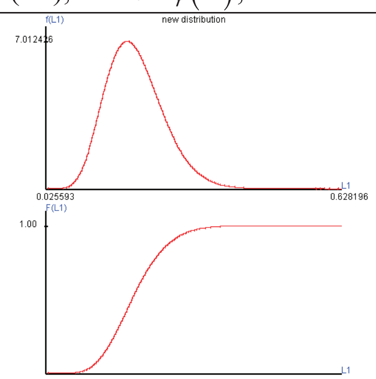
(5-2-3) n=40, λ=0.2,

f(L1),F(L1),L1= $\hat{\lambda}=\phi(\overline{X})$,	Coefficinet	
	Mathematical Mean:	0.21160
	Geometrical Mean :	0.19129
	Harmonic Mean :	0.16997
	Variance :	0.00856
	S.D. :	0.09250
	Skewed Coef. :	0.74971
	Kurtosis Coef. :	3.58262
	MAD :	0.07350
	Range :	0.83185
	Mid_range :	0.41865
	Median :	0.19886
	Q1 :	0.14313
	Q2 :	0.19886
	Q3 :	0.26680
	IQR :	0.12367
	C.V. :	0.43716

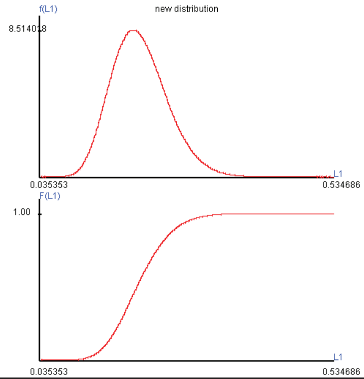
(5-2-4) n=70, λ=0.2,

f(L1),F(L1),L1= $\hat{\lambda}=\phi(\overline{X})$,	Coefficinet	
	Mathematical Mean:	0.20680
	Geometrical Mean :	0.19502
	Harmonic Mean :	0.18290
	Variance :	0.00487
	S.D. :	0.06981
	Skewed Coef. :	0.60669
	Kurtosis Coef. :	3.41690
	MAD :	0.05551
	Range :	0.72504
	Mid_range :	0.37734
	Median :	0.19935
	Q1 :	0.15608
	Q2 :	0.19935
	Q3 :	0.24961
	IQR :	0.09353
	C.V. :	0.33757

(5-3-5) n=100, λ=0.2,

f(L1),F(L1),L1= $\hat{\lambda}=\phi(\overline{X})$,	Coefficinet	
	Mathematical Mean:	0.20481
	Geometrical Mean :	0.19650
	Harmonic Mean :	0.18804
	Variance :	0.00340
	S.D. :	0.05833
	Skewed Coef. :	0.52275
	Kurtosis Coef. :	3.32176
	MAD :	0.04642
	Range :	0.60484
	Mid_range :	0.32689
	Median :	0.19952
	Q1 :	0.16284
	Q2 :	0.19952
	Q3 :	0.24113
	IQR :	0.07829
	C.V. :	0.28482

(5-4-6) n=150, λ =0.2,

f(L1),F(L1),L1= $\hat{\lambda} = \phi\left(\overline{X}\right)$,	Coefficinet
	Mathematical Mean: 0.20323 Geometrical Mean : 0.19766 Harmonic Mean : 0.19203 Variance : 0.00226 S.D. : 0.04757 Skewed Coef. : 0.43533 Kurtosis Coef. : 3.22559 MAD : 0.03788 Range : 0.50119 Mid_range : 0.28502 Median : 0.19968 Q1 : 0.16934 Q2 : 0.19968 Q3 : 0.23329 IQR : 0.06395 C.V. : 0.23406

(5-3),

$X_1, X_2, ..., X_n \overset{iid}{\sim} CB(\lambda)$, λ =0.3, E(X)= 0.43033, sigma(X)= 0.28365, Var(X)= 0.08046,

sample size	$E\left(\hat{\lambda}\right)$	$Var\left(\hat{\lambda}\right)$	$E\left(\left(\hat{\lambda}-\lambda\right)^2\right)$	$Var\left(\overline{X}\right)$
n=10	0.330882733	0.0438956694	0.0448494126	0.008046
n=20	0.3175563401	0.0244554511	0.0247636762	0.004023
n=40	0.3094689186	0.0129434440	0.0130331045	0.0020115
n=70	0.3055935638	0.0075789217	0.0076102096	0.001149
n=100	0.3039595950	0.0053585155	0.0053741939	0.0008046
n=150	0.3026521277	0.0035982319	0.0036052657	0.0005364
n=500	0.3007796649	0.0010899490	0.0010905569	0.00016092

(5-4),

$X_1, X_2, ..., X_n \overset{iid}{\sim} CB(\lambda)$, λ =0.4, E(X)= 0.46633, sigma(X)= 0.28751, Var(X)= 0.08266,

sample size	$E\left(\hat{\lambda}\right)$	$Var\left(\hat{\lambda}\right)$	$E\left(\left(\hat{\lambda}-\lambda\right)^2\right)$	$Var\left(\overline{X}\right)$
n=10	0.4165747103	0.0502580047	0.0505327257	0.008266
n=20	0.4095618209	0.0290600732	0.0291515016	0.004133
n=40	0.4052120362	0.0158184881	0.0158456534	0.0020665
n=70	0.4030819931	0.0094032042	0.0094127029	0 0.001181
n=100	0.4021657080	0.0066885366	0.0066932269	0.0008266
n=150	0.4014417469	0.0045177751	0.0045198537	0.0005511
n=500	0.4004011412	0.0013804965	0.0013806574	0.00016532

example 5-5,

$X_1, X_2, ..., X_n \overset{iid}{\sim} CB(\lambda)$, λ =0.5, E(X)= 0.50002 sigma(X)= 0.28869, Var(X)= 0.08334,

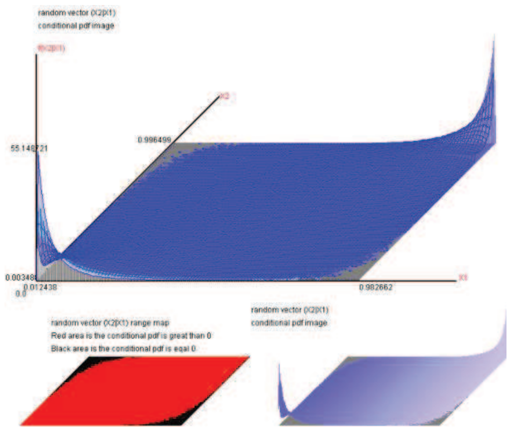
sample size	$E\left(\hat{\lambda}\right)$	$Var\left(\hat{\lambda}\right)$	$E\left(\left(\hat{\lambda}-\lambda\right)^2\right)$	$Var\left(\overline{X}\right)$
n=10	0.4999456967	0.0524223792	0.0524223822	0.008334
n=20	0.4999339358	0.0306200021	0.0306200064	0.004167
n=40	0.4999608371	0.0168066054	0.0168066069	0.0020835
n=70	0.4999558386	0.0100385116	0.0100385135	0.0011906
n=100	0.4999539881	0.0071605540	0.0071605561	0.0008334
n=150	0.4999515346	0.0048435585	0.0048435608	0.0005556
n=500	0.4999531707	0.0014845850	0.0014845872	0.00016668

Section 6, $\hat{\lambda} = \phi(\overline{X}) \xrightarrow[n \to \infty]{} Normal(E(\hat{\lambda}), Var(\hat{\lambda}))$**,**

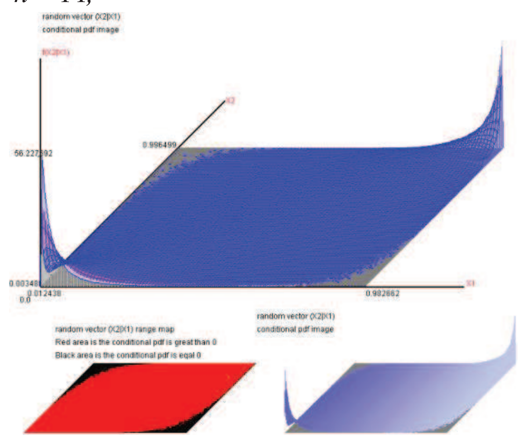
The simulator and transformation can get $\hat{\lambda} = \phi(\overline{X})$ sampling distribution and conditional probability density function in λ to be explained.
Let X2= $\hat{\lambda} = \phi(\overline{X})$ and f(X2|X1= λ), the simulated data number=100,000,000.
(5-2-1)$0.01 \le \lambda \le 0.99$ for $E(\hat{\lambda}) \xrightarrow[n \to \infty]{} \lambda$ and $Var(\hat{\lambda}) \xrightarrow[n \to \infty]{} 0$.

The diagram is (X1= λ , f(X2|X1)).

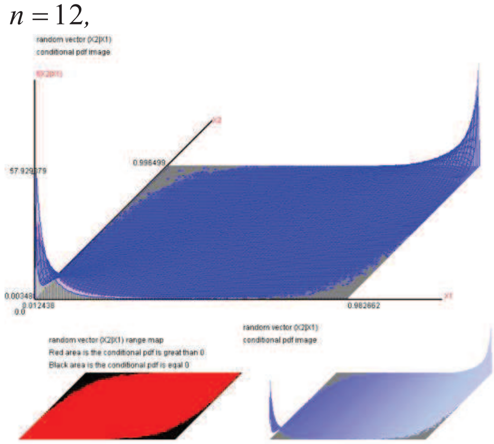
$n = 10,$



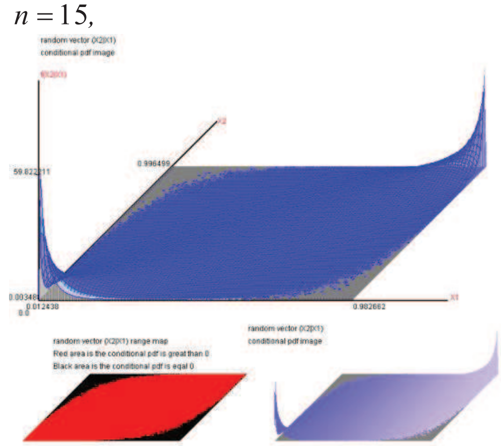
$n = 11,$



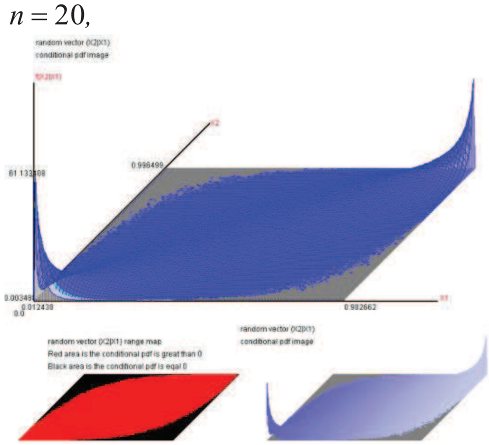
$n = 12,$



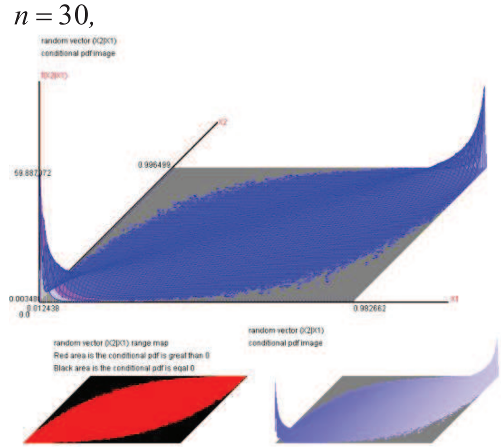
$n = 15,$



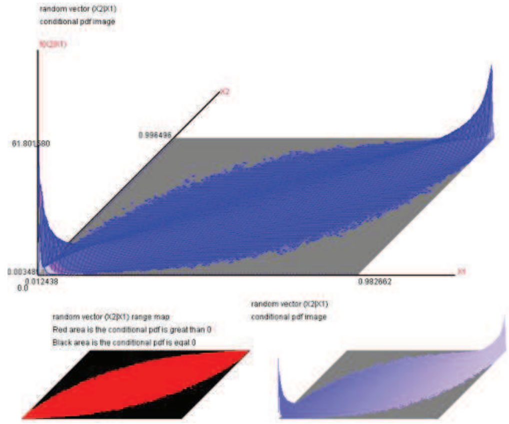
$n = 20,$



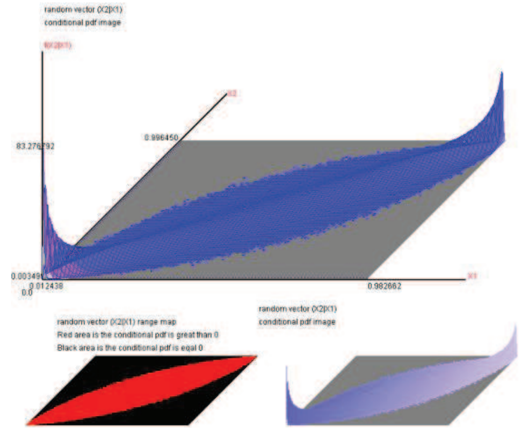
$n = 30,$



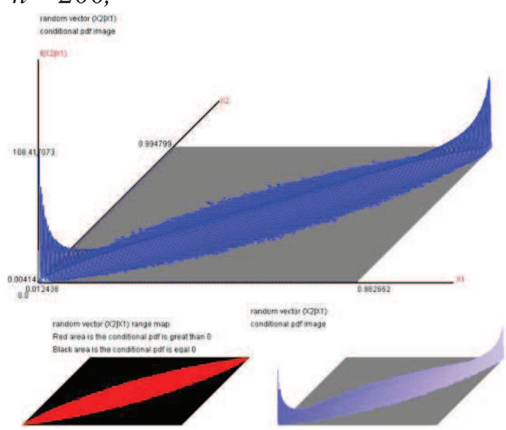
$n = 50,$



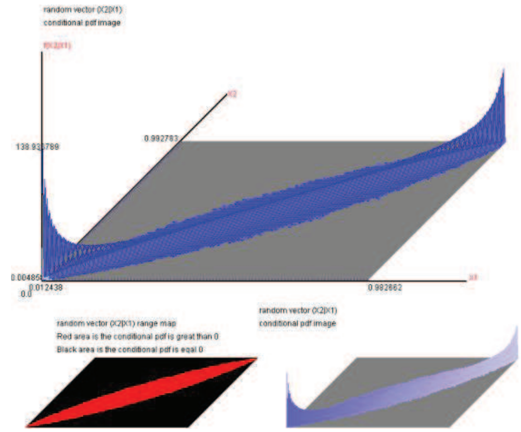
$n = 100,$



$n = 200,$



$n = 400,$



The red area is the range of ($\hat{\lambda} = \phi\left(\overline{X}\right), \lambda$).

From *n*=10,11,12,15,20,30,50, $\left(\lambda, E\left(\hat{\lambda}\right)\right)$ diagram is not 45^0 line.

From *n*=100, 200, $\left(\lambda, E\left(\hat{\lambda}\right)\right)$ diagram is approaching to 45^0 line.

n=400, $\left(\lambda, E\left(\hat{\lambda}\right)\right)$ diagram is close to 45^0 line,

$E\left(\hat{\lambda}\right) \xrightarrow[n\to\infty]{} \lambda$ and $Var\left(\hat{\lambda}\right) \xrightarrow[n\to\infty]{} 0$, but $E\left(\overline{X}\right) = \lambda$ =0.5 if λ =0.5 in any sample size.

Chapter 4, The test statistic of Continuous Bernoulli distribution

$X_1, X_2, ..., X_n \overset{iid}{\sim} CB(\lambda)$, n random samples from $CB(\lambda)$.
There are two test statistic,
one is $\frac{\sqrt{n}(\overline{X}-\mu(X))}{\sigma(X)}$, the other is $\frac{\hat{\lambda}-E(\hat{\lambda})}{\sqrt{Var(\hat{\lambda})}}$, but $\frac{\sqrt{n}(\overline{X}-\mu(X))}{\sigma(X)}$ is better than $\frac{\hat{\lambda}-E(\hat{\lambda})}{\sqrt{Var(\hat{\lambda})}}$.

Section 1, The difference of and $\frac{\hat{\lambda}-E(\hat{\lambda})}{\sqrt{Var(\hat{\lambda})}}$,

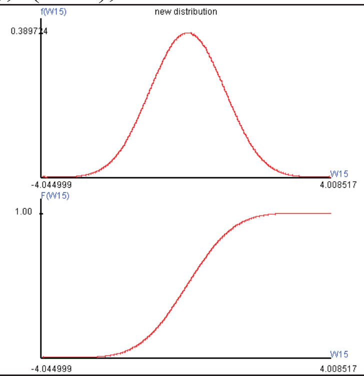
The $\frac{\sqrt{n}(\overline{X}-\mu(X))}{\sigma(X)}$ and $\frac{\hat{\lambda}-E(\hat{\lambda})}{\sqrt{Var(\hat{\lambda})}}$ sampling distributions

when $X_1, X_2, ..., X_n \overset{iid}{\sim} CB(\lambda)$, $\overline{X} = \frac{\sum_{i=1}^{n} X_i}{n}, \hat{\lambda} = \phi(\overline{X})$ (chapter 3, section 3).
The $\frac{\sqrt{n}(\overline{X}-\mu(X))}{\sigma(X)} \xrightarrow[n \geq n(\overline{X})]{} Normal(0,1)$ and $\frac{\hat{\lambda}-E(\hat{\lambda})}{\sqrt{Var(\hat{\lambda})}} \xrightarrow[n \geq n(\lambda)]{} Normal(0,1)$,
because $\hat{\lambda} = \phi(\overline{X})$ is the non-linear function of $\overline{X}$ and $E(\hat{\lambda}) \neq \lambda$, $n(\overline{X})$ is less than $n(\lambda)$, $\frac{\sqrt{n}(\overline{X}-\mu(X))}{\sigma(X)}$ is the good test statistic.

(1) $n(\overline{X}) = ?$ when $\frac{\sqrt{n}(\overline{X}-\mu(X))}{\sigma(X)} \xrightarrow[n \geq n(\overline{X})]{} Normal(0,1)$,
W15=$\frac{\sqrt{n}(\overline{X}-\mu(X))}{\sigma(X)} \xrightarrow[n \geq n(\overline{X})]{} Normal(0,1)$,
Getting the simulated data of W15 and standard normal distribution using the simulator and the simulated data number=100,000,000.
Calculating the $n(\overline{X})$ using the Strong Law of Large Number, the requirement is
$P\{|F_{W15}(W15) - \Phi(W15)| < 0.1\} = 1$, $P\{|F_{W15}(W15) - \Phi(W15)| < 0.05\} = 1$,
$P\{|F_{W15}(W15) - \Phi(W15)| < 0.01\} = 1$, $P\{|F_{W15}(W15) - \Phi(W15)| < 0.005\} = 1$,
when $\frac{\sqrt{n}(\overline{X}-\mu(X))}{\sigma(X)} \to Normal(0,1)$.
$\Phi(z) = P(Z \leq z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} exp\left(-\frac{z^2}{2}\right) dz$ is the distribution function of standard normal distribution.

(1-1) $\lambda = 0.5, n\left(\overline{X}\right) = 6,$

f(W15),F(W15),



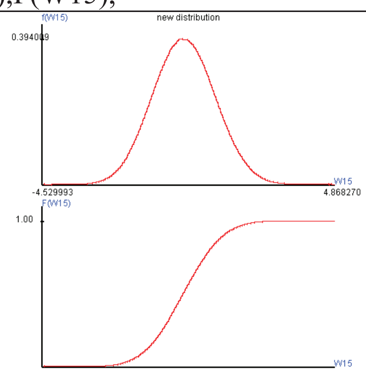
Coefficinet

Mathematical Mean:		0.00020
Geometrical Mean	:	none
Harmonic Mean	:	none
Variance	:	1.00007
S.D.	:	1.00004
Skewed Coef.	:	0.00025
Kurtosis Coef.	:	2.80095
MAD	:	0.80472
Range	:	8.08346
Mid_range	:	-0.01824
Median	:	0.00008
Q1	:	-0.68912
Q2	:	0.00008
Q3	:	0.68947
IQR	:	1.37859
C.V.	:	none

The almost surely limiting theory
E(| W15 distribution function - Z distribution function|^2)=0.0000109107,
Pr(| W15 distribution function - Z distribution function|<0.1000000000)= 1.000000,
Pr(| W15 distribution function - Z distribution function|<0.0500000000)= 1.000000,
Pr(| W15 distribution function - Z distribution function|<0.0100000000)= 1.000000,
Pr(| W15 distribution function - Z distribution function|<0.0050000000)= 1.000000,
Pr(| W15 distribution function - Z distribution function|<0.0010000000)= 0.164155,
Pr(| W15 distribution function - Z distribution function|<0.0005000000)= 0.078938,
Pr(| W15 distribution function - Z distribution function|<0.0001000000)= 0.015383,

(1-2) $\lambda = 0.4, n\left(\overline{X}\right) = 11,$

f(W15),F(W15),

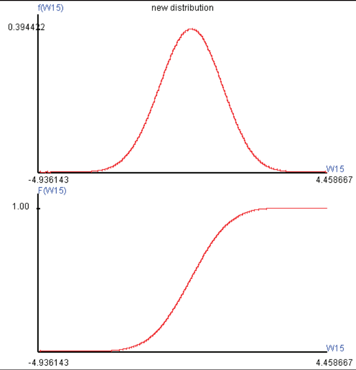


Coefficinet

Mathematical Mean:		0.00029
Geometrical Mean	:	none
Harmonic Mean	:	none
Variance	:	1.00043
S.D.	:	1.00021
Skewed Coef.	:	0.04193
Kurtosis Coef.	:	2.89399
MAD	:	0.80179
Range	:	9.43320
Mid_range	:	0.16914
Median	:	-0.00678
Q1	:	-0.68659
Q2	:	-0.00678
Q3	:	0.67890
IQR	:	1.36549
C.V.	:	none

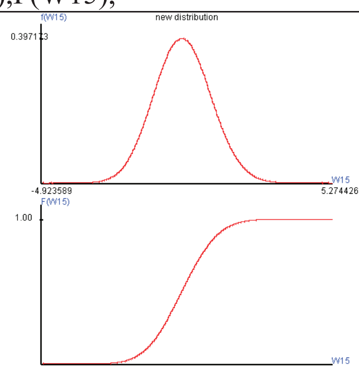
The almost surely limiting theory
E(| W15 distribution function - Z distribution function|^2)=0.0000064069,
Pr(| W15 distribution function - Z distribution function|<0.1000000000)= 1.000000,
Pr(| W15 distribution function - Z distribution function|<0.0500000000)= 1.000000,
Pr(| W15 distribution function - Z distribution function|<0.0100000000)= 1.000000,
Pr(| W15 distribution function - Z distribution function|<0.0050000000)= 1.000000,
Pr(| W15 distribution function - Z distribution function|<0.0010000000)= 0.219017,
Pr(| W15 distribution function - Z distribution function|<0.0005000000)= 0.105682,
Pr(| W15 distribution function - Z distribution function|<0.0001000000)= 0.023000,

(1-3) $\lambda = 0.6, n(\overline{X}) = 11,$

f(w15),F(w15)	Coefficient
	Mathematical Mean: -0.00011 Geometrical Mean : none Harmonic Mean : none Variance : 1.00030 S.D. : 1.00015 Skewed Coef. : -0.04186 Kurtosis Coef. : 2.89406 MAD : 0.80161 Range : 9.42974 Mid_range : -0.23874 Median : 0.00698 Q1 : -0.67838 Q2 : 0.00698 Q3 : 0.68630 IQR : 1.36468 C.V. : none

The almost surely limiting theory
E(\| W15 distribution function - Z distribution function\|^2)=0.0000060964,
Pr(\| W15 distribution function - Z distribution function\|<0.1000000000)= 1.000000,
Pr(\| W15 distribution function - Z distribution function\|<0.0500000000)= 1.000000,
Pr(\| W15 distribution function - Z distribution function\|<0.0100000000)= 1.000000,
Pr(\| W15 distribution function - Z distribution function\|<0.0050000000)= 1.000000,
Pr(\| W15 distribution function - Z distribution function\|<0.0010000000)= 0.209822,

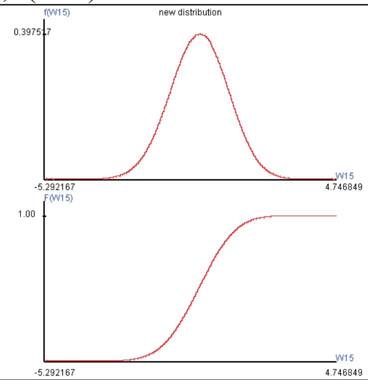
(1-4) $\lambda = 0.3, n(\overline{X}) = 25,$

f(W15),F(W15),	Coefficinet
	Mathematical Mean: -0.00006 Geometrical Mean : none Harmonic Mean : none Variance : 0.99993 S.D. : 0.99996 Skewed Coef. : 0.06069 Kurtosis Coef. : 2.95441 MAD : 0.79950 Range : 10.23593 Mid_range : 0.17542 Median : -0.01031 Q1 : -0.68362 Q2 : -0.01031 Q3 : 0.67262 IQR : 1.35624 C.V. : none

The almost surely limiting theory
E(\| W15 distribution function - Z distribution function\|^2)=0.0000069808,
Pr(\| W15 distribution function - Z distribution function\|<0.1000000000)= 1.000000,
Pr(\| W15 distribution function - Z distribution function\|<0.0500000000)= 1.000000,
Pr(\| W15 distribution function - Z distribution function\|<0.0100000000)= 1.000000,
Pr(\| W15 distribution function - Z distribution function\|<0.0050000000)= 1.000000,
Pr(\| W15 distribution function - Z distribution function\|<0.0010000000)= 0.197691,
Pr(\| W15 distribution function - Z distribution function\|<0.0005000000)= 0.105150,
Pr(\| W15 distribution function - Z distribution function\|<0.0001000000)= 0.017595,

(1-5) $\lambda = 0.7, n(\overline{X}) = 24,$

f(w15),F(w15)

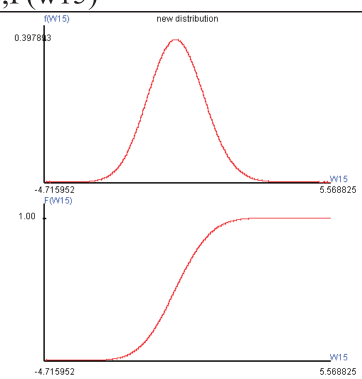


Coefficient		
Mathematical Mean:		0.00001
Geometrical Mean	:	none
Harmonic Mean	:	none
Variance	:	1.00017
S.D.	:	1.00009
Skewed Coef.	:	-0.06003
Kurtosis Coef.	:	2.95459
MAD	:	0.79958
Range	:	10.07634
Mid_range	:	-0.27266
Median	:	0.01029
Q1	:	-0.67251
Q2	:	0.01029
Q3	:	0.68351
IQR	:	1.35602
C.V.	:	none

E(| W15 distribution - Z distribution |^2)=0.0002347298
*********** | W15 distribution function - Z distribution function| ***********
The almost surely limiting theory
E(| W15 distribution function - Z distribution function|^2)=0.0000070229,
Pr(| W15 distribution function - Z distribution function|<0.1000000000)= 1.000000,
Pr(| W15 distribution function - Z distribution function|<0.0500000000)= 1.000000,
Pr(| W15 distribution function - Z distribution function|<0.0100000000)= 1.000000,
Pr(| W15 distribution function - Z distribution function|<0.0050000000)= 1.000000,
Pr(| W15 distribution function - Z distribution function|<0.0010000000)= 0.191289,

(1-6) $\lambda = 0.2, n(\overline{X}) = 45,$

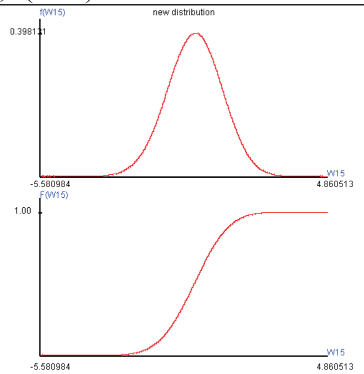
f(w15),F(w15)



Coefficient		
Mathematical Mean:		0.00027
Geometrical Mean	:	none
Harmonic Mean	:	none
Variance	:	1.00030
S.D.	:	1.00015
Skewed Coef.	:	0.07156
Kurtosis Coef.	:	2.98005
MAD	:	0.79888
Range	:	10.32301
Mid_range	:	0.42644
Median	:	-0.01140
Q1	:	-0.68296
Q2	:	-0.01140
Q3	:	0.67016
IQR	:	1.35312
C.V.	:	none

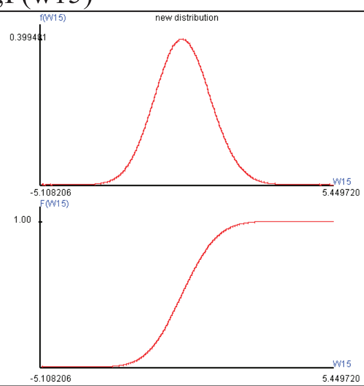
The almost surely limiting theory
E(| W15 distribution function - Z distribution function|^2)=0.0000089662,
Pr(| W15 distribution function - Z distribution function|<0.1000000000)= 1.000000,
Pr(| W15 distribution function - Z distribution function|<0.0500000000)= 1.000000,
Pr(| W15 distribution function - Z distribution function|<0.0100000000)= 1.000000,
Pr(| W15 distribution function - Z distribution function|<0.0050000000)= 1.000000,
Pr(| W15 distribution function - Z distribution function|<0.0010000000)= 0.174623,
Pr(| W15 distribution function - Z distribution function|<0.0005000000)= 0.089743,
Pr(| W15 distribution function - Z distribution function|<0.0001000000)= 0.015884,

(1-7) $\lambda = 0.8, n(\overline{X}) = 50,$

f(w15),F(w15)	Coefficient
	Mathematical Mean: 0.00010 Geometrical Mean : none Harmonic Mean : none Variance : 1.00003 S.D. : 1.00002 Skewed Coef. : -0.06710 Kurtosis Coef. : 2.98280 MAD : 0.79865 Range : 10.48031 Mid_range : -0.36024 Median : 0.01144 Q1 : -0.67018 Q2 : 0.01144 Q3 : 0.68215 IQR : 1.35233 C.V. : none

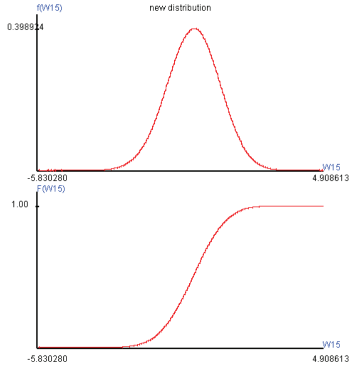
The almost surely limiting theory
E(| W15 distribution function - Z distribution function|^2)=0.0000079026,
Pr(| W15 distribution function - Z distribution function|<0.1000000000)= 1.000000,
Pr(| W15 distribution function - Z distribution function|<0.0500000000)= 1.000000,
Pr(| W15 distribution function - Z distribution function|<0.0100000000)= 1.000000,
Pr(| W15 distribution function - Z distribution function|<0.0050000000)= 1.000000,
Pr(| W15 distribution function - Z distribution function|<0.0010000000)= 0.194868,
Pr(| W15 distribution function - Z distribution function|<0.0005000000)= 0.092056,
Pr(| W15 distribution function - Z distribution function|<0.0001000000)= 0.016767,

(1-8) $\lambda = 0.1, n(\overline{X}) = 100,$

f(w15),F(w15)	Coefficient
	Mathematical Mean: 0.00000 Geometrical Mean : none Harmonic Mean : none Variance : 0.99968 S.D. : 0.99984 Skewed Coef. : 0.07413 Kurtosis Coef. : 2.99644 MAD : 0.79804 Range : 10.59717 Mid_range : 0.17076 Median : -0.01230 Q1 : -0.68177 Q2 : -0.01230 Q3 : 0.66822 IQR : 1.35000 C.V. : none

The almost surely limiting theory
E(| W15 distribution function - Z distribution function|^2)=0.0000093035,
Pr(| W15 distribution function - Z distribution function|<0.1000000000)= 1.000000,
Pr(| W15 distribution function - Z distribution function|<0.0500000000)= 1.000000,
Pr(| W15 distribution function - Z distribution function|<0.0100000000)= 1.000000,
Pr(| W15 distribution function - Z distribution function|<0.0050000000)= 1.000000,
Pr(| W15 distribution function - Z distribution function|<0.0010000000)= 0.174883,
Pr(| W15 distribution function - Z distribution function|<0.0005000000)= 0.084831,
Pr(| W15 distribution function - Z distribution function|<0.0001000000)= 0.017134,

(1-9) $\lambda = 0.9, n(\overline{X}) = 100,$

f(w15),F(w15)	Coefficient
	Mathematical Mean: -0.00004 Geometrical Mean : none Harmonic Mean : none Variance : 1.00030 S.D. : 1.00015 Skewed Coef. : -0.07337 Kurtosis Coef. : 2.99442 MAD : 0.79833 Range : 10.77881 Mid_range : -0.46083 Median : 0.01243 Q1 : -0.66874 Q2 : 0.01243 Q3 : 0.68179 IQR : 1.35052 C.V. : none

The almost surely limiting theory
E(| W15 distribution function - Z distribution function|^2)=0.0000094424,
Pr(| W15 distribution function - Z distribution function|<0.1000000000)= 1.000000,
Pr(| W15 distribution function - Z distribution function|<0.0500000000)= 1.000000,
Pr(| W15 distribution function - Z distribution function|<0.0100000000)= 1.000000,
Pr(| W15 distribution function - Z distribution function|<0.0050000000)= 0.976842,
Pr(| W15 distribution function - Z distribution function|<0.0010000000)= 0.172794,
Pr(| W15 distribution function - Z distribution function|<0.0005000000)= 0.087209,
Pr(| W15 distribution function - Z distribution function|<0.0001000000)= 0.015426,

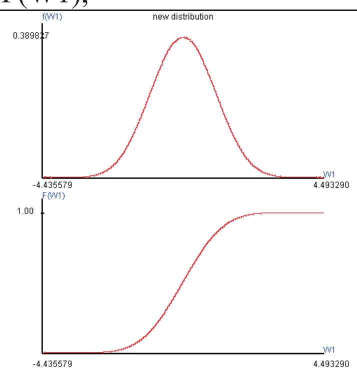
(2) $n(\lambda)=?$ W1= $\frac{\hat{\lambda}-E(\hat{\lambda})}{\sqrt{Var(\hat{\lambda})}} \xrightarrow[n(\lambda)\to\infty]{} Normal(0,1)$,

Getting the simulated data of W1 and standard normal distribution using the simulator and the simulated data number=100,000,000.

Calculating the $n(\lambda)$ using the Strong Law of Large Number, the requirement is
$P\{|F_{W1}(W1)-\Phi(W1)|<0.1\}=1,\ P\{|F_{W1}(W1)-\Phi(W1)|<0.05\}=1,$
$P\{|F_{W1}(W1)-\Phi(W1)|<0.01\}=1,\ P\{|F_{W1}(W1)-\Phi(W1)|<0.005\}=1,$
when $\frac{\hat{\lambda}-E(\hat{\lambda})}{\sqrt{Var(\hat{\lambda})}} \to Normal(0,1)$.

$\Phi(z)=P(Z\le z)=\int_{-\infty}^{z}\frac{1}{\sqrt{2\pi}}exp\left(-\frac{z^2}{2}\right)dz$ is the distribution function of standard normal distribution.

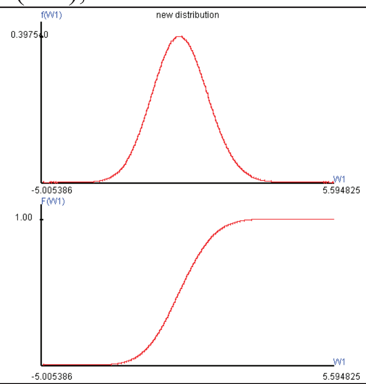
(2-1) $n(\lambda=0.5)=100$,

<table>
<tr><th>f(W1),F(W1),</th><th>Coefficinet</th></tr>
<tr><td>
</td><td>Mathematical Mean: 0.00000
Geometrical Mean : none
Harmonic Mean : none
Variance : 1.00000
S.D. : 1.00000
Skewed Coef. : 0.00074
Kurtosis Coef. : 2.82063
MAD : 0.80427
Range : 8.96206
Mid_range : 0.02886
Median : -0.00006
Q1 : -0.68846
Q2 : -0.00006
Q3 : 0.68861
IQR : 1.37707
C.V. : none</td></tr>
</table>

The almost surely limiting theory
E(| W1 distribution function - Z distribution function|^2)= 0.0000097307,
Pr(| W1 distribution function - Z distribution function|<0.1000000000)= 1.000000,
Pr(| W1 distribution function - Z distribution function|<0.0500000000)= 1.000000,
Pr(| W1 distribution function - Z distribution function|<0.0100000000)= 1.000000,
Pr(| W1 distribution function - Z distribution function|<0.0050000000)= 1.000000,
Pr(| W1 distribution function - Z distribution function|<0.0010000000)= 0.172468,
Pr(| W1 distribution function - Z distribution function|<0.0005000000)= 0.082425,
Pr(| W1 distribution function - Z distribution function|<0.0001000000)= 0.016093,

(2-2) $n(\lambda = 0.4) = 900$,

f(W1),F(W1),



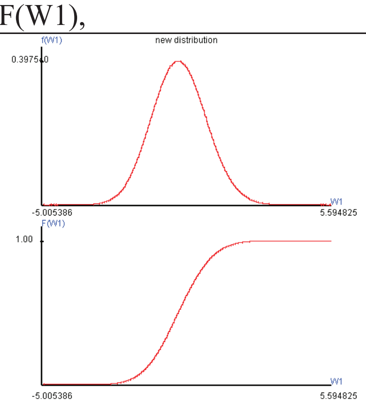
Coefficinet

Mathematical Mean:		-0.00000
Geometrical Mean	:	none
Harmonic Mean	:	none
Variance	:	1.00000
S.D.	:	1.00000
Skewed Coef.	:	0.05973
Kurtosis Coef.	:	2.98411
MAD	:	0.79862
Range	:	10.30001
Mid_range	:	0.42119
Median	:	-0.01003
Q1	:	-0.68174
Q2	:	-0.01003
Q3	:	0.67056
IQR	:	1.35230
C.V.	:	none

E(| W1 distribution - Z distribution |^2)=0.0002080453
*********** | W1 distribution function - Z distribution function| ***********
The almost surely limiting theory
E(| W1 distribution function - Z distribution function|^2)=0.0000066983,
Pr(| W1 distribution function - Z distribution function|<0.1000000000)= 1.000000,
Pr(| W1 distribution function - Z distribution function|<0.0500000000)= 1.000000,
Pr(| W1 distribution function - Z distribution function|<0.0100000000)= 1.000000,
Pr(| W1 distribution function - Z distribution function|<0.0050000000)= 1.000000,
Pr(| W1 distribution function - Z distribution function|<0.0010000000)= 0.208039,
Pr(| W1 distribution function - Z distribution function|<0.0005000000)= 0.101090,
Pr(| W1 distribution function - Z distribution function|<0.0001000000)= 0.019154,

(2-3) $n(\lambda = 0.6) = 1000$,

f(W1),F(W1),

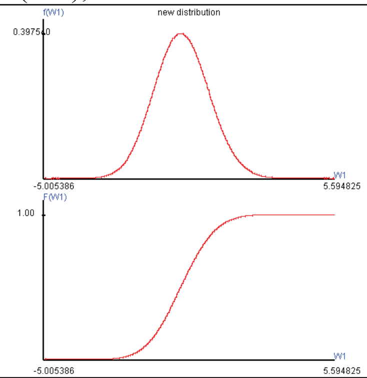


Coefficinet

Mathematical Mean:		0.00000
Geometrical Mean	:	none
Harmonic Mean	:	none
Variance	:	1.00000
S.D.	:	1.00000
Skewed Coef.	:	-0.05636
Kurtosis Coef.	:	2.98622
MAD	:	0.79855
Range	:	10.30557
Mid_range	:	-0.35668
Median	:	0.00937
Q1	:	-0.67102
Q2	:	0.00937
Q3	:	0.68115
IQR	:	1.35217
C.V.	:	none

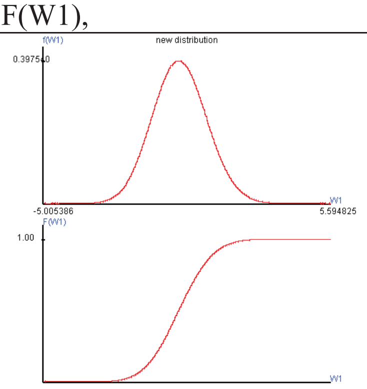
E(| W1 distribution - Z distribution |^2)=0.0001808311
*********** | W1 distribution function - Z distribution function| ***********
The almost surely limiting theory
E(| W1 distribution function - Z distribution function|^2)=0.0000053620,
Pr(| W1 distribution function - Z distribution function|<0.1000000000)= 1.000000,
Pr(| W1 distribution function - Z distribution function|<0.0500000000)= 1.000000,
Pr(| W1 distribution function - Z distribution function|<0.0100000000)= 1.000000,
Pr(| W1 distribution function - Z distribution function|<0.0050000000)= 1.000000,
Pr(| W1 distribution function - Z distribution function|<0.0010000000)= 0.244898,
Pr(| W1 distribution function - Z distribution function|<0.0005000000)= 0.118409,
Pr(| W1 distribution function - Z distribution function|<0.0001000000)= 0.023120,

(2-4) $n(\lambda = 0.3) = 2400$,

f(W1),F(W1),	Coefficinet
	Mathematical Mean: -0.00000 Geometrical Mean : none Harmonic Mean : none Variance : 1.00000 S.D. : 1.00000 Skewed Coef. : 0.07410 Kurtosis Coef. : 3.00307 MAD : 0.79794 Range : 10.49939 Mid_range : 0.42904 Median : -0.01258 Q1 : -0.68095 Q2 : -0.01258 Q3 : 0.66808 IQR : 1.34903 C.V. : none

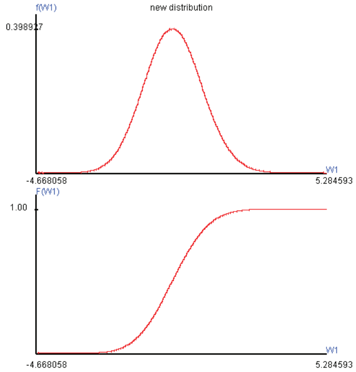
E(| W1 distribution - Z distribution |^2)=0.0003012279
*********** | W1 distribution function - Z distribution function| ***********
The almost surely limiting theory
E(| W1 distribution function - Z distribution function|^2)=0.0000098990,
Pr(| W1 distribution function - Z distribution function|<0.1000000000)= 1.000000,
Pr(| W1 distribution function - Z distribution function|<0.0500000000)= 1.000000,
Pr(| W1 distribution function - Z distribution function|<0.0100000000)= 1.000000,
Pr(| W1 distribution function - Z distribution function|<0.0050000000)= 0.897146,
Pr(| W1 distribution function - Z distribution function|<0.0010000000)= 0.174774,
Pr(| W1 distribution function - Z distribution function|<0.0005000000)= 0.085842,
Pr(| W1 distribution function - Z distribution function|<0.0001000000)= 0.016369,

(2-5) $n(\lambda = 0.7) = 2600$,

f(W1),F(W1),	Coefficinet
	Mathematical Mean: 0.00000 Geometrical Mean : none Harmonic Mean : none Variance : 1.00000 S.D. : 1.00000 Skewed Coef. : -0.07069 Kurtosis Coef. : 3.00036 MAD : 0.79808 Range : 10.15801 Mid_range : -0.27947 Median : 0.01130 Q1 : -0.66846 Q2 : 0.01130 Q3 : 0.68181 IQR : 1.35027 C.V. : none

E(| W1 distribution - Z distribution |^2)=0.0002813453
*********** | W1 distribution function - Z distribution function| ***********
The almost surely limiting theory
E(| W1 distribution function - Z distribution function|^2)=0.0000081742,
Pr(| W1 distribution function - Z distribution function|<0.1000000000)= 1.000000,
Pr(| W1 distribution function - Z distribution function|<0.0500000000)= 1.000000,
Pr(| W1 distribution function - Z distribution function|<0.0100000000)= 1.000000,
Pr(| W1 distribution function - Z distribution function|<0.0050000000)= 1.000000,
Pr(| W1 distribution function - Z distribution function|<0.0010000000)= 0.176549,
Pr(| W1 distribution function - Z distribution function|<0.0005000000)= 0.086442,
Pr(| W1 distribution function - Z distribution function|<0.0001000000)= 0.016321,

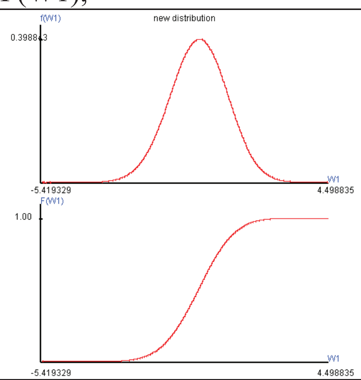
(2-6) $n(\lambda = 0.2) = 6000$,

<table>
<tr><th>F(W1),F(W1),</th><th>Coefficinet</th></tr>
<tr><td>


</td><td>
Mathematical Mean: -0.00000
Geometrical Mean : none
Harmonic Mean : none
Variance : 1.00000
S.D. : 1.00000
Skewed Coef. : 0.07422
Kurtosis Coef. : 3.00972
MAD : 0.79779
Range : 9.98965
Mid_range : 0.30827
Median : -0.01161
Q1 : -0.68095
Q2 : -0.01161
Q3 : 0.66732
IQR : 1.34827
C.V. : none
</td></tr>
</table>

E(| W1 distribution - Z distribution |^2)=0.0003061605
*********** | W1 distribution function - Z distribution function| ***********
The almost surely limiting theory
E(| W1 distribution function - Z distribution function|^2)=0.0000089389,
Pr(| W1 distribution function - Z distribution function|<0.1000000000)= 1.000000,
Pr(| W1 distribution function - Z distribution function|<0.0500000000)= 1.000000,
Pr(| W1 distribution function - Z distribution function|<0.0100000000)= 1.000000,
Pr(| W1 distribution function - Z distribution function|<0.0050000000)= 1.000000,
Pr(| W1 distribution function - Z distribution function|<0.0010000000)= 0.176264,
Pr(| W1 distribution function - Z distribution function|<0.0005000000)= 0.087118,
Pr(| W1 distribution function - Z distribution function|<0.0001000000)= 0.015256,

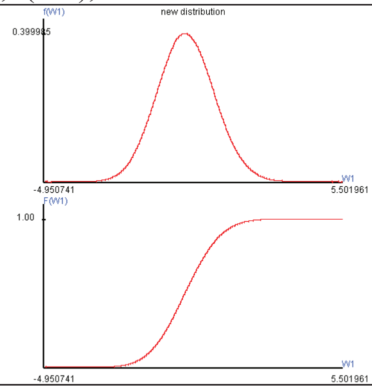
(2-7) $n(\lambda = 0.8) = 5800$,

<table>
<tr><th>f(W1),F(W1),</th><th>Coefficinet</th></tr>
<tr><td>


</td><td>
Mathematical Mean: 0.00000
Geometrical Mean : none
Harmonic Mean : none
Variance : 1.00000
S.D. : 1.00000
Skewed Coef. : -0.06899
Kurtosis Coef. : 3.00322
MAD : 0.79809
Range : 9.95503
Mid_range : -0.46025
Median : 0.01204
Q1 : -0.66815
Q2 : 0.01204
Q3 : 0.68150
IQR : 1.34965
C.V. : none
</td></tr>
</table>

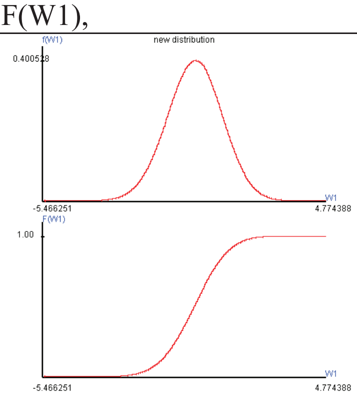
The almost surely limiting theory
E(| W1 distribution function - Z distribution function|^2)=0.0000087477,
Pr(| W1 distribution function - Z distribution function|<0.1000000000)= 1.000000,
Pr(| W1 distribution function - Z distribution function|<0.0500000000)= 1.000000,
Pr(| W1 distribution function - Z distribution function|<0.0100000000)= 1.000000,
Pr(| W1 distribution function - Z distribution function|<0.0050000000)= 1.000000,
Pr(| W1 distribution function - Z distribution function|<0.0010000000)= 0.167440,
Pr(| W1 distribution function - Z distribution function|<0.0005000000)= 0.081499,
Pr(| W1 distribution function - Z distribution function|<0.0001000000)= 0.013535,

(2-8) $n(\lambda = 0.1) = 10000$,

F(W1),F(W1),	Coefficinet
	Mathematical Mean: -0.00000 Geometrical Mean : none Harmonic Mean : none Variance : 1.00000 S.D. : 1.00000 Skewed Coef. : 0.07874 Kurtosis Coef. : 3.01369 MAD : 0.79758 Range : 10.49156 Mid_range : 0.27561 Median : -0.01294 Q1 : -0.68058 Q2 : -0.01294 Q3 : 0.66668 IQR : 1.34726 C.V. : none

E(| W1 distribution - Z distribution |^2)=0.0003423204
*********** | W1 distribution function - Z distribution function| ***********
The almost surely limiting theory
E(| W1 distribution function - Z distribution function|^2)=0.0000099227,
Pr(| W1 distribution function - Z distribution function|<0.1000000000)= 1.000000,
Pr(| W1 distribution function - Z distribution function|<0.0500000000)= 1.000000,
Pr(| W1 distribution function - Z distribution function|<0.0100000000)= 1.000000,
Pr(| W1 distribution function - Z distribution function|<0.0050000000)= 0.892561,
Pr(| W1 distribution function - Z distribution function|<0.0010000000)= 0.181459,
Pr(| W1 distribution function - Z distribution function|<0.0005000000)= 0.085919,
Pr(| W1 distribution function - Z distribution function|<0.0001000000)= 0.018578,

(2-9) $n(\lambda = 0.9) = 120000$,

f(W1),F(W1),	Coefficinet
	Mathematical Mean: 0.00000 Geometrical Mean : none Harmonic Mean : none Variance : 1.00000 S.D. : 1.00000 Skewed Coef. : -0.07167 Kurtosis Coef. : 3.01126 MAD : 0.79761 Range : 10.27871 Mid_range : -0.34593 Median : 0.01234 Q1 : -0.66727 Q2 : 0.01234 Q3 : 0.67985 IQR : 1.34712 C.V. : none

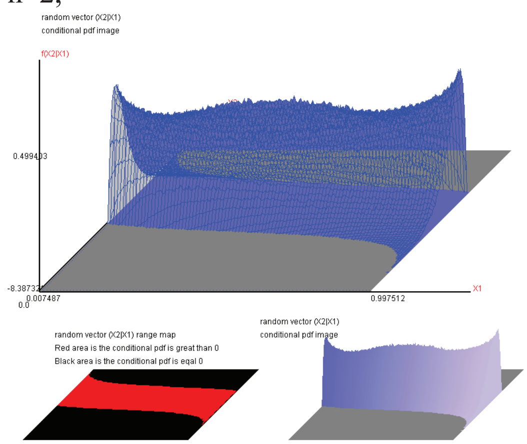
E(| W1 distribution - Z distribution |^2)=0.0002921814
*********** | W1 distribution function - Z distribution function| ***********
The almost surely limiting theory
E(| W1 distribution function - Z distribution function|^2)=0.0000090084,
Pr(| W1 distribution function - Z distribution function|<0.1000000000)= 1.000000,
Pr(| W1 distribution function - Z distribution function|<0.0500000000)= 1.000000,
Pr(| W1 distribution function - Z distribution function|<0.0100000000)= 1.000000,
Pr(| W1 distribution function - Z distribution function|<0.0050000000)= 1.000000,
Pr(| W1 distribution function - Z distribution function|<0.0010000000)= 0.179855,
Pr(| W1 distribution function - Z distribution function|<0.0005000000)= 0.085605,
Pr(| W1 distribution function - Z distribution function|<0.0001000000)= 0.018494,

Section 2, $f\left(\frac{\sqrt{n}\left(\overline{X}-\mu(X)\right)}{\sigma(X)} \mid \lambda\right)$,

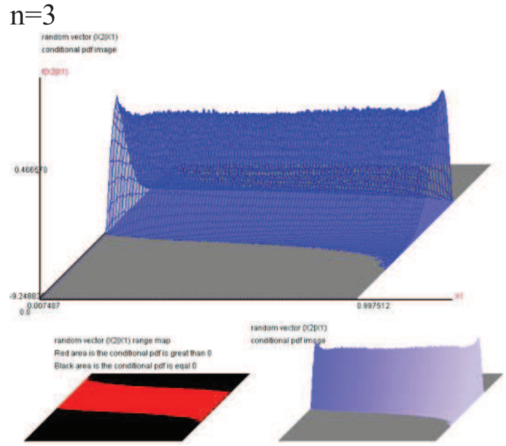
$X2=\frac{\sqrt{n}\left(\overline{X}-\mu(X)\right)}{\sigma(X)}$, the simulator and transformation can get f(X2|X1= λ), $0<\lambda<1$, the simulated data number=100,000,000.

The probability distribution shape is affected by sample size and λ .

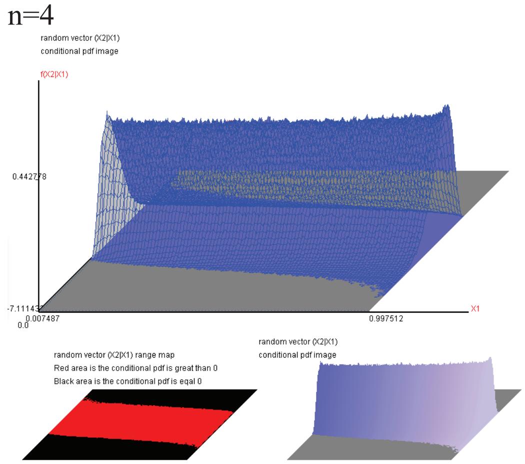
n=2,



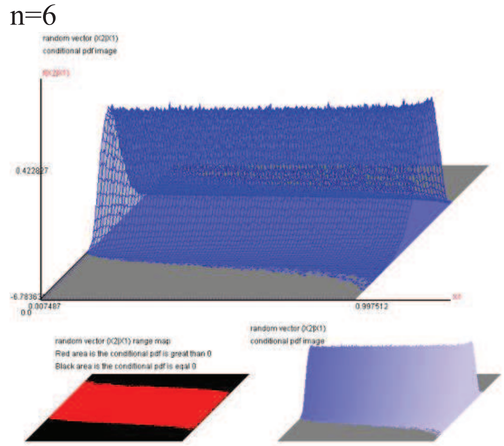
n=3



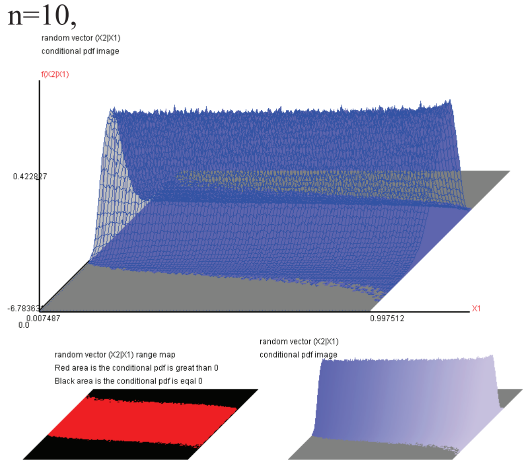
n=4



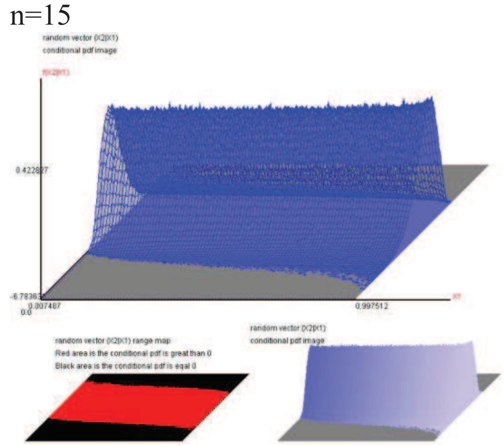
n=6



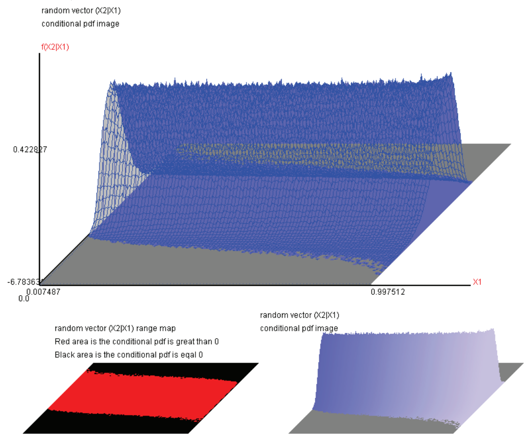
n=10,



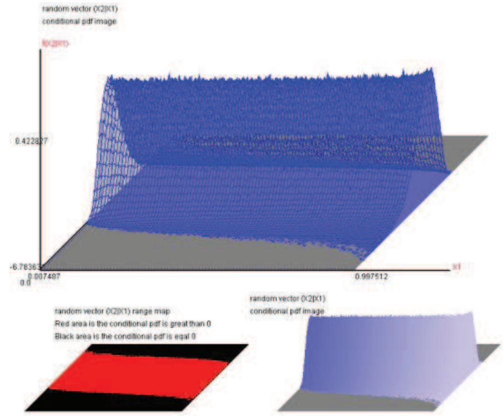
n=15



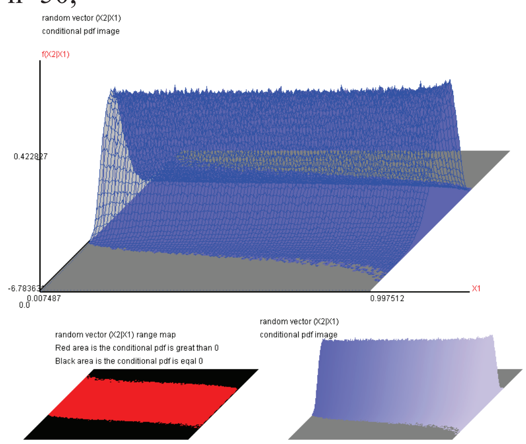
n=20,



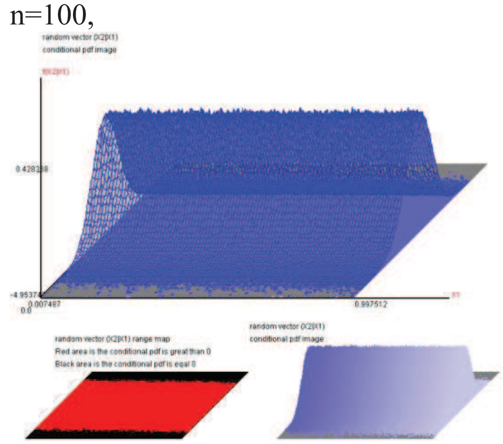
n=25



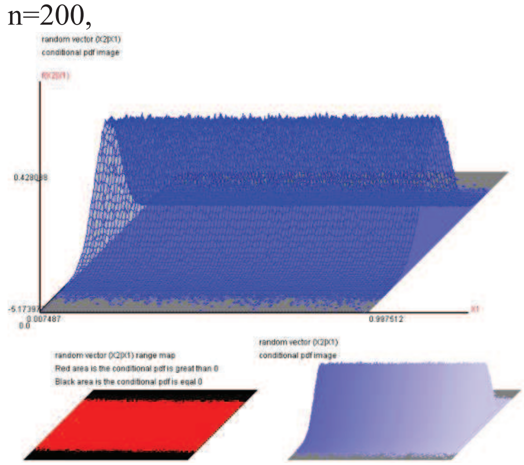
n=50,



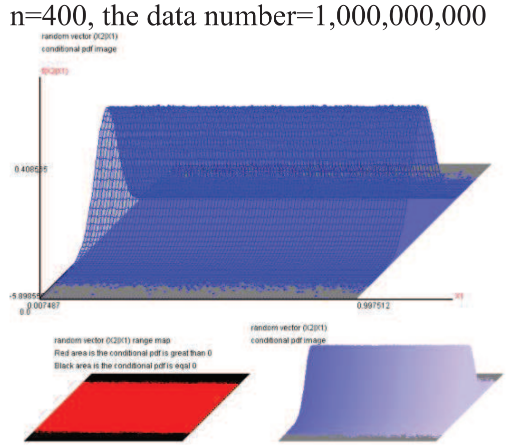
n=100,



n=200,



n=400, the data number=1,000,000,000

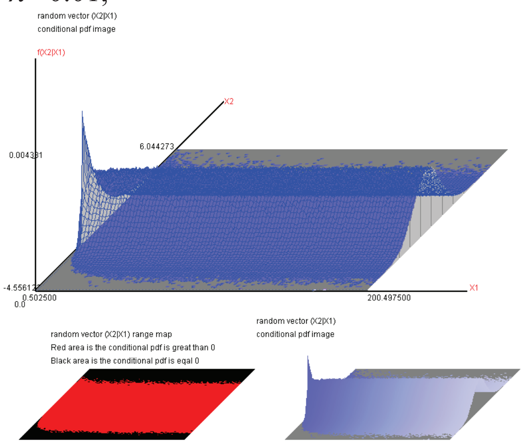


Section 3, $f(\frac{\sqrt{n}\left(\overline{X}-\mu(X)\right)}{\sigma(X)}$ **|n=sample size),**

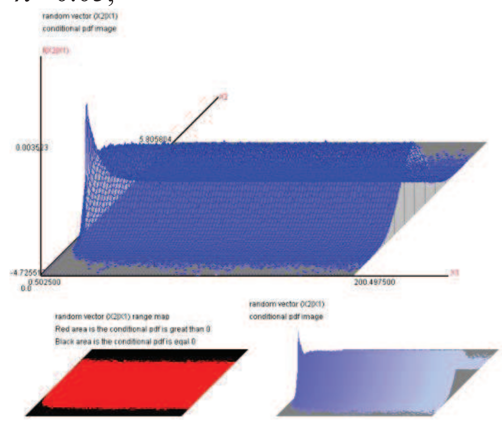
$f(\frac{\sqrt{n}\left(\overline{X}-\mu(X)\right)}{\sigma(X)}$ |n),

X2= $\frac{\sqrt{n}\left(\overline{X}-\mu(X)\right)}{\sigma(X)}$ and X1=n=sample size and n=1,2,..,200, the simulated data number=1,000,000,000,the shape of f(X2|X1) can show the sample size effect. The skewed coefficient will more away from 0 when |lamda-0.5| is increased. The sample size is increasing if test statistic approaching standard normal distribution.

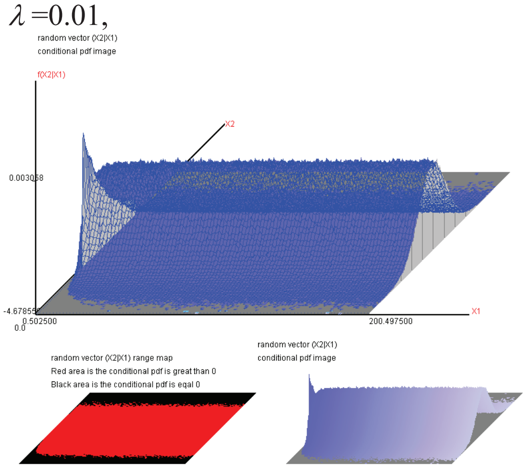
λ =0.01,



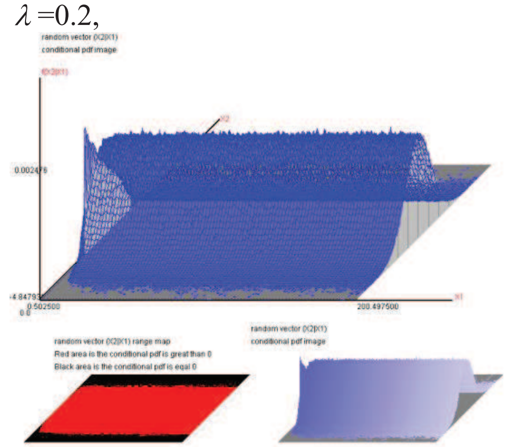
λ =0.05,



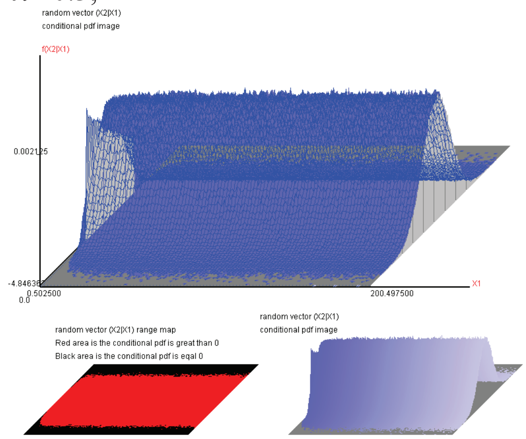
λ =0.01,



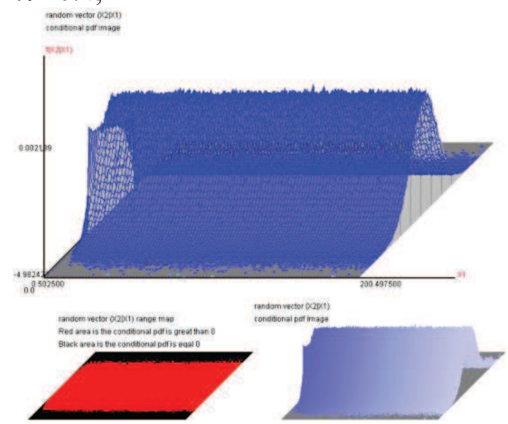
λ =0.2,



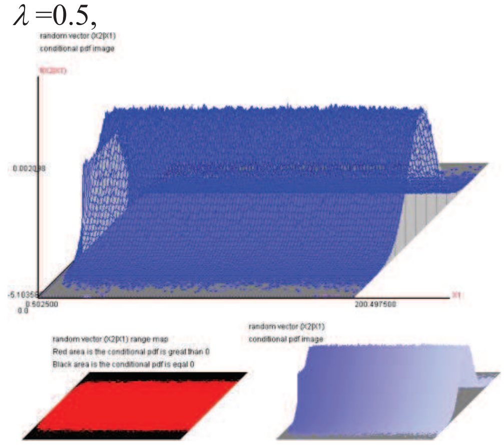
λ =0.3,



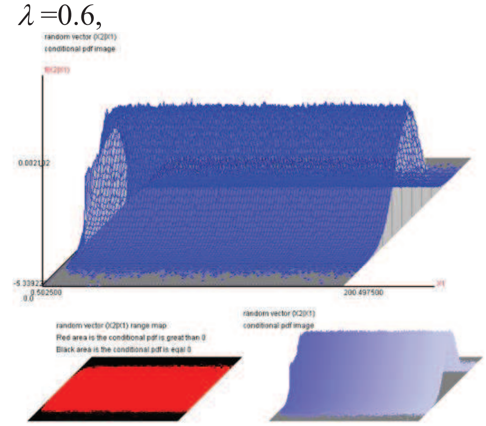
λ =0.4,



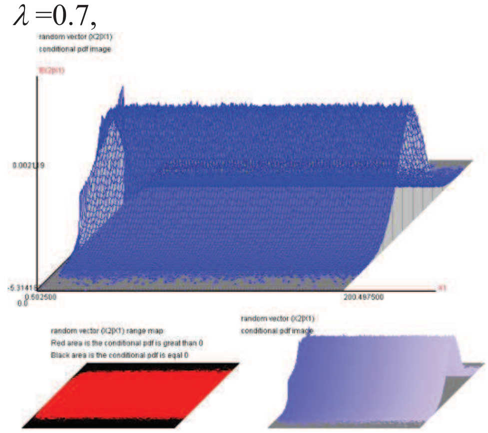
λ =0.5,



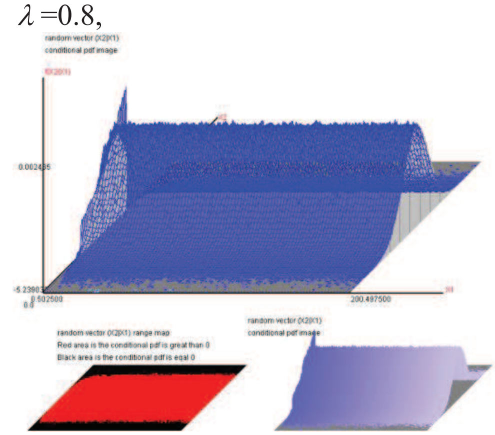
λ =0.6,



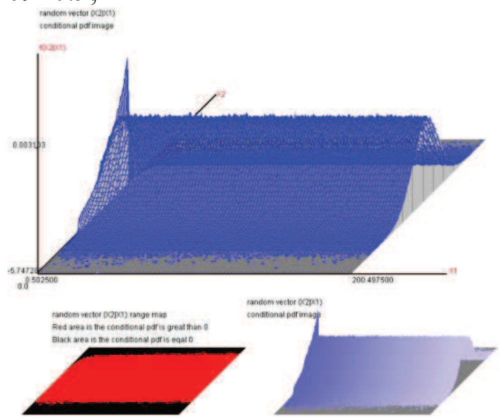
λ =0.7,



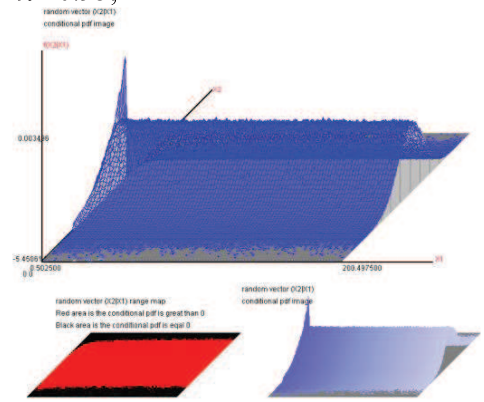
λ =0.8,



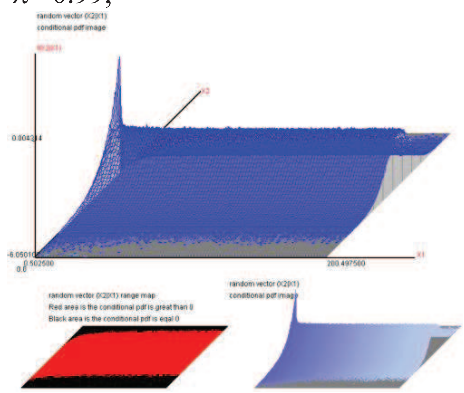
λ =0.9,



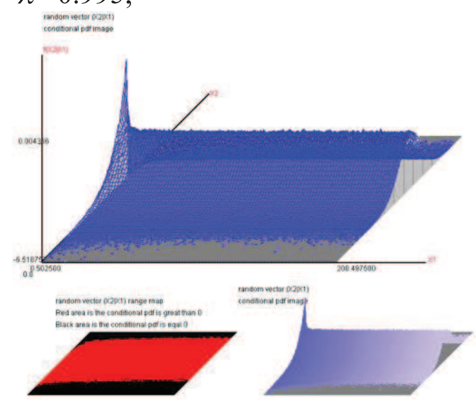
λ =0.95,



λ =0.99,



λ =0.995,



Section 4, The parameter λ test statistic when $X_1, X_2, ..., X_n \overset{iid}{\sim} CB(\lambda)$,

(1) The Z test statistic for large sample,
$n \geq 6+250\times|\lambda$-0.5|, if $0.1 \leq \lambda \leq 0.9$,
$n \geq 100+2000\times(\lambda$-0.1), if λ<0.1,
$n \geq 100+2000\times(\lambda$-0.9),if λ>0.9,

$$\frac{\sqrt{n}\left(\overline{X}-\mu(X)\right)}{\sigma(X)} \longrightarrow Normal(0,1),$$

$H_0:\lambda=c \quad H_0:\lambda=c$,

$$Z^* = \frac{\overline{X}-G_1(c)}{\sqrt{\frac{G_2(c)}{n}}} \rightarrow Z \sim Normal(0,1), \left|Z^*\right| > Z_{\alpha/2} \text{ rejected } H_0 \text{ and } P\left(Z > Z_{\alpha/2}\right) = \frac{\alpha}{2}.$$

$G_1(\lambda)$ is $E(X)$ estimated equation and $G_2(\lambda)$ is $Var(X)$ estimated equation.
$G_1(\lambda)$ and $G_2(\lambda)$ please see chapter 1, section 3.

The test statistic distribution to computing the P($H_0|H_0$),
pr(1-α)=P(doesn't rejected H_0 $|H_0:\lambda=\lambda_0$)=1-α, α=significant level=0.1,0.05,0.01 and pr(1-α) =(the times right test result)/100,000, each probability is from 100,000 times simulated and each time simulated data is the sample size. The simulated data is from Continuous Bernoulli(λ) simulator.

	n	pr(90%)	,pr(95%)	pr(99%)
λ=0.01				
E(X)=0.207514	400	0.901270	0.950920	0.989900
Var(X)=0.037087	500	0.901350	0.950620	0.989690
	600	0.898450	0.949300	0.989780
	1,000	0.899510	0.950370	0.990090
	5,000	0.900170	0.950940	0.990500
	10,000	0.899170	0.949220	0.989930
λ=0.05				
E(X)=0.283806	210	0.899670	0.950210	0.989900
Var(X)=0.056654	300	0.901510	0.950950	0.990060
	500	0.900320	0.950260	0.989820
	1,000	0.900810	0.950750	0.989770
	5,000	0.898540	0.950460	0.990170
	10,000	0.895140	0.946430	0.989330
λ=0.1				
E(X)=0.329809	100	0.900700	0.950820	0.989910
Var(X)=0.066461	200	0.901030	0.950390	0.989740
	400	0.898730	0.949230	0.989730
	600	0.899860	0.950230	0.990100
	1,000	0.898840	0.948990	0.990440
	10,000	0.897180	0.947060	0.989190

	n	pr(90%)	,pr(95%)	pr(99%)
λ=0.2				
E(X)=0.387832	50	0.900730	0.951580	0.990470
Var(X)=0.075884	100	0.901610	0.950830	0.990080
	200	0.900560	0.949700	0.989740
	500	0.899290	0.949630	0.989850
	1,000	0.898650	0.950200	0.990020
	10,000	0.897680	0.948620	0.989100
λ=0.3				
E(X)=0.430251	25	0.901120	0.951770	0.990770
Var(X)=0.080441	40	0.900970	0.951300	0.990860
	50	0.898790	0.949480	0.990160
	100	0.898340	0.950300	0.989930
	1,000	0.900160	0.951080	0.989840
	10,000	0.900280	0.949150	0.989940
λ=0.4				
E(X)=0.466538	12	0.902500	0.952870	0.991340
Var(X)=0.082677	20	0.899200	0.950250	0.990560
	30	0.900090	0.951240	0.990610
	50	0.900430	0.949730	0.990700
	100	0.899370	0.950800	0.990370
	1,000	0.901830	0.951070	0.990070
	10,000	0.898590	0.949070	0.989920
	n	pr(90%)	,pr(95%)	pr(99%)
λ=0.5				
E(X)=0.500057	10	0.901550	0.953000	0.991380
Var(X)=0.083346	20	0.899020	0.950100	0.990750
	30	0.900090	0.950110	0.990020
	50	0.899000	0.950340	0.990650
	100	0.898840	0.950440	0.990670
	1,000	0.900320	0.949950	0.990170
	10,000	0.901130	0.951080	0.990490
λ=0.6				
E(X)=0.533567	12	0.899030	0.950130	0.991150
Var(X)=0.082673	20	0.900840	0.950440	0.990970
	30	0.899500	0.950020	0.990590
	50	0.901080	0.951550	0.990790
	100	0.899800	0.950220	0.990780
	1,000	0.900360	0.950150	0.990220
	10,000	0.898490	0.949730	0.990280
λ=0.7				
E(X)=0.569850	25	0.900730	0.951260	0.991100
Var(X)=0.080434	40	0.900240	0.951790	0.990380
	50	0.900210	0.949890	0.990880
	100	0.899950	0.950380	0.990340
	1,000	0.900170	0.951070	0.990090
	10,000	0.900540	0.950900	0.990260

	n	pr(90%)	,pr(95%)	pr(99%)
λ=0.8				
E(X)=0.612235	50	0.900520	0.950540	0.989800
Var(X)=0.075875	100	0.899560	0.950060	0.989830
	200	0.899480	0.949820	0.990020
	500	0.902130	0.951240	0.990410
	1,000	0.900730	0.950280	0.990510
	10,000	0.898910	0.949800	0.989580
λ=0.9				
E(X)=0.670253	100	0.900170	0.949210	0.989910
Var(X)=0.066451	200	0.900730	0.950720	0.990360
	400	0.900080	0.950210	0.989730
	600	0.899610	0.950270	0.990150
	1,000	0.899590	0.949320	0.989420
	10,000	0.898450	0.949720	0.989650
λ=0.99				
E(X)=0.792923	400	0.900020	0.949940	0.990110
Var(X)=0.036975	500	0.899650	0.949330	0.990030
	600	0.899690	0.950160	0.989600
	1,000	0.899920	0.950330	0.989790
	5,000	0.897040	0.947930	0.989480
	10,000	0.894170	0.946400	0.988960

(2) The test statistic sampling distribution from simulator for small sample,
$n<6+250\times|\lambda-0.5|$, if $0.1\le\lambda\le0.9$,
$n<100+2000\times(\lambda-0.1)$, if $\lambda<0.1$,
$n<100+2000\times(\lambda-0.9)$,if $\lambda>0.9$,
The critical value of test statistic is computed by the simulated sampling distribution of $\frac{\sqrt{n}\left(\overline{X}-\mu(X)\right)}{\sigma(X)}$.

$H_0:\lambda=c$ $H_0:\lambda=c$, the test statistic value= $\frac{\overline{X}-G_1(c)}{\sqrt{\frac{G_2(c)}{n}}}$,

$G_1(\lambda)$ is $E(X)$ estimated equation and $G_2(\lambda)$ is $Var(X)$ estimated equation.
(2-4) The test statistic distribution to computing the P($H_0|H_0$),
pr(1- α)=P(doesn't rejected H_0 $|H_0:\lambda=\lambda_0$)=1- α , α =significant level=0.1,0.05,0.01 and pr(1- α) =(the times right test result)/100,000, each probability is from 100,000 times simulated and each time simulated data is the sample size. The simulated data is from Continuous Bernoulli(λ) simulator.

	n	pr(90%)	,pr(95%)	pr(99%)
λ =0.01				
E(X)=0.207514	5	0.899920	0.950160	0.990650
Var(X)=0.037087	10	0.899220	0.950070	0.989420
	30	0.899780	0.950870	0.989760
	50	0.900250	0.950480	0.990180
	100	0.899770	0.949790	0.990010
	250	0.899360	0.949510	0.989880
λ =0.05				
E(X)=0.283806	5	0.901300	0.949930	0.990690
Var(X)=0.056654	10	0.900010	0.949400	0.989400
	20	0.899670	0.949880	0.989690
	30	0.898830	0.950860	0.989900
	50	0.900220	0.950820	0.989980
	100	0.900160	0.949200	0.989990
	190	0.901250	0.950520	0.989750
λ =0.1				
E(X)=0.329809	5	0.900970	0.949890	0.990610
Var(X)=0.066461	20	0.899580	0.950130	0.989640
	30	0.898750	0.950660	0.990000
	40	0.898360	0.948760	0.989640
	80	0.899540	0.949370	0.989460
	100	0.899570	0.949700	0.990220
λ =0.2				
E(X)=0.387832	5	0.901110	0.950330	0.990610
Var(X)=0.075884	10	0.899830	0.949390	0.989590
	20	0.899490	0.950080	0.989820
	30	0.898890	0.949960	0.989780
	40	0.898660	0.948730	0.989820
	70	0.900970	0.950750	0.990480

	n	pr(90%)	,pr(95%)	pr(99%)
λ=0.3	5	0.900820	0.950350	0.990440
E(X)=0.430251	10	0.900150	0.949480	0.989590
Var(X)=0.080441	15	0.901220	0.950820	0.990430
	20	0.900120	0.950060	0.990060
λ=0.4				
E(X)=0.466538	5	0.900700	0.950430	0.990450
Var(X)=0.082677	8	0.900010	0.951130	0.989670
	10	0.900590	0.949510	0.989840
λ=0.5				
E(X)=0.500057	2	0.899090	0.949920	0.989680
Var(X)=0.083346	5	0.900740	0.950550	0.990440
	8	0.898010	0.950420	0.991350
λ=0.6				
E(X)=0.533567	5	0.901090	0.950800	0.990390
Var(X)=0.082673	8	0.900670	0.951360	0.989670
	10	0.900610	0.949810	0.989730
λ=0.7				
E(X)=0.569850	5	0.901300	0.950780	0.990440
Var(X)=0.080434	10	0.900610	0.949470	0.989520
	20	0.900640	0.950130	0.989850
λ=0.8				
E(X)=0.612235	5	0.901260	0.950580	0.990430
Var(X)=0.075875	10	0.900670	0.949350	0.989420
	20	0.900710	0.950070	0.989760
	30	0.899230	0.948630	0.989930
	40	0.898500	0.948990	0.989640
	70	0.901220	0.951200	0.990440
λ=0.9				
E(X)=0.670253	5	0.901190	0.950590	0.990300
Var(X)=0.066451	10	0.900700	0.949300	0.989680
	20	0.900620	0.949950	0.989690
	30	0.898880	0.949140	0.989720
	50	0.900280	0.950360	0.990260
	80	0.898800	0.949970	0.989740
	100	0.900490	0.950770	0.989940
λ=0.99				
E(X)=0.792923	5	0.901590	0.950590	0.990210
Var(X)=0.036975	10	0.900390	0.949260	0.989740
	30	0.898980	0.948610	0.990020
	50	0.899220	0.950230	0.990470
	100	0.900970	0.950680	0.990280
	250	0.897500	0.949580	0.989850

Note: The computer program is C:\C_Bernoulli\C_Bernoulli_05.exe, which is the testing of λ when population is Continuous Bernoulli population.

Chapter 5, The confidence interval of Continuous Bernoulli distribution

The statistic = $\frac{\sqrt{n}\left(\overline{X}-\mu(X)\right)}{S(X)}, \overline{X}=\frac{\sum_{i=1}^{n} X_i}{n}, S(X)=\sqrt{\frac{\sum_{i=1}^{n}\left(X_i-\overline{X}\right)^2}{n-1}}$, E(X),Var(X) cannot get the value when λ is unknown, the statistic could infer the confidence interval of λ. $\frac{\sqrt{n}\left(\overline{X}-\mu(X)\right)}{S(X)} \xrightarrow[n \geq n(\overline{X})]{} Normal(0,1)$.

The sample size must very large when this statistic approaching standard normal distribution, because the λ is shape parameter. The exception of this statistic is not 0 and variance is not 1 when λ is not 0.5. The sample size is infinite, the exception is 0 and variance is 1.

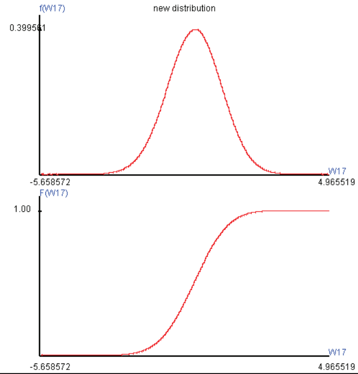
Section 1, $n(\overline{X})=?$ **W17=**$\frac{\sqrt{n}\left(\overline{X}-\mu(X)\right)}{S(X)} \xrightarrow[n \geq n(\overline{X})]{} Normal(0,1)$**,**

Getting the simulated data of W17 and standard normal distribution using the simulator and the simulated data number=100,000,000.
Calculating the $n(\overline{X})$ using the Strong Law of Large Number, the requirement is
$P\{|F_{W17}(W17)-\Phi(W17)|<0.1\}=1, P\{|F_{W17}(W17)-\Phi(W17)|<0.05\}=1,$
$P\{|F_{W17}(W17)-\Phi(W17)|<0.01\}=1, P\{|F_{W17}(W17)-\Phi(W17)|<0.005\}=1,$
when $\frac{\sqrt{n}\left(\overline{X}-\mu(X)\right)}{S(X)} \to Normal(0,1)$.

$\Phi(z)=P(Z \leq z)=\int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} exp\left(-\frac{z^2}{2}\right)dz$ is the distribution function of standard normal distribution.

(1-1) $\lambda = 0.01, n\left(\overline{X}\right) = 2000$,

f(w17),F(w17)

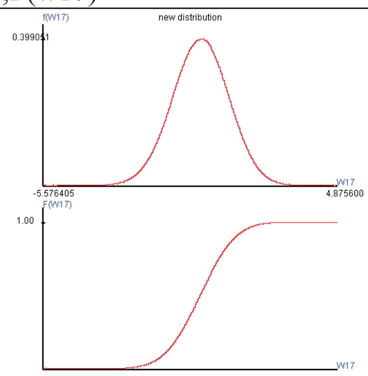


Coefficient		
Mathematical Mean:		-0.01584
Geometrical Mean	:	none
Harmonic Mean	:	none
Variance	:	1.00286
S.D.	:	1.00143
Skewed Coef.	:	-0.06361
Kurtosis Coef.	:	3.01442
MAD	:	0.79862
Range	:	10.66359
Mid_range	:	-0.34653
Median	:	-0.00544
Q1	:	-0.68481
Q2	:	-0.00544
Q3	:	0.66454
IQR	:	1.34935
C.V.	:	none

The almost surely limiting theory
E(| W17 distribution function - Z distribution function|^2)=0.0000094662,
Pr(| W17 distribution function - Z distribution function|<0.1000000000)= 1.000000,
Pr(| W17 distribution function - Z distribution function|<0.0500000000)= 1.000000,
Pr(| W17 distribution function - Z distribution function|<0.0100000000)= 1.000000,
Pr(| W17 distribution function - Z distribution function|<0.0050000000)= 1.000000,
Pr(| W17 distribution function - Z distribution function|<0.0010000000)= 0.008583,
Pr(| W17 distribution function - Z distribution function|<0.0005000000)= 0.003156,
Pr(| W17 distribution function - Z distribution function|<0.0001000000)= 0.000403,

(1-2) $\lambda = 0.03, n\left(\overline{X}\right) = 1550$,

f(w17),F(w17)

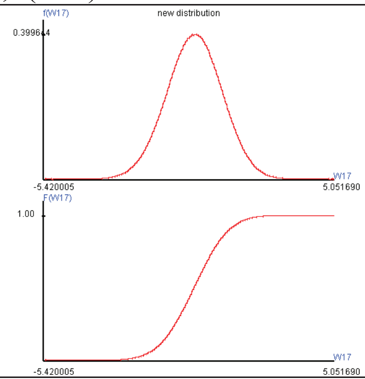


Coefficient		
Mathematical Mean:		-0.01437
Geometrical Mean	:	none
Harmonic Mean	:	none
Variance	:	1.00310
S.D.	:	1.00155
Skewed Coef.	:	-0.05744
Kurtosis Coef.	:	3.01418
MAD	:	0.79879
Range	:	10.49086
Mid_range	:	-0.35040
Median	:	-0.00506
Q1	:	-0.68398
Q2	:	-0.00506
Q3	:	0.66593
IQR	:	1.34991
C.V.	:	none

The almost surely limiting theory
E(| W17 distribution function - Z distribution function|^2)=0.0000079172,
Pr(| W17 distribution function - Z distribution function|<0.1000000000)= 1.000000,
Pr(| W17 distribution function - Z distribution function|<0.0500000000)= 1.000000,
Pr(| W17 distribution function - Z distribution function|<0.0100000000)= 1.000000,
Pr(| W17 distribution function - Z distribution function|<0.0050000000)= 1.000000,
Pr(| W17 distribution function - Z distribution function|<0.0010000000)= 0.010497,
Pr(| W17 distribution function - Z distribution function|<0.0005000000)= 0.003637,
Pr(| W17 distribution function - Z distribution function|<0.0001000000)= 0.000452,

(1-3) $\lambda = 0.05, n(\overline{X}) = 1250,$

f(w17),F(w17)



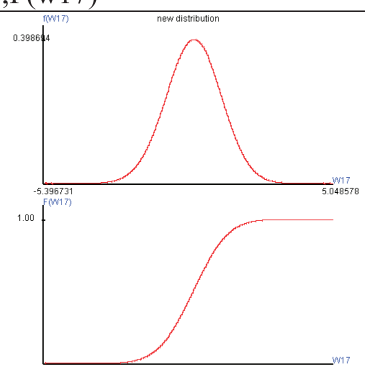
Coefficient	
Mathematical Mean:	-0.01383
Geometrical Mean : none	
Harmonic Mean : none	
Variance :	1.00330
S.D. :	1.00165
Skewed Coef. :	-0.05606
Kurtosis Coef. :	3.01392
MAD :	0.79885
Range :	10.51062
Mid_range :	-0.18416
Median :	-0.00508
Q1 :	-0.68330
Q2 :	-0.00508
Q3 :	0.66660
IQR :	1.34990
C.V. : none	

The almost surely limiting theory

E(| W17 distribution function - Z distribution function|^2)=0.0000072482,
Pr(| W17 distribution function - Z distribution function|<0.1000000000)= 1.000000,
Pr(| W17 distribution function - Z distribution function|<0.0500000000)= 1.000000,
Pr(| W17 distribution function - Z distribution function|<0.0100000000)= 1.000000,
Pr(| W17 distribution function - Z distribution function|<0.0050000000)= 1.000000,
Pr(| W17 distribution function - Z distribution function|<0.0010000000)= 0.009918,
Pr(| W17 distribution function - Z distribution function|<0.0005000000)= 0.003701,
Pr(| W17 distribution function - Z distribution function|<0.0001000000)= 0.000484,

(1-4) $\lambda = 0.06, n(\overline{X}) = 1100,$

f(w17),F(w17)



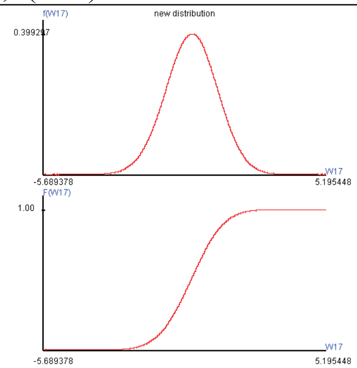
Coefficient	
Mathematical Mean:	-0.01392
Geometrical Mean : none	
Harmonic Mean : none	
Variance :	1.00306
S.D. :	1.00153
Skewed Coef. :	-0.05546
Kurtosis Coef. :	3.01492
MAD :	0.79878
Range :	10.48414
Mid_range :	-0.17408
Median :	-0.00518
Q1 :	-0.68371
Q2 :	-0.00518
Q3 :	0.66612
IQR :	1.34982
C.V. : none	

The almost surely limiting theory

E(| W17 distribution function - Z distribution function|^2)=0.0000076046,
Pr(| W17 distribution function - Z distribution function|<0.1000000000)= 1.000000,
Pr(| W17 distribution function - Z distribution function|<0.0500000000)= 1.000000,
Pr(| W17 distribution function - Z distribution function|<0.0100000000)= 1.000000,
Pr(| W17 distribution function - Z distribution function|<0.0050000000)= 1.000000,
Pr(| W17 distribution function - Z distribution function|<0.0010000000)= 0.010557,
Pr(| W17 distribution function - Z distribution function|<0.0005000000)= 0.003876,
Pr(| W17 distribution function - Z distribution function|<0.0001000000)= 0.000477,

(1-5) $\lambda = 0.08, n(\overline{X}) = 800$,

f(w17),F(w17)

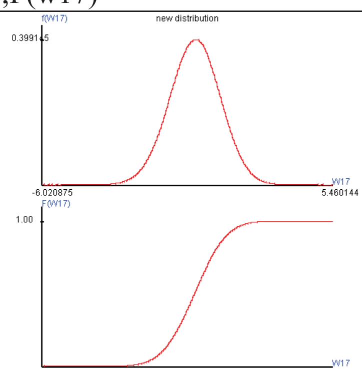


Coefficient	
Mathematical Mean:	-0.01455
Geometrical Mean : none	
Harmonic Mean : none	
Variance :	1.00400
S.D. :	1.00200
Skewed Coef. :	-0.05885
Kurtosis Coef. :	3.01682
MAD :	0.79905
Range :	10.92529
Mid_range :	-0.24697
Median :	-0.00485
Q1 :	-0.68419
Q2 :	-0.00485
Q3 :	0.66581
IQR :	1.34999
C.V. : none	

The almost surely limiting theory
E(| W17 distribution function - Z distribution function|^2)=0.0000080681,
Pr(| W17 distribution function - Z distribution function|<0.1000000000)= 1.000000,
Pr(| W17 distribution function - Z distribution function|<0.0500000000)= 1.000000,
Pr(| W17 distribution function - Z distribution function|<0.0100000000)= 1.000000,
Pr(| W17 distribution function - Z distribution function|<0.0050000000)= 1.000000,
Pr(| W17 distribution function - Z distribution function|<0.0010000000)= 0.010068,
Pr(| W17 distribution function - Z distribution function|<0.0005000000)= 0.003465,
Pr(| W17 distribution function - Z distribution function|<0.0001000000)= 0.000442,

(1-6) $\lambda = 0.1, n(\overline{X}) = 528$,

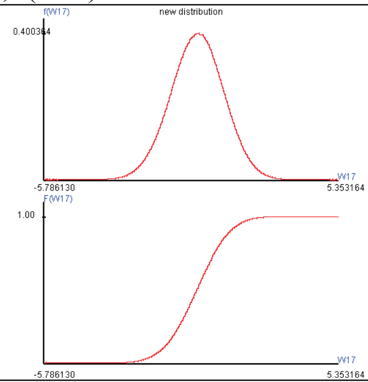
f(w17),F(w17)



Coefficient	
Mathematical Mean:	-0.01629
Geometrical Mean : none	
Harmonic Mean : none	
Variance :	1.00566
S.D. :	1.00283
Skewed Coef. :	-0.06572
Kurtosis Coef. :	3.02463
MAD :	0.79947
Range :	11.52370
Mid_range :	-0.28037
Median :	-0.00514
Q1 :	-0.68547
Q2 :	-0.00514
Q3 :	0.66440
IQR :	1.34988
C.V. : none	

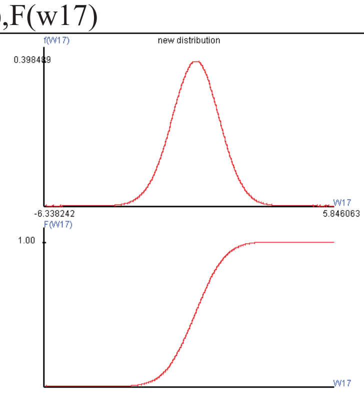
The almost surely limiting theory
E(| W17 distribution function - Z distribution function|^2)=0.0000102790,
Pr(| W17 distribution function - Z distribution function|<0.1000000000)= 1.000000,
Pr(| W17 distribution function - Z distribution function|<0.0500000000)= 1.000000,
Pr(| W17 distribution function - Z distribution function|<0.0100000000)= 1.000000,
Pr(| W17 distribution function - Z distribution function|<0.0050000000)= 1.000000,
Pr(| W17 distribution function - Z distribution function|<0.0010000000)= 0.008997,
Pr(| W17 distribution function - Z distribution function|<0.0005000000)= 0.003328,
Pr(| W17 distribution function - Z distribution function|<0.0001000000)= 0.000399,

(1-7) $\lambda = 0.2, n(\overline{X}) = 264$,

f(w17),F(w17)	Coefficient
	Mathematical Mean: -0.01487 Geometrical Mean : none Harmonic Mean : none Variance : 1.00903 S.D. : 1.00451 Skewed Coef. : -0.06051 Kurtosis Coef. : 3.04095 MAD : 0.80025 Range : 11.18070 Mid_range : -0.21648 Median : -0.00462 Q1 : -0.68436 Q2 : -0.00462 Q3 : 0.66533 IQR : 1.34969 C.V. : none

The almost surely limiting theory
E(| W17 distribution function - Z distribution function|^2)=0.0000085055,
Pr(| W17 distribution function - Z distribution function|<0.1000000000)= 1.000000,
Pr(| W17 distribution function - Z distribution function|<0.0500000000)= 1.000000,
Pr(| W17 distribution function - Z distribution function|<0.0100000000)= 1.000000,
Pr(| W17 distribution function - Z distribution function|<0.0050000000)= 1.000000,
Pr(| W17 distribution function - Z distribution function|<0.0010000000)= 0.011238,
Pr(| W17 distribution function - Z distribution function|<0.0005000000)= 0.004477,
Pr(| W17 distribution function - Z distribution function|<0.0001000000)= 0.000616,

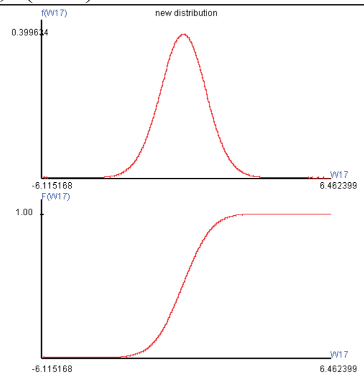
(1-8) $\lambda = 0.3, n(\overline{X}) = 132$,

f(w17),F(w17)	Coefficient
	Mathematical Mean: -0.01310 Geometrical Mean : none Harmonic Mean : none Variance : 1.01690 S.D. : 1.00841 Skewed Coef. : -0.05465 Kurtosis Coef. : 3.07245 MAD : 0.80230 Range : 12.22960 Mid_range : -0.24609 Median : -0.00433 Q1 : -0.68346 Q2 : -0.00433 Q3 : 0.66719 IQR : 1.35065 C.V. : none

The almost surely limiting theory
E(| W17 distribution function - Z distribution function|^2)=0.0000068368,
Pr(| W17 distribution function - Z distribution function|<0.1000000000)= 1.000000,
Pr(| W17 distribution function - Z distribution function|<0.0500000000)= 1.000000,
Pr(| W17 distribution function - Z distribution function|<0.0100000000)= 1.000000,
Pr(| W17 distribution function - Z distribution function|<0.0050000000)= 1.000000,
Pr(| W17 distribution function - Z distribution function|<0.0010000000)= 0.024788,
Pr(| W17 distribution function - Z distribution function|<0.0005000000)= 0.009720,
Pr(| W17 distribution function - Z distribution function|<0.0001000000)= 0.002386,

(1-9) $\lambda = 0.4, n\left(\overline{X}\right) = 66,$

f(w17),F(w17)

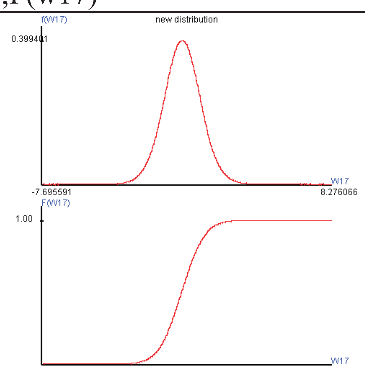


Coefficient		
Mathematical Mean:		-0.00905
Geometrical Mean	:	none
Harmonic Mean	:	none
Variance	:	1.03332
S.D.	:	1.01653
Skewed Coef.	:	-0.03977
Kurtosis Coef.	:	3.14851
MAD	:	0.80642
Range	:	12.62432
Mid_range	:	0.17362
Median	:	-0.00294
Q1	:	-0.68123
Q2	:	-0.00294
Q3	:	0.67055
IQR	:	1.35179
C.V.	:	none

The almost surely limiting theory
E(| W17 distribution function - Z distribution function|^2)=0.0000043226,
Pr(| W17 distribution function - Z distribution function|<0.1000000000)= 1.000000,
Pr(| W17 distribution function - Z distribution function|<0.0500000000)= 1.000000,
Pr(| W17 distribution function - Z distribution function|<0.0100000000)= 1.000000,
Pr(| W17 distribution function - Z distribution function|<0.0050000000)= 1.000000,
Pr(| W17 distribution function - Z distribution function|<0.0010000000)= 0.158343,
Pr(| W17 distribution function - Z distribution function|<0.0005000000)= 0.068154,
Pr(| W17 distribution function - Z distribution function|<0.0001000000)= 0.013522,

(1-10) $\lambda = 0.5, n\left(\overline{X}\right) = 33,$

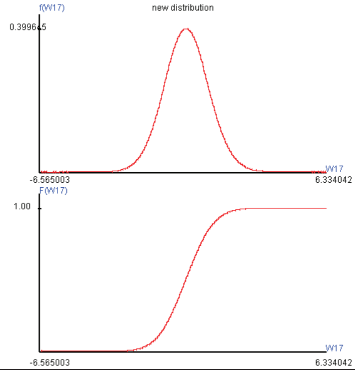
f(w17),F(w17)



Coefficient		
Mathematical Mean:		-0.00008
Geometrical Mean	:	none
Harmonic Mean	:	none
Variance	:	1.06952
S.D.	:	1.03418
Skewed Coef.	:	-0.00071
Kurtosis Coef.	:	3.32993
MAD	:	0.81510
Range	:	16.03103
Mid_range	:	0.29024
Median	:	0.00010
Q1	:	-0.67684
Q2	:	0.00010
Q3	:	0.67669
IQR	:	1.35353
C.V.	:	none

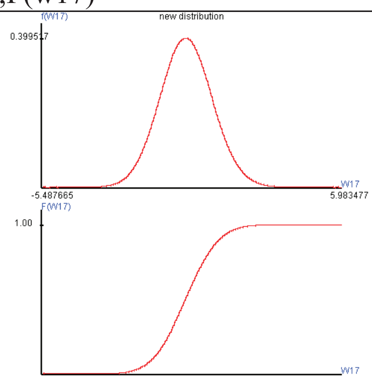
The almost surely limiting theory
E(| W17 distribution function - Z distribution function|^2)=0.0000051275,
Pr(| W17 distribution function - Z distribution function|<0.1000000000)= 1.000000,
Pr(| W17 distribution function - Z distribution function|<0.0500000000)= 1.000000,
Pr(| W17 distribution function - Z distribution function|<0.0100000000)= 1.000000,
Pr(| W17 distribution function - Z distribution function|<0.0050000000)= 1.000000,
Pr(| W17 distribution function - Z distribution function|<0.0010000000)= 0.541121,
Pr(| W17 distribution function - Z distribution function|<0.0005000000)= 0.448275,
Pr(| W17 distribution function - Z distribution function|<0.0001000000)= 0.311869,

(1-11) $\lambda = 0.6, n\left(\overline{X}\right) = 66,$

f(w17),F(w17)	Coefficient
	Mathematical Mean: 0.00889 Geometrical Mean : none Harmonic Mean : none Variance : 1.03293 S.D. : 1.01633 Skewed Coef. : 0.03923 Kurtosis Coef. : 3.14945 MAD : 0.80620 Range : 12.94700 Mid_range : -0.11548 Median : 0.00266 Q1 : -0.67004 Q2 : 0.00266 Q3 : 0.68114 IQR : 1.35119 C.V. : none

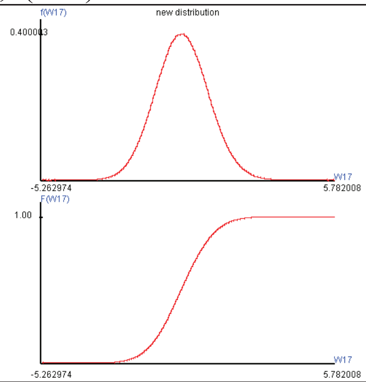
The almost surely limiting theory
E(| W17 distribution function - Z distribution function|^2)=0.0000040656,
Pr(| W17 distribution function - Z distribution function|<0.1000000000)= 1.000000,
Pr(| W17 distribution function - Z distribution function|<0.0500000000)= 1.000000,
Pr(| W17 distribution function - Z distribution function|<0.0100000000)= 1.000000,
Pr(| W17 distribution function - Z distribution function|<0.0050000000)= 1.000000,
Pr(| W17 distribution function - Z distribution function|<0.0010000000)= 0.177088,
Pr(| W17 distribution function - Z distribution function|<0.0005000000)= 0.065803,
Pr(| W17 distribution function - Z distribution function|<0.0001000000)= 0.012367,

(1-12) $\lambda = 0.7, n\left(\overline{X}\right) = 132,$

f(w17),F(w17)	Coefficient
	Mathematical Mean: 0.01303 Geometrical Mean : none Harmonic Mean : none Variance : 1.01659 S.D. : 1.00826 Skewed Coef. : 0.05436 Kurtosis Coef. : 3.07218 MAD : 0.80218 Range : 11.51379 Mid_range : 0.24791 Median : 0.00426 Q1 : -0.66690 Q2 : 0.00426 Q3 : 0.68338 IQR : 1.35028 C.V. : 77.40926

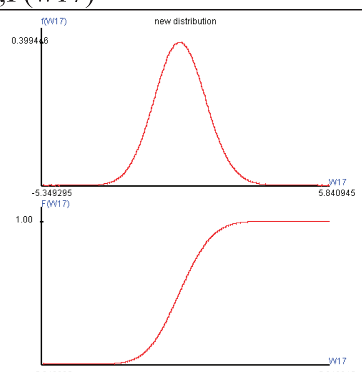
E(| W17 distribution - Z distribution |^2)=0.0004408079
*********** | W17 distribution function - Z distribution function| ***********
The almost surely limiting theory
E(| W17 distribution function - Z distribution function|^2)=0.0000063504,
Pr(| W17 distribution function - Z distribution function|<0.1000000000)= 1.000000,
Pr(| W17 distribution function - Z distribution function|<0.0500000000)= 1.000000,
Pr(| W17 distribution function - Z distribution function|<0.0100000000)= 1.000000,
Pr(| W17 distribution function - Z distribution function|<0.0050000000)= 1.000000,
Pr(| W17 distribution function - Z distribution function|<0.0010000000)= 0.028075,
Pr(| W17 distribution function - Z distribution function|<0.0005000000)= 0.011185,
Pr(| W17 distribution function - Z distribution function|<0.0001000000)= 0.002317,

(1-13) $\lambda = 0.8, n\left(\overline{X}\right) = 264,$

f(w17),F(w17)	Coefficient
	Mathematical Mean: 0.01480 Geometrical Mean : none Harmonic Mean : none Variance : 1.00902 S.D. : 1.00450 Skewed Coef. : 0.06034 Kurtosis Coef. : 3.04041 MAD : 0.80026 Range : 11.08604 Mid_range : 0.25952 Median : 0.00500 Q1 : -0.66543 Q2 : 0.00500 Q3 : 0.68441 IQR : 1.34984 C.V. : 67.87240

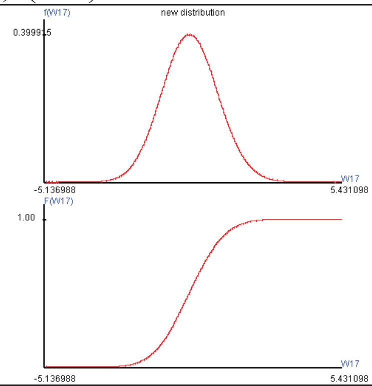
E(| W17 distribution - Z distribution |^2)=0.0004513547
*********** | W17 distribution function - Z distribution function| ***********
The almost surely limiting theory
E(| W17 distribution function - Z distribution function|^2)=0.0000079659,
Pr(| W17 distribution function - Z distribution function|<0.1000000000)= 1.000000,
Pr(| W17 distribution function - Z distribution function|<0.0500000000)= 1.000000,
Pr(| W17 distribution function - Z distribution function|<0.0100000000)= 1.000000,
Pr(| W17 distribution function - Z distribution function|<0.0050000000)= 1.000000,
Pr(| W17 distribution function - Z distribution function|<0.0010000000)= 0.011788,
Pr(| W17 distribution function - Z distribution function|<0.0005000000)= 0.004322,
Pr(| W17 distribution function - Z distribution function|<0.0001000000)= 0.000513,

(1-14) $\lambda = 0.9, n\left(\overline{X}\right) = 528,$

f(w17),F(w17)	Coefficient
	Mathematical Mean: 0.01628 Geometrical Mean : none Harmonic Mean : none Variance : 1.00576 S.D. : 1.00288 Skewed Coef. : 0.06598 Kurtosis Coef. : 3.02388 MAD : 0.79951 Range : 11.23184 Mid_range : 0.24582 Median : 0.00535 Q1 : -0.66478 Q2 : 0.00535 Q3 : 0.68535 IQR : 1.35013 C.V. : 61.60868

E(| W17 distribution - Z distribution |^2)=0.0005164726
*********** | W17 distribution function - Z distribution function| ***********
The almost surely limiting theory
E(| W17 distribution function - Z distribution function|^2)=0.0000096211,
Pr(| W17 distribution function - Z distribution function|<0.1000000000)= 1.000000,
Pr(| W17 distribution function - Z distribution function|<0.0500000000)= 1.000000,
Pr(| W17 distribution function - Z distribution function|<0.0100000000)= 1.000000,
Pr(| W17 distribution function - Z distribution function|<0.0050000000)= 1.000000,
Pr(| W17 distribution function - Z distribution function|<0.0010000000)= 0.008793,
Pr(| W17 distribution function - Z distribution function|<0.0005000000)= 0.003211,
Pr(| W17 distribution function - Z distribution function|<0.0001000000)= 0.000362,

(1-15) $\lambda = 0.99, n(\overline{X}) = 2000,$

f(w17),F(w17)	Coefficient		
	Mathematical Mean:		0.01586
	Geometrical Mean : none		
	Harmonic Mean	: none	
	Variance	:	1.00287
	S.D.	:	1.00143
	Skewed Coef.	:	0.06289
	Kurtosis Coef.	:	3.01256
	MAD	:	0.79884
	Range	:	10.60737
	Mid_range	:	0.14705
	Median	:	0.00538
	Q1	:	-0.66499
	Q2	:	0.00538
	Q3	:	0.68549
	IQR	:	1.35048
	C.V.	:	63.14381

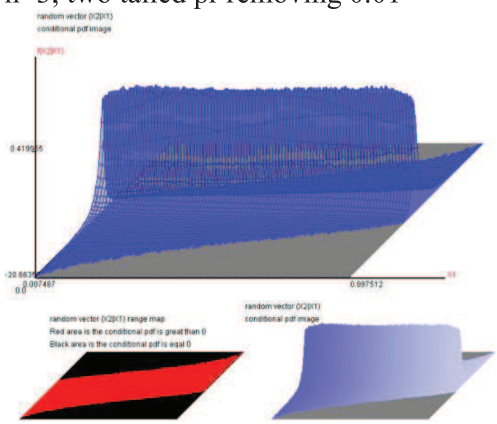
The almost surely limiting theory
E(| W17 distribution function - Z distribution function|^2)=0.0000093140,
Pr(| W17 distribution function - Z distribution function|<0.1000000000)= 1.000000,
Pr(| W17 distribution function - Z distribution function|<0.0500000000)= 1.000000,
Pr(| W17 distribution function - Z distribution function|<0.0100000000)= 1.000000,
Pr(| W17 distribution function - Z distribution function|<0.0050000000)= 1.000000,
Pr(| W17 distribution function - Z distribution function|<0.0010000000)= 0.009328,
Pr(| W17 distribution function - Z distribution function|<0.0005000000)= 0.003253,
Pr(| W17 distribution function - Z distribution function|<0.0001000000)= 0.000389,

Section 2, f($\frac{\sqrt{n}\left(\overline{X}-\mu(X)\right)}{S(X)}|\lambda$),

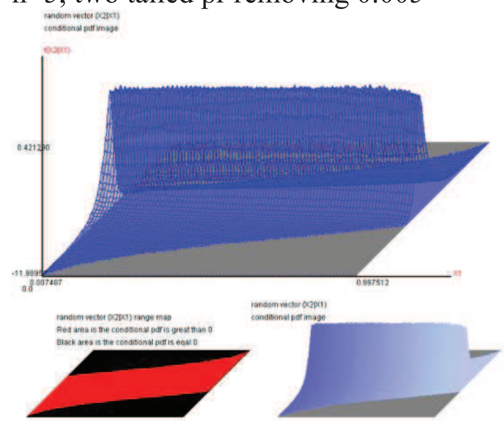
X2=$\frac{\sqrt{n}\left(\overline{X}-\mu(X)\right)}{S(X)}$, the simulator and transformation can get f(X2|X1=λ), $0<\lambda<1$, the simulated data number=100,000,000.

The probability distribution shape is affected by sample size and λ.

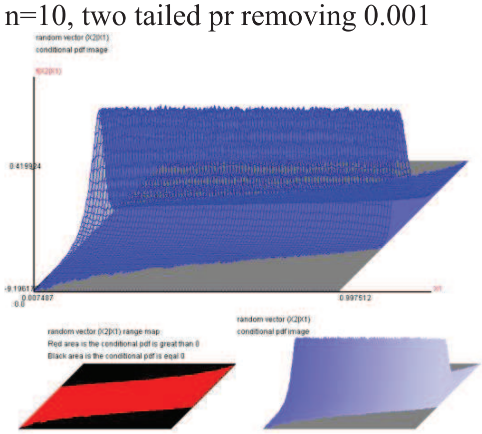
n=3, two tailed pr removing 0.01



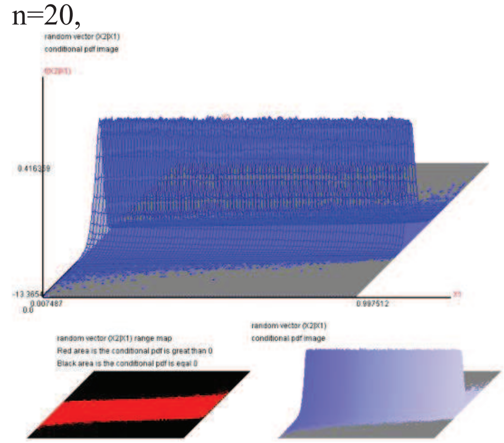
n=5, two tailed pr removing 0.005



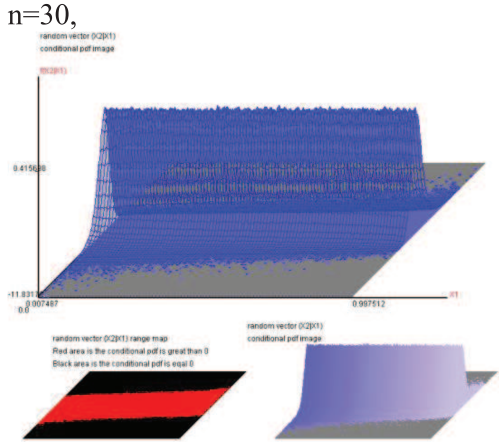
n=10, two tailed pr removing 0.001



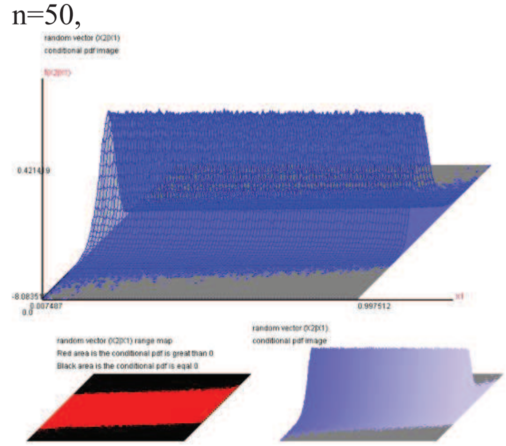
n=20,



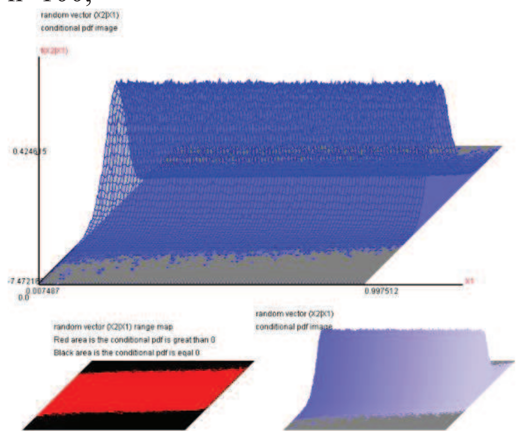
n=30,



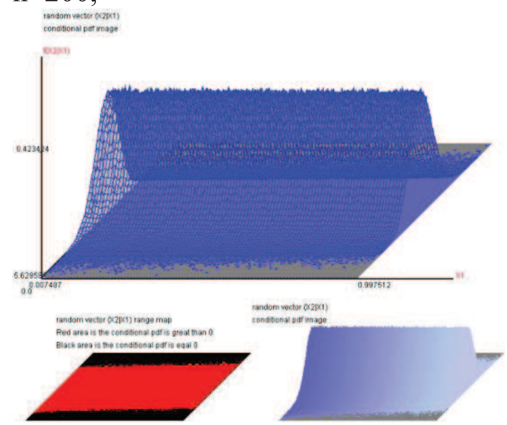
n=50,



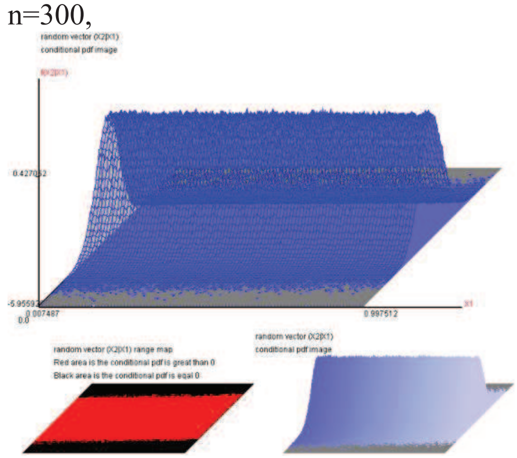
n=100,



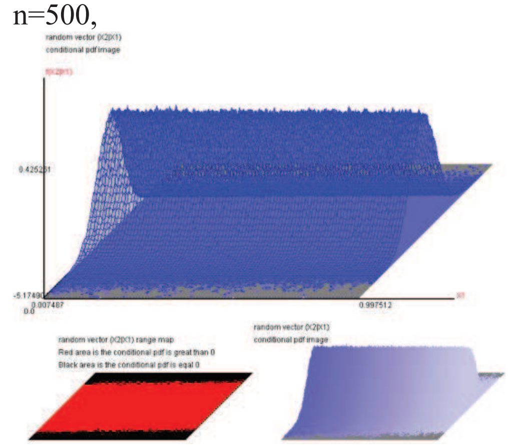
n=200,



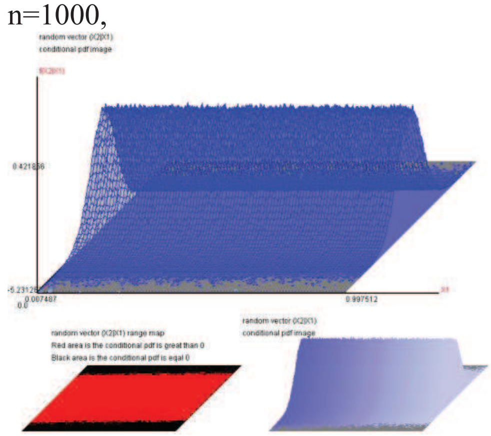
n=300,



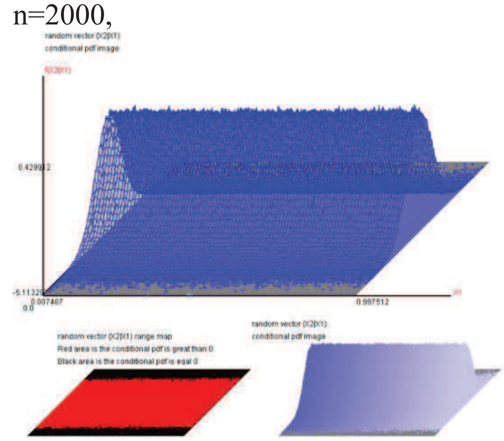
n=500,



n=1000,



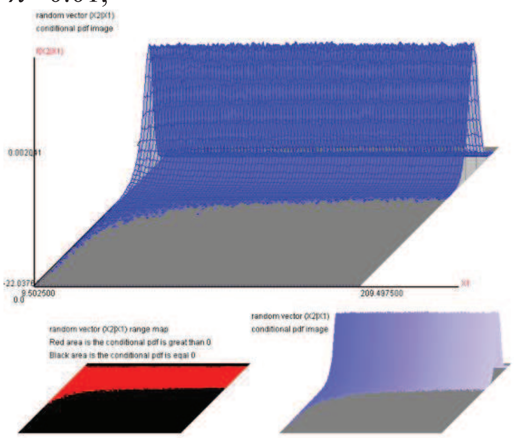
n=2000,



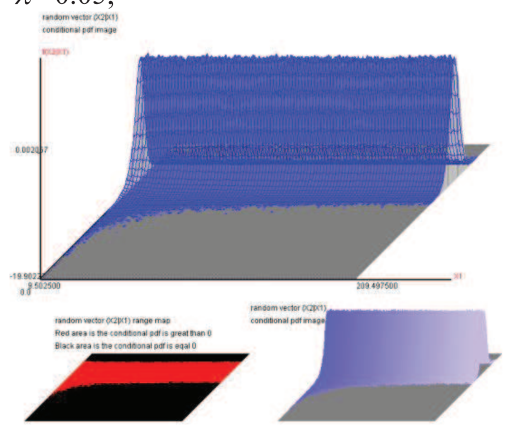
Section 3, f($\frac{\sqrt{n}\left(\overline{X}-\mu(X)\right)}{S(X)}$|n=sample size),

X2=$\frac{\sqrt{n}\left(\overline{X}-\mu(X)\right)}{S(X)}$ and X1=n=sample size, n=10,11,12,…,208,209, the simulated data number=1,000,000,000,the shape of f(X2|X1) can show the sample size effect. The λ is more far from 0.5 and the skewed coefficient of this statistic is more far from 0 when sample size is small. The statistic will be approaching to the symmetric when n is very large. The following each diagram two tailed probabilities are removing 0.00001.

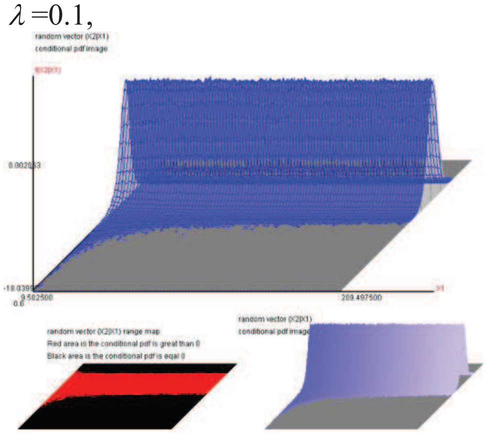
λ=0.01,



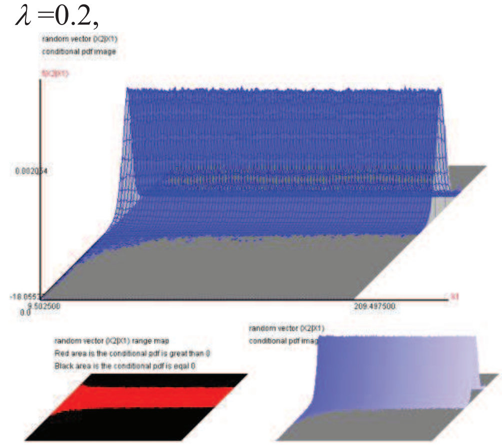
λ=0.05,



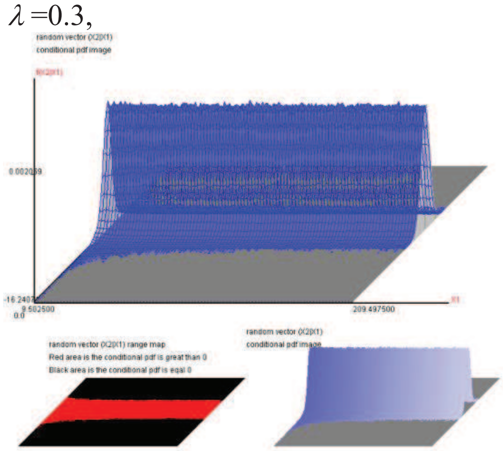
λ=0.1,



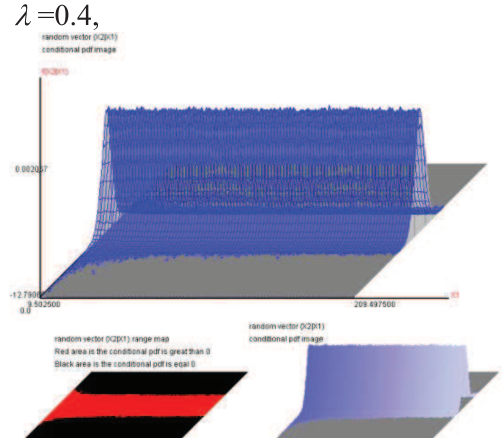
λ=0.2,



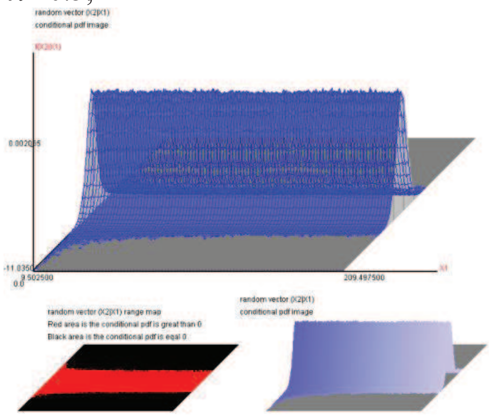
λ=0.3,



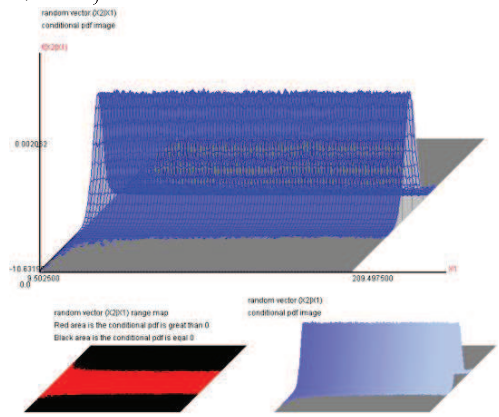
λ=0.4,



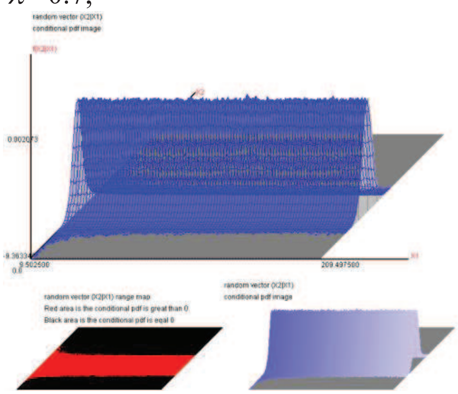
λ =0.5,



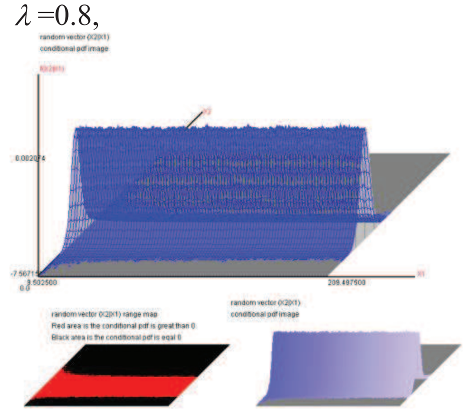
λ =0.6,



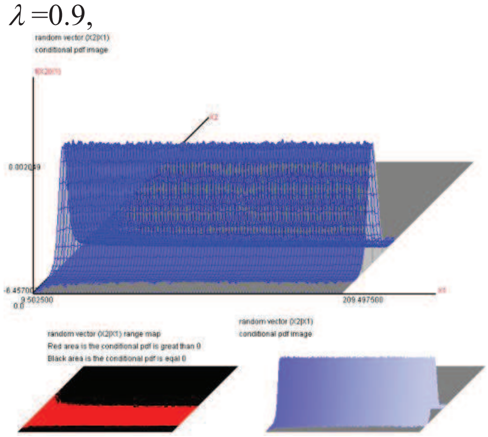
λ =0.7,



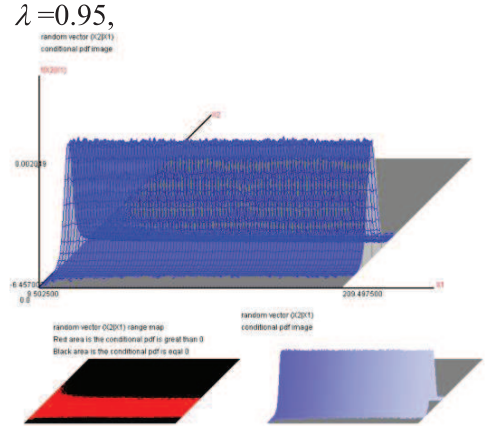
λ =0.8,



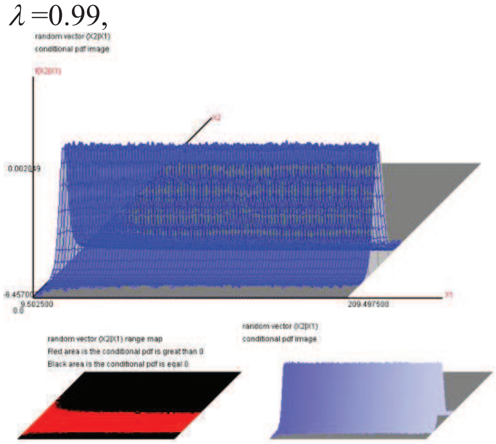
λ =0.9,



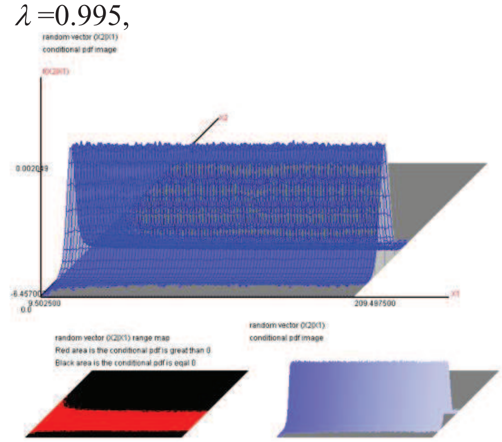
λ =0.95,



λ =0.99,



λ =0.995,



Section 4, The Confidence interval of λ,

(1) The confidence interval of λ for large sample,
The sample size is affected by the λ when this statistic approaching standard normal distribution.
$\hat{\lambda}=\phi(\overline{X})$, $0.143853919 \le \overline{X} \le 0.856221427$ and $0.001 \le \hat{\lambda} \le 0.999$.
$n \ge 33+350 \times |\hat{\lambda}=\phi(\overline{X})-0.5|$, if $0.1 \le \hat{\lambda} \le 0.9$,
$n \ge 500+15000 \times (0.1-\hat{\lambda}=\phi(\overline{X}))$, if $\hat{\lambda}=\phi(\overline{X})<0.1$,
$n \ge 500+15000 \times (\hat{\lambda}=\phi(\overline{X})-0.9)$,if $\hat{\lambda}=\phi(\overline{X})>0.9$,

$$\frac{(\overline{X}-\mu(X))}{S(\overline{X})} \longrightarrow Normal(0,1), \overline{X}=\frac{\sum_{i=1}^{n} X_i}{n}, S(X)=\sqrt{\frac{\sum_{i=1}^{n}(X_i-\overline{X})^2}{n-1}}, S(\overline{X})=\frac{S(X)}{\sqrt{n}},$$

$(1-\alpha)\times 100\%$ C.I. for $E(\overline{X})=\mu$
$\overline{X}-Z_{\alpha/2}\times S(\overline{X}) \le \mu \le \overline{X}+Z_{\alpha/2}\sqrt{S^2(\overline{X})}$,
$P(Z>Z_{\alpha})=\alpha$, Z is the standard normal distribution,

$(1-\alpha)\times 100\%$ C.I. for λ
$\phi(\overline{X}-Z_{\alpha/2}\times S(\overline{X})) \le \lambda \le \phi(\overline{X}+Z_{\alpha/2}\times S(\overline{X}))$

Checking the right probability when the C.I. for λ at the confidence interval, computing the right probability of confirming and the simulated times is changed to 1,000,000 for the accurate when using Z distribution to do confidence interval.
P(C.I. containing λ)$=1-\alpha$, the C.I. is the confidence interval of λ at $1-\alpha$, α=0.1,0.05,0.01.
(1-1) The λ is continuous bernoulli parameter value and computing the sample size requirement for CLT,
$n \ge 33+350 \times |\lambda-0.5|$, if $0.1 \le \lambda \le 0.9$,
$n \ge 500+15000 \times (0.1-\lambda)$, if $\lambda<0.1$,
$n \ge 500+15000 \times (\lambda-0.9)$,if $\lambda>0.9$,

	n	90% C.I.	95% C.I.	99% C.I.
λ=0.01				
	3,000	0.900505	0.950215	0.989620
	4,000	0.901065	0.949845	0.990065
	5,000	0.899645	0.949640	0.990120
	8,000	0.899790	0.949340	0.989860
	10,000	0.900485	0.949685	0.989845
λ=0.05				
	2,000	0.900240	0.950140	0.989960
	4,000	0.898095	0.948985	0.989640
	5,000	0.900680	0.949720	0.989860
	6,000	0.901025	0.951080	0.989895
	8,000	0.899695	0.950215	0.989920
	10,000	0.898615	0.949430	0.989370

	n	90% C.I.	95% C.I.	99% C.I.
λ =0.1				
	600	0.899075	0.948880	0.989335
	800	0.898905	0.949465	0.989505
	1,000	0.899815	0.949840	0.989500
	2,000	0.899545	0.949715	0.989455
	5,000	0.899170	0.949555	0.989715
	10,000	0.899660	0.949290	0.989855
λ =0.2				
	270	0.899140	0.949290	0.989145
	400	0.897890	0.948295	0.989275
	800	0.899545	0.950140	0.989660
	1,000	0.898120	0.948460	0.989420
	5,000	0.899440	0.949610	0.989960
	10,000	0.900520	0.950720	0.990195
λ =0.3				
	150	0.898825	0.948715	0.988950
	200	0.898350	0.948335	0.989060
	500	0.900060	0.950120	0.990155
	1,000	0.898745	0.949560	0.989920
	5,000	0.899595	0.949775	0.989905
	10,000	0.900160	0.950070	0.990300
λ =0.4				
	70	0.895365	0.945905	0.987150
	100	0.897145	0.947800	0.988630
	200	0.898160	0.948260	0.988735
	500	0.899235	0.949195	0.989555
	1,000	0.899085	0.948885	0.989775
	5,000	0.901930	0.949910	0.989815
	10,000	0.898610	0.949410	0.989975
λ =0.5				
	35	0.891346	0.941718	0.984796
	50	0.893659	0.943555	0.986068
	100	0.898384	0.947118	0.988253
	200	0.899027	0.948804	0.989157
	500	0.899124	0.949427	0.989530
	1,000	0.899107	0.949654	0.989860
	10,000	0.899831	0.949755	0.990077
λ =0.6				
	70	0.895914	0.945756	0.987593
	100	0.897033	0.947035	0.988277
	200	0.898369	0.948562	0.988939
	500	0.899249	0.949378	0.989691
	1,000	0.899834	0.950020	0.989984
	5,000	0.899699	0.949652	0.990061
	10,000	0.900199	0.950187	0.989956

	n	90% C.I.	95% C.I.	99% C.I.
λ =0.7				
	150	0.897281	0.947588	0.988693
	200	0.898266	0.948077	0.988858
	500	0.899269	0.949266	0.989517
	1,000	0.899945	0.949908	0.989812
	5,000	0.900028	0.949633	0.989840
	10,000	0.900385	0.950264	0.990192
λ =0.8				
	270	0.898917	0.948731	0.989265
	400	0.898715	0.948703	0.989392
	1,000	0.899571	0.949728	0.989903
	2,000	0.899534	0.949790	0.989785
	5,000	0.899893	0.949936	0.989942
	10,000	0.899537	0.949818	0.989918
λ =0.9				
	600	0.899140	0.949255	0.989555
	800	0.899185	0.949140	0.989556
	1,000	0.899433	0.949262	0.989836
	2,000	0.899854	0.949810	0.989978
	5,000	0.900380	0.950224	0.990090
	10,000	0.898989	0.949133	0.989589
λ =0.99				
	3,000	0.899048	0.949600	0.989948
	4,000	0.899391	0.949511	0.989852
	5,000	0.899889	0.949950	0.989842
	8,000	0.900068	0.950004	0.989882
	10,000	0.900071	0.950004	0.990086

(1-2) The computing the sample size by $\hat{\lambda}=\phi(\overline{X})$,

The confidence interval is from Z distribution when the sample size is large sample and the confidence interval is from sampling distribution of $\overline{X}$ when sample size is small sample.

$\hat{\lambda}=\phi(\overline{X})$, $0.143853919 \le \overline{X} \le 0.856221427$ and $0.001 \le \hat{\lambda} \le 0.999$.

The large sample is $n \ge 33+350\times|\hat{\lambda}=\phi(\overline{X})-0.5|$, if $0.1 \le \hat{\lambda} \le 0.9$,

$n \ge 500+15000\times(0.1-\hat{\lambda}=\phi(\overline{X}))$, if $\hat{\lambda}=\phi(\overline{X})<0.1$,

$n \ge 500+15000\times(\hat{\lambda}=\phi(\overline{X})-0.9)$, if $\hat{\lambda}=\phi(\overline{X})>0.9$,

	n	90% C.I.	95% C.I.	99% C.I.
λ=0.01				
	2,000	0.899814	0.949779	0.989928
	3,000	0.899950	0.949887	0.989921
	5,000	0.899825	0.950076	0.990035
	10,000	0.899739	0.949491	0.989734
λ=0.05				
	1,500(*)	0.900188	0.949617	0.989673
	3,000	0.900190	0.950375	0.990107
	5,000	0.900071	0.950028	0.989892
	10,000	0.900127	0.949951	0.989982
λ=0.1				
	1080(*)	0.899839	0.949582	0.989695
	1,500	0.900126	0.949749	0.989797
	3,000	0.900331	0.950092	0.989928
	5,000	0.899697	0.949476	0.989911
λ=0.2				
	400(*)	0.900024	0.949660	0.989641
	800	0.899517	0.949793	0.989745
	1,000	0.899321	0.9499296	0.989530
	2,000	0.900125	0.949884	0.989761
λ=0.3				
	170(*)	0.898181	0.948315	0.988909
	300	0.899103	0.948971	0.989288
	500	0.899457	0.949406	0.989773
	1,000	0.899878	0.949736	0.989885
λ=0.4				
	140(*)	0.898391	0.948058	0.988730
	300	0.898572	0.948802	0.989357
	500	0.899625	0.949655	0.989614
	1,000	0.899966	0.949842	0.9897811
λ=0.5				
	120(*)	0.897300	0.947570	0.988611
	200	0.898299	0.948476	0.989197
	500	0.899565	0.949496	0.989712
	1,000	0.900089	0.949858	0.989823

	n	90% C.I.	95% C.I.	99% C.I.
λ=0.6				
	150(*)	0.897906	0.948188	0.988869
	500	0.899677	0.949765	0.989730
	1,000	0.899784	0.949707	0.989904
λ=0.7				
	168(*)	0.898060	0.948317	0.989046
	500	0.898918	0.949052	0.989493
	1,000	0.899404	0.949638	0.989665
λ=0.8				
	405(*)	0.899247	0.949639	0.989602
	1,000	0.899669	0.949710	0.989831
	2,000	0.899972	0.949771	0.989888
λ=0.9				
	1,050(*)	0.899274	0.949087	0.989627
	3,000	0.899349	0.949414	0.989901
	5,000	0.900296	0.950281	0.989971
	10,000	0.900017	0.950119	0.989804
λ=0.99				
	2,000	0.899705	0.949349	0.989560
	3,000	0.899470	0.949437	0.989469
	5,000	0.899218	0.949684	0.989927
	10,000	0.899529	0.949559	0.989802

(*) is the part of confidence interval critical value is used to the sampling distribution, part is from the standard normal distribution.

(2)The small sample,

n<33+350$\times$| $\hat{\lambda}=\phi(\overline{X})$-0.5|, if $0.1\leq\hat{\lambda}\leq 0.9$,

n<500+15000$\times$(0.1- $\hat{\lambda}=\phi(\overline{X})$), if $\hat{\lambda}=\phi(\overline{X})<0.1$,

n<500+15000$\times$($\hat{\lambda}=\phi(\overline{X})$-0.9),if $\hat{\lambda}=\phi(\overline{X})$>0.9,

$(1-\alpha)\times 100\%$ C.I. for $E(\overline{X})=\mu$

$\overline{X}-W_{\alpha/2}\times S(\overline{X})\leq\mu\leq\overline{X}+W_{\alpha/2}\sqrt{S^2(\overline{X})}$,

$P(W>W_\alpha)=\alpha,$ W is the sampling distribution of $\frac{(\overline{X}-\mu(X))}{S(\overline{X})}$ which can be simulated using the continuous bernoulli distribution simulator. The λ and sample size will be a specific sampling distribution, the software computing critical value is a essentially way.

Warning:

Because the sample size too small that $\hat{\lambda}=\phi(\overline{X})$ might be not used when $\overline{X}$ is not in [0.143853919, 0.856221427], the minimum sample number requirement as follows. The simulated times=100,000, $\hat{\lambda}=\phi(\overline{X})$ cannot work which is “error”.

λ=0.01, n>=270, P(error)=0.001098,
λ=0.1, n>=55, P(error)=0.001420,
λ=0.2, n>=38, P(error)=0.001198,
λ=0.3, n>=30, P(error)=0.001250,
λ=0.4, n>=25, P(error)=0.001296,
λ=0.5, n>=22, P(error)=0.001613,
λ=0.6, n>=25, P(error)=0.001289,
λ=0.7, n>=30, P(error)=0.001238,
λ=0.8, n>=38, P(error)=0.001119,
λ=0.9, n>=55, P(error)=0.001425,
λ=0.99, n>=260, P(error)=0.001399,

Note: The computer program is C:\C_Bernoulli\C_Bernoulli_6.exe.

Chapter 6, The test statistic and confidence interval of two Continuous Bernoulli populations,

The test statistic is about two independent continuous Bernoulli populations $\mu_1 - \mu_2$ and inferring to $\lambda_1 - \lambda_2$, which is in according to the chapter 5 and chapter 6.

$$X_{1,1}, X_{1,2}, ..., X_{1,n_1} \overset{iid}{\sim} CB(\lambda_1), \overline{X}_1 = \frac{\sum_{i=1}^{n_1} X_{1,i}}{n_1}, S_1 = \sqrt{\frac{\sum_{i=1}^{n_1}\left(X_{1,i} - \overline{X}_1\right)^2}{n_1 - 1}},$$

$\mu_1 = G_1(\lambda_1)$, ($G_1(\ )$, chapter 1, section 3).

$$X_{2,1}, X_{2,2}, ..., X_{2,n_2} \overset{iid}{\sim} CB(\lambda_2), \overline{X}_2 = \frac{\sum_{j=1}^{n_2} X_{2,j}}{n_2}, S_2 = \sqrt{\frac{\sum_{j=1}^{n_2}\left(X_{2,j} - \overline{X}_2\right)^2}{n_2 - 1}},$$

$\mu_2 = G_1(\lambda_2)$, ($G_1(\ )$,chapter 1, section 3).

Section 1, The test statistic of $H_0: \mu_1 = \mu_2 + c$, $c \neq 0$,

λ_1 and λ_2 are unknown, $\hat{\lambda}_1 = \phi(\overline{X}_1), \hat{\lambda}_2 = \phi(\overline{X}_2)$, and $\lambda_1 = \phi(\mu_1), \lambda_2 = \phi(\mu_2)$.
($\phi(\ )$,chapter 3, section 3).

If $\mu_1 \neq \mu_2$, $\lambda_1 \neq \lambda_2$,

the test statistic= $\dfrac{\overline{X}_1 - \overline{X}_2 - (\mu_1 - \mu_2)}{\sqrt{\dfrac{S_1^2}{n_1} + \dfrac{S_2^2}{n_2}}}$

(1) The large sample,

$\hat{\lambda}_1 = \phi(\overline{X}_1)$, $0.143853919 \leq \overline{X}_1 \leq 0.856221427$ and $0.001 \leq \hat{\lambda}_1 \leq 0.999$.

The large sample is $n_1 \geq 33 + 350 \times |\hat{\lambda}_1 - 0.5|$, if $0.1 \leq \hat{\lambda}_1 \leq 0.9$,

$n_1 \geq 500 + 15000 \times (0.1 - \hat{\lambda}_1)$, if $\hat{\lambda}_1 < 0.1$,

$n_1 \geq 500 + 15000 \times (\hat{\lambda}_1 - 0.9)$,if $\hat{\lambda}_1 > 0.9$,

and

$\hat{\lambda}_2 = \phi(\overline{X}_2)$, $0.143853919 \leq \overline{X}_2 \leq 0.856221427$ and $0.001 \leq \hat{\lambda}_2 \leq 0.999$.

The large sample is $n_2 \geq 33 + 350 \times |\hat{\lambda}_2 - 0.5|$, if $0.1 \leq \hat{\lambda}_2 \leq 0.9$,

$n_2 \geq 500 + 15000 \times (0.1 - \hat{\lambda}_2)$, if $\hat{\lambda}_2 < 0.1$,

$n_2 \geq 500 + 15000 \times (\hat{\lambda}_2 - 0.9)$,if $\hat{\lambda}_2 > 0.9$,

$H_0: \mu_1 = \mu_2 + c$, $c \neq 0$,

$$Z^* = \frac{\overline{X}_1 - \overline{X}_2 - c}{\sqrt{\dfrac{S_1^2}{n_1} + \dfrac{S_2^2}{n_2}}} \longrightarrow Z \text{ (standard normal distribution)},$$

$Z^* > Z_{\alpha/2}$, H_0 is rejected.

p value= $2 \times P(Z \leq Z^*)$, if $P(Z \leq Z^*) < 0.5$

p value= $2 \times (1 - P(Z \leq Z^*))$, if $P(Z \leq Z^*) \geq 0.5$

(2) The small sample,

$\hat{\lambda}_1 = \phi(\overline{X}_1)$, $0.143853919 \le \overline{X}_1 \le 0.856221427$ and $0.001 \le \hat{\lambda}_1 \le 0.999$.

The large sample is $n_1 < 33+350\times|\hat{\lambda}_1 - 0.5|$, if $0.1 \le \hat{\lambda}_1 \le 0.9$,

$n_1 < 500+15000\times(0.1-\hat{\lambda}_1)$, if $\hat{\lambda}_1 < 0.1$,

$n_1 < 500+15000\times(\hat{\lambda}_1 - 0.9)$, if $\hat{\lambda}_1 > 0.9$,

or

$\hat{\lambda}_2 = \phi(\overline{X}_2)$, $0.143853919 \le \overline{X}_2 \le 0.856221427$ and $0.001 \le \hat{\lambda}_2 \le 0.999$.

The large sample is $n_2 < 33+350\times|\hat{\lambda}_2 - 0.5|$, if $0.1 \le \hat{\lambda}_2 \le 0.9$,

$n_2 < 500+15000\times(0.1-\hat{\lambda}_2)$, if $\hat{\lambda}_2 < 0.1$,

$n_2 < 500+15000\times(\hat{\lambda}_2 - 0.9)$, if $\hat{\lambda}_2 > 0.9$,

$H_0 : \mu_1 = \mu_2 + c,\ c \ne 0$,

$$W^* = \frac{\overline{X}_1 - \overline{X}_2 - c}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}},$$

the sampling distribution of $W = \frac{\overline{X}_1 - \overline{X}_2 - (\hat{\mu}_1 - \hat{\mu}_2)}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}}$ will be simulated using the probability simulator and $\hat{\mu}_1 = G_1(\hat{\lambda}_1)$ and $\hat{\mu}_2 = G_1(\hat{\lambda}_2)$,

the simulated data is based on

$$X_{1,1}, X_{1,2}, ..., X_{1,n_1} \overset{iid}{\sim} CB(\hat{\lambda}_1), \overline{X}_1 = \frac{\sum_{i=1}^{n_1} X_{1,i}}{n_1}, S_1 = \sqrt{\frac{\sum_{i=1}^{n_1}(X_{1,i} - \overline{X}_1)^2}{n_1 - 1}},$$

$$X_{2,1}, X_{2,2}, ..., X_{2,n_2} \overset{iid}{\sim} CB(\hat{\lambda}_2), \overline{X}_2 = \frac{\sum_{j=1}^{n_2} X_{2,j}}{n_2}, S_2 = \sqrt{\frac{\sum_{j=1}^{n_2}(X_{2,j} - \overline{X}_2)^2}{n_2 - 1}},$$

p value= $2\times P(W \le W^*)$, if $P(Z \le Z^*) < 0.5$

p value= $2\times(1 - P(W \le W^*))$, if $P(Z \le Z^*) \ge 0.5$

(3)The λ_1 and λ_2 estimated value,

(i) $H_0 : \mu_1 = \mu_2 + c,\ c \ne 0$ is rejected,

$\hat{\lambda}_1 = \phi(\overline{X}_1)$, $\hat{\lambda}_2 = \phi(\overline{X}_2)$.

(ii) $H_0 : \mu_1 = \mu_2 + c,\ c \ne 0$ is not rejected,

$$\hat{\lambda}_1 = \phi\left(\frac{\sum_{i=1}^{n_1} X_{1,i} + \sum_{j=1}^{n_2}(X_{2,j} + c)}{n_1 + n_2}\right), \quad \hat{\lambda}_2 = \phi\left(\frac{\sum_{i=1}^{n_1}(X_{1,i} - c) + \sum_{j=1}^{n_2} X_{2,j}}{n_1 + n_2}\right).$$

Note: The computer program is C:\C_Bernoulli\C_Bernoulli_09.exe.

Section 2, The test statistic of $H_0: \mu_1 = \mu_2$,

λ_1 and λ_2 are unknown, $\hat{\lambda}_1 = \phi(\overline{X}_1), \hat{\lambda}_2 = \phi(\overline{X}_2)$, and $\lambda_1 = \phi(\mu_1), \lambda_2 = \phi(\mu_2)$.

If $\mu_1 = \mu_2$, $\lambda_1 = \lambda_2 = \lambda$,

$$\overline{\overline{X}} = \frac{n_1 \overline{X}_1 + n_2 \overline{X}_2}{n_1 + n_2}, S_p^2 = \frac{\sum_{i=1}^{n_1}\left(X_{1,i} - \overline{\overline{X}}\right)^2 + \sum_{j=1}^{n_2}\left(X_{2,j} - \overline{\overline{X}}\right)^2}{n_1 + n_2 - 1},$$

the test statistic= $\dfrac{\overline{X}_1 - \overline{X}_2 - (\mu_1 - \mu_2)}{\sqrt{\dfrac{S_p^2}{n_1} + \dfrac{S_p^2}{n_2}}}$

(1) The large sample,

$\hat{\lambda} = \phi(\overline{\overline{X}})$,$0.143853919 \le \overline{\overline{X}} \le 0.856221427$ and $0.001 \le \hat{\lambda} \le 0.999$.

The large sample is $n_1 + n_2 \ge 33 + 350 \times |\hat{\lambda} - 0.5|$, if $0.1 \le \hat{\lambda} \le 0.9$,

$n_1 + n_2 \ge 500 + 15000 \times (0.1 - \hat{\lambda}_1)$, if $\hat{\lambda} < 0.1$,

$n_1 + n_2 \ge 500 + 15000 \times (\hat{\lambda}_1 - 0.9)$,if $\hat{\lambda} > 0.9$,

$H_0: \mu_1 = \mu_2$,

$$Z^* = \frac{\overline{X}_1 - \overline{X}_2}{\sqrt{\dfrac{S_p^2}{n_1} + \dfrac{S_p^2}{n_2}}} \longrightarrow Z \text{ (standard normal distribution)},$$

$Z^* > Z_{\alpha/2}$, H_0 is rejected.

p value= $2 \times P(Z \le Z^*)$, if $P(Z \le Z^*) < 0.5$

p value= $2 \times (1 - P(Z \le Z^*))$, if $P(Z \le Z^*) \ge 0.5$

(2) The small sample,

$\hat{\lambda} = \phi(\overline{\overline{X}})$,$0.143853919 \le \overline{\overline{X}} \le 0.856221427$ and $0.001 \le \hat{\lambda} \le 0.999$.

The large sample is $n_1 + n_2 < 33 + 350 \times |\hat{\lambda} - 0.5|$, if $0.1 \le \hat{\lambda} \le 0.9$,

$n_1 + n_2 < 500 + 15000 \times (0.1 - \hat{\lambda}_1)$, if $\hat{\lambda} < 0.1$,

$n_1 + n_2 < 500 + 15000 \times (\hat{\lambda}_1 - 0.9)$,if $\hat{\lambda} > 0.9$,

$$H_0: \mu_1 = \mu_2, \quad W^* = \frac{\overline{X}_1 - \overline{X}_2}{\sqrt{\dfrac{S_p^2}{n_1} + \dfrac{S_p^2}{n_2}}},$$

the sampling distribution of $W = \dfrac{\overline{X}_1 - \overline{X}_2}{\sqrt{\dfrac{S_p^2}{n_1} + \dfrac{S_p^2}{n_2}}}$ will be simulated using the probability simulator and $\hat{\lambda} = \phi(\overline{\overline{X}})$, the simulated data is based on

$$X_{1,1}, X_{\backslash 1,2}, ..., X_{1,n_1} \overset{iid}{\sim} CB(\hat{\lambda}), X_{2,1}, X_{\backslash 2,2}, ..., X_{2,n_2} \overset{iid}{\sim} CB(\hat{\lambda}),$$

$$\overline{\overline{X}} = \frac{n_1 \overline{X}_1 + n_2 \overline{X}_2}{n_1 + n_2,}, S_p^2 = \frac{\sum_{i=1}^{n_1}\left(X_{1,i} - \overline{\overline{X}}\right)^2 + \sum_{j=1}^{n_2}\left(X_{2,j} - \overline{\overline{X}}\right)^2}{n_1 + n_2 - 1},$$

p value= $2 \times P(W \le W^*)$, if $P(Z \le Z^*) < 0.5$
p value= $2 \times (1 - P(W \le W^*))$, if $P(Z \le Z^*) \ge 0.5$

(3)The λ_1 and λ_2 estimated value,
(i) $H_0 : \mu_1 = \mu_2$ is rejected,
$\hat{\lambda}_1 = \phi(\overline{X}_1)$, $\hat{\lambda}_2 = \phi(\overline{X}_2)$.
(ii) $H_0 : \mu_1 = \mu_2 \neq$ is not rejected,
$\hat{\lambda}_1 = \hat{\lambda}_2 = \hat{\lambda} = \phi(\overline{\overline{X}})$,.

Note: The computer program is C:\C_Bernoulli\C_Bernoulli_10.exe.

Section 3, The confidence interval of $\mu_1-\mu_2$ and $\lambda_1-\lambda_2$

λ_1 and λ_2 are unknown, $\hat{\lambda}_1=\phi(\overline{X}_1), \hat{\lambda}_2=\phi(\overline{X}_2)$, and $\lambda_1=\phi(\mu_1), \lambda_2=\phi(\mu_2)$.

If $\mu_1 \neq \mu_2$, $\lambda_1 \neq \lambda_2$,

the statistic= $\dfrac{\overline{X}_1-\overline{X}_2-(\mu_1-\mu_2)}{\sqrt{\dfrac{S_1^2}{n_1}+\dfrac{S_2^2}{n_2}}}$

(1) The large sample,

$\hat{\lambda}_1=\phi(\overline{X}_1)$,$0.143853919 \le \overline{X}_1 \le 0.856221427$ and $0.001 \le \hat{\lambda}_1 \le 0.999$.

The large sample is $n_1 \ge 33+350\times|\hat{\lambda}_1-0.5|$, if $0.1 \le \hat{\lambda}_1 \le 0.9$,

$n_1 \ge 500+15000\times(0.1-\hat{\lambda}_1)$, if $\hat{\lambda}_1 < 0.1$,

$n_1 \ge 500+15000\times(\hat{\lambda}_1-0.9)$,if $\hat{\lambda}_1>0.9$,

and

$\hat{\lambda}_2=\phi(\overline{X}_2)$,$0.143853919 \le \overline{X}_2 \le 0.856221427$ and $0.001 \le \hat{\lambda}_2 \le 0.999$.

The large sample is $n_2 \ge 33+350\times|\hat{\lambda}_2-0.5|$, if $0.1 \le \hat{\lambda}_2 \le 0.9$,

$n_2 \ge 500+15000\times(0.1-\hat{\lambda}_2)$, if $\hat{\lambda}_2 < 0.1$,

$n_2 \ge 500+15000\times(\hat{\lambda}_2-0.9)$,if $\hat{\lambda}_2>0.9$,

$$\frac{\overline{X}_1-\overline{X}_2-(\mu_1-\mu_2)}{\sqrt{\dfrac{S_1^2}{n_1}+\dfrac{S_2^2}{n_2}}}\longrightarrow Z \text{ (standard normal distribution)},$$

$(1-\alpha)\times 100\%$ C.I. of $\mu_1-\mu_2$

$$\overline{X}_1-\overline{X}_2-Z_{\alpha/2}\sqrt{\frac{S_1^2}{n_1}+\frac{S_2^2}{n_2}} \le \mu_1-\mu_2 \le \overline{X}_1-\overline{X}_2+Z_{\alpha/2}\sqrt{\frac{S_1^2}{n_1}+\frac{S_2^2}{n_2}}$$

(2) The small sample,

$\hat{\lambda}_1=\phi(\overline{X}_1)$,$0.143853919 \le \overline{X}_1 \le 0.856221427$ and $0.001 \le \hat{\lambda}_1 \le 0.999$.

The large sample is $n_1 < 33+350\times|\hat{\lambda}_1-0.5|$, if $0.1 \le \hat{\lambda}_1 \le 0.9$,

$n_1 < 500+15000\times(0.1-\hat{\lambda}_1)$, if $\hat{\lambda}_1 < 0.1$,

$n_1 < 500+15000\times(\hat{\lambda}_1-0.9)$,if $\hat{\lambda}_1>0.9$,

or

$\hat{\lambda}_2=\phi(\overline{X}_2)$,$0.143853919 \le \overline{X}_2 \le 0.856221427$ and $0.001 \le \hat{\lambda}_2 \le 0.999$.

The large sample is $n_2 < 33+350\times|\hat{\lambda}_2-0.5|$, if $0.1 \le \hat{\lambda}_2 \le 0.9$,

$n_2 < 500+15000\times(0.1-\hat{\lambda}_2)$, if $\hat{\lambda}_2 < 0.1$,

$n_2 < 500+15000\times(\hat{\lambda}_2-0.9)$,if $\hat{\lambda}_2>0.9$, the statistic= $\dfrac{\overline{X}_1 - \overline{X}_2 - (\mu_1 - \mu_2)}{\sqrt{\dfrac{S_1^2}{n_1} + \dfrac{S_2^2}{n_2}}}$,

the sampling distribution of $W = \dfrac{\overline{X}_1 - \overline{X}_2 - (\hat{\mu}_1 - \hat{\mu}_2)}{\sqrt{\dfrac{S_1^2}{n_1} + \dfrac{S_2^2}{n_2}}}$ will be simulated using the probability simulator and $\hat{\mu}_1 = G_1(\hat{\lambda}_1)$ and $\hat{\mu}_2 = G_1(\hat{\lambda}_2)$,

the simulated data is based on

$$X_{1,1}, X_{1,2}, \ldots, X_{1,n_1} \overset{iid}{\sim} CB(\hat{\lambda}_1), \overline{X}_1 = \frac{\sum_{i=1}^{n_1} X_{1,i}}{n_1}, S_1 = \sqrt{\frac{\sum_{i=1}^{n_1} (X_{1,i} - \overline{X}_1)^2}{n_1 - 1}},$$

$$X_{2,1}, X_{2,2}, \ldots, X_{2,n_2} \overset{iid}{\sim} CB(\hat{\lambda}_2), \overline{X}_2 = \frac{\sum_{j=1}^{n_2} X_{2,j}}{n_2}, S_2 = \sqrt{\frac{\sum_{j=1}^{n_2} (X_{2,j} - \overline{X}_2)^2}{n_2 - 1}},$$

$(1-\alpha)\times 100\%$ C.I. of $\mu_1 - \mu_2$

$$\overline{X}_1 - \overline{X}_2 + W_{1-\alpha}\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}} \le \mu_1 - \mu_2 \le \overline{X}_1 - \overline{X}_2 + W_{\alpha}\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}$$

$P(W > W_\alpha) = \alpha,$

Note: $(1-\alpha)\times 100\%$ C.I. of $\mu_1 - \mu_2$ cannot convert to
$(1-\alpha)\times 100\%$ C.I. of $\lambda_1 - \lambda_2$.

Let $\hat{\lambda}_2 = \phi(\overline{X}_2), \hat{\lambda}_{L,1} = \phi\left(\overline{X}_1 + W_{1-\alpha}\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}\right), \hat{\lambda}_{U,1} = \phi\left(\overline{X}_1 + W_{\alpha}\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}\right),$

$(1-\alpha)\times 100\%$ C.I. of $\lambda_1 - \lambda_2$

$\hat{\lambda}_{L,1} - \hat{\lambda}_2 \le \lambda_1 - \lambda_2 \le \hat{\lambda}_{U,1} - \hat{\lambda}_2$

Note: The computer program is C:\C_Bernoulli\C_Bernoulli_11.exe.

Chapter 7, Goodness of fit about Continuous Bernoulli distribution,

$X_1, X_2, ..., X_n \overset{iid}{\sim} CB(\lambda)$, n random samples is from $CB(\lambda)$, the frequency table of sample is getting and suppose population is $CB(\lambda)$. The goodness of fit will be applied to determine the samples is from $CB(\lambda)$ population.

Section 1, λ is known,

(1)The goodness of fit,

$X_1, X_2, ..., X_n \overset{iid}{\sim} CB(\lambda)$,.

H_0:X~Continuous Bernoulli(λ) and λ is known,

H_1:against H_0,

The test process,

The frequency distribution setting,

(i)The class number and the probability of each class,

The class number= $k = log_2(n) + 1$, each class probability is setting to $\frac{1}{k}$.

(ii)The class limit,

The first class lower limit=0 and the last class upper limit=1.

$$c_j = \begin{cases} \dfrac{log_e\left(\frac{j}{k} \times (2\lambda - 1) - (\lambda - 1)\right) - log_e(1 - \lambda)}{log_e\left(\frac{\lambda}{1 - \lambda}\right)}, \hat{\lambda} \neq \frac{1}{2} \\ \frac{j}{k}, \hat{\lambda} = \frac{1}{2} \end{cases}, j = 1,2,...,k-1,,$$

The first class upper limit= c_1=the second class lower limit,…..,

The $j-th$ class upper limit= c_j=the $(j+1)-th$ class lower limit, $j = 1,2,...,k-1$.

(iii)The frequency table for testing and computing the observed number and expected number,

class	class limit	frequency= O	$E = n \times \frac{1}{k}$
1	$0 \sim c_1$	O_1	E_1
2	$c_1 \sim c_2$	O_2	E_2
…			
k	$c_{k-1} \sim 1$	O_k	E_k

The chi square test statistic,

$\chi^2_{k-1} = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i} > \chi^2_{\alpha,k-1}$, rejected H_0.

(2)Confirming the test,

H_0: X~Continuous Bernoulli($\lambda = \lambda_0$), H_1:against H_0,

The chi square test statistic,

$$\chi^2_{k-1} = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i} > \chi^2_{\alpha,k-1}\text{, rejected } H_0 .$$

pr(1- α)=P(doesn't rejected H_0 | H_0:X~ Continuous Bernoulli(λ))=1- α ,

The pr(1- α) =(the times right test result)/100,000, each probability is from 100,000 times simulated and each time simulated data is the sample size. The simulated data is from Continuous Bernoulli(λ) simulator.

	sample size	pr(90%)	,pr(95%)	pr(99%)
λ =0.1= λ_0				
	10	0.891491	0.963930	0.988730
	20	0.901961	0.949261	0.990490
	100	0.902461	0.952280	0.990100
	1,000	0.898181	0.949021	0.990010
	10,000	0.898951	0.948471	0.989840
λ =0.2= λ_0				
	10	0.891491	0.963930	0.988730
	20	0.901771	0.949391	0.991020
	100	0.901991	0.951860	0.990080
	1,000	0.900401	0.949901	0.989730
	10,000	0.899321	0.949491	0.989910
λ =0.3= λ_0				
	10	0.891491	0.963930	0.988730
	20	0.901771	0.9499391	0.991020
	100	0.901991	0.951860	0.990080
	1,000	0.900401	0.949901	0.989730
	10,000	0.899321	0.949491	0.989910
λ =0.4= λ_0				
	10	0.891491	0.963930	0.988730
	20	0.901771	0.949391	0.991020
	100	0.901991	0.951860	0.990080
	1,000	0.900401	0.949901	0.989730
	10,000	0.899321	0.949491	0.989910
λ =0.5= λ_0				
	10	0.891491	0.963930	0.988730
	20	0.901771	0.949391	0.991020
	100	0.901991	0.951860	0.990080
	1,000	0.900401	0.949901	0.989730
	10,000	0.899321	0.949491	0.989910

Note: The computer program is C:\C_Bernoulli\C_Bernoulli_7.exe.

Section 2, λ is unknown,

(1)The goodness of fit,

$$X_1, X_2, \ldots, X_n \overset{iid}{\sim} CB(\lambda), \overline{X} = \frac{\sum_{i=1}^{n} X_i}{n},$$

H_0:Continuous Bernoulli($\hat{\lambda}$), H_1:against H_0,

$\hat{\lambda} = \phi(\overline{X})$ is the estimated equation of the λ (chapter 3,section 3).

The test process,

(i)The class number and the probability of each class,

The class number= $k = log_2(n) + 1$, each class probability is setting to $\frac{1}{k}$.

(ii)The class limit,

The first class lower limit=0 and the last class upper limit=1.

$$c_j = \begin{cases} \dfrac{log_e\left(\dfrac{j}{k} \times (2\hat{\lambda} - 1) - (\hat{\lambda} - 1)\right) - log_e(1 - \hat{\lambda})}{log_e\left(\dfrac{\hat{\lambda}}{1 - \hat{\lambda}}\right)}, \hat{\lambda} \neq \dfrac{1}{2} \\ \dfrac{j}{k}, \hat{\lambda} = \dfrac{1}{2} \end{cases}, j = 1,2,\ldots, k-1, ,$$

The first class upper limit= c_1 =the second class lower limit,.....,

The $j-th$ class upper limit= c_j =the $(j+1)-th$ class lower limit, $j = 1,2,\ldots, k-1$.

(iii)The frequency table for testing and computing the observed number and expected number,

class	class limit	frequency= O	$E = n \times \frac{1}{k}$
1	$0 \sim c_1$	O_1	E_1
2	$c_1 \sim c_2$	O_2	E_2
…			
k	$c_{k-1} \sim 1$	O_k	E_k

The chi square test statistic,

$$\chi^2_{k-2} = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i} > \chi^2_{\alpha,k-2}, \text{ rejected } H_0.$$

(2)Confirming,
pr(1-α)=P(doesn't rejected H_0 $|H_0$:X~ Continuous Bernoulli(λ))=1-α,
The pr(1-α) =(the times right test result)/100,000, each probability is from 100,000 times simulated and each time simulated data is the sample size. The simulated data is from Continuous Bernoulli(λ) simulator.

H_0: X~Continuous Bernoulli($\hat{\lambda}=\phi(\overline{X})$), H_1:against H_0,

	sample size	pr(90%)	,pr(95%)	pr(99%)
λ=0.1				
	10	0.90995	0.93887	0.987210
	20	0.894591	0.947381	0.988890
	100	0.901551	0.949801	0.989830
	1,000	0.901041	0.950400	0.990150
	10,000	0.898031	0.948891	0.989680
λ=0.2				
	10	0.918301	0.943211	0.991730
	20	0.895291	0.947921	0.989130
	100	0.901351	0.950730	0.989550
	1,000	0.900781	0.950630	0.990130
	10,000	0.898831	0.949031	0.989670
λ=0.3				
	10	0.922111	0.944091	0.992030
	20	0.895911	0.947831	0.989160
	100	0.901831	0.951140	0.989660
	1,000	0.901561	0.950240	0.990000
	10,000	0.898721	0.949161	0.989530
λ=0.4				
	10	0.923581	0.944241	0.991690
	20	0.896271	0.948331	0.989000
	100	0.901141	0.949891	0.989760
	1,000	0.901551	0.950450	0.990260
	10,000	0.898311	0.949501	0.989490
λ=0.5				
	10	0.923761	0.944291	0.991690
	20	0.896471	0.948801	0.989190
	100	0.901001	0.949941	0.989760
	1,000	0.902111	0.950620	0.990090
	10,000	0.898431	0.950130	0.989790

Note: The computer program is C:\C_Bernoulli\C_Bernoulli_8.exe.

Chapter 8, One way analysis when population is Continuous Bernoulli distribution

Section 1, The one way analysis,
There are k independent Continuous Bernoulli distributions, the random samples from each population and the same size.

$X_{1,1}, X_{1,2}, ..., X_{1,n} \overset{iid}{\sim} CB(\lambda_1),,$

$X_{2,1}, X_{2,2}, ..., X_{2,n} \overset{iid}{\sim} CB(\lambda_2), ,$

……………………,

$X_{k,1}, X_{k,2}, ..., X_{k,n} \overset{iid}{\sim} CB(\lambda_k), ,$

$X_{i,j} = \mu + \alpha_i + \varepsilon_{ij}, i = 1,2,...,k, j = 1,2,....,n, , ,$

$X_{1,1}, X_{1,2}, ..., X_{1,n} \overset{iid}{\sim} CB(\lambda_1),$

$X_{2,1}, X_{2,2}, ..., X_{2,n} \overset{iid}{\sim} CB(\lambda_2),$

……………………,

$X_{k,1}, X_{k,2}, ..., X_{k,n} \overset{iid}{\sim} CB(\lambda_k),$

$X_{i,j} = \mu + \alpha_i + \varepsilon_{ij}, i = 1,2,...,k, j = 1,2,....,n,$

$X_{i,j} \overset{iid}{\sim} CB(E(X_{ij})), \quad E(X_{ij}) = \mu_i = \mu + \alpha_i = \mathrm{G}_1(\lambda_i), i = 1,2,...,k,$

$H_0 : \lambda_1 = \lambda_2 = ... = \lambda_k, (\mu_1 = \mu_2 = ... = \mu_k = \mu), (\alpha_1 = \alpha_2 = ... = \alpha_k = 0),$

$$\overline{X}_i = \frac{\sum_{j=1}^{n} X_{i,j}}{n}, S_1^2 = \frac{\sum_{j=1}^{n}\left(X_{i,j} - \overline{X}_i\right)^2}{n-1}, i = 1,2,...,k,$$

The grand mean $\overline{\overline{X}} = \frac{\sum_{i=1}^{k}\sum_{j=1}^{n} X_{i,j}}{n_{\mathrm{T}}}, n_{\mathrm{T}} = n \times k,$

$$SST = \sum_{i=1}^{k}\sum_{j=1}^{n}\left(X_{i,j} - \overline{\overline{X}}\right)^2 = \sum_{i=1}^{k}\sum_{j=1}^{n}\left(X_{i,j} - \overline{X}_i + \overline{X}_i - \overline{\overline{X}}\right)^2$$

$$= \sum_{i=1}^{k}\sum_{j=1}^{n}\left(X_{i,j} - \overline{X}_i\right)^2 + \sum_{i=1}^{k}\sum_{j=1}^{n}\left(\overline{X}_i - \overline{\overline{X}}\right)^2,$$

$$SSTR = \sum_{i=1}^{k}\sum_{j=1}^{n}\left(\overline{X}_i - \overline{\overline{X}}\right)^2, SSE = \sum_{i=1}^{k}\sum_{j=1}^{n}\left(X_{i,j} - \overline{X}_i\right)^2,$$

SST degree of freedom = n_{T}-1, SSTR degree of freedom=k-1,
SSE degree of free= n_{T} -k, MSTR=SSTR/(k-1), MSE=SSE/(n_{T} -k).

Section 2, ANOVA and test statistic,

ANOVA

Source	SS	df	MS
Treatment	SSTR	k-1	MSTR=SSTR/(k-1)
Error	SSE	n_T-k	MSE=SSE/(n_T-k)
C Total	SST	n_T-1	

The test statistic=MSTR/MSE and the rejected region is the right region.
The p vlaue=P(MSTR/MSE>W), p vlaue< α , rejected H0.
W~MSTR/MSE probability distribution.

the sampling distribution of W will be simulated using the probability simulator and the simulated data is based on

$$X_{2,1}, X_{2,2}, ..., X_{2,n} \overset{iid}{\sim} CB(\hat{\lambda}), \hat{\lambda} = \phi(\overline{\overline{X}}),$$

$$X_{2,1}, X_{2,2}, ..., X_{2,n} \overset{iid}{\sim} CB(\hat{\lambda}), \ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$X_{k,1}, X_{k,2}, ..., X_{k,n} \overset{iid}{\sim} CB(\hat{\lambda}),$$

$$SST = \sum_{i=1}^{k}\sum_{j=1}^{n}\left(X_{i,j} - \overline{\overline{X}}\right)^2 = \sum_{i=1}^{k}\sum_{j=1}^{n}\left(X_{i,j} - \overline{X}_i + \overline{X}_i - \overline{\overline{X}}\right)^2$$

$$= \sum_{i=1}^{k}\sum_{j=1}^{n}\left(X_{i,j} - \overline{X}_i\right)^2 + \sum_{i=1}^{k}\sum_{j=1}^{n}\left(\overline{X}_i - \overline{\overline{X}}\right)^2,$$

$$SSTR = \sum_{i=1}^{k}\sum_{j=1}^{n}\left(\overline{X}_i - \overline{\overline{X}}\right)^2, SSE = \sum_{i=1}^{k}\sum_{j=1}^{n}\left(X_{i,j} - \overline{X}_i\right)^2, W = \frac{MSTR}{MSE}$$

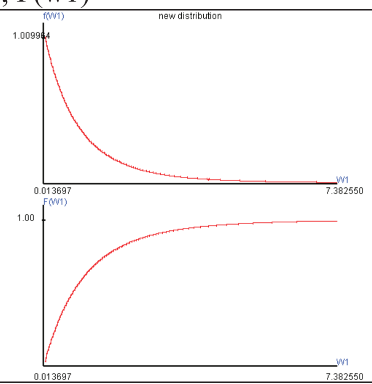
Section 3, The sampling distribution of MSTR/MSE,

Let W1= MSTR/MSE,

(3-1) $H_0 : \lambda_1 = \lambda_2 = ... = \lambda_k$=0.2,

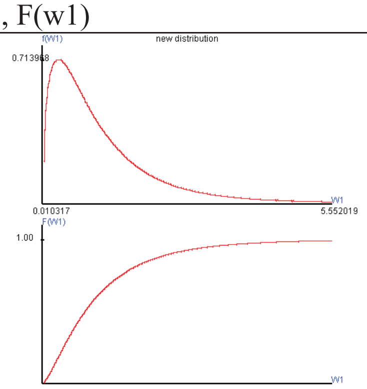
(3-1-1)k=3, $H_0 : \lambda_1 = \lambda_2 = ... = \lambda_k$=0.2, n=5,

The right tailed probability is removing 0.01,

f(w1), F(w1)	Coefficient	
	Mathematical Mean:	1.21681
	Geometrical Mean :	0.60534
	Harmonic Mean :	0.04355
	Variance :	2.58886
	S.D. :	1.60899
	Skewed Coef. :	5.56376
	Kurtosis Coef. :	101.93343
	MAD :	0.99676
	Range :	167.55616
	Mid_range :	83.77808
	Median :	0.72408
	Q1 :	0.28834
	Q2 :	0.72408
	Q3 :	1.55208
	IQR :	1.26374
	C.V. :	1.32230

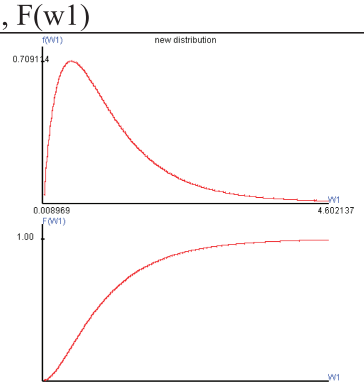
(3-1-2)k=4, $H_0 : \lambda_1 = \lambda_2 = ... = \lambda_k$=0.2, n=5,

The right tailed probability is removing 0.01,

f(w1), F(w1)	Coefficient	
	Mathematical Mean:	1.15101
	Geometrical Mean :	0.73240
	Harmonic Mean :	0.32904
	Variance :	1.37442
	S.D. :	1.17235
	Skewed Coef. :	3.39320
	Kurtosis Coef. :	30.98498
	MAD :	0.79263
	Range :	66.01058
	Mid_range :	33.00530
	Median :	0.81608
	Q1 :	0.40147
	Q2 :	0.81608
	Q3 :	1.50546
	IQR :	1.10398
	C.V. :	1.01854

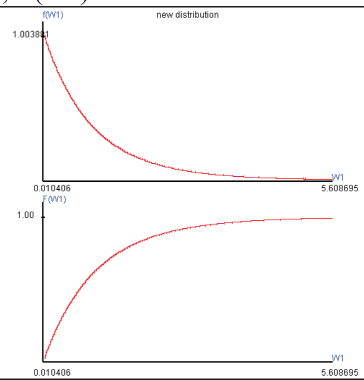
(3-1-3)k=5, $H_0 : \lambda_1 = \lambda_2 = ... = \lambda_k$=0.2, n=5,

The right tailed probability is removing 0.01,

f(w1), F(w1)	Coefficient	
	Mathematical Mean:	1.11625
	Geometrical Mean :	0.80000
	Harmonic Mean :	0.49614
	Variance :	0.91651
	S.D. :	0.95735
	Skewed Coef. :	2.62111
	Kurtosis Coef. :	18.49106
	MAD :	0.67527
	Range :	45.45590
	Mid_range :	22.72838
	Median :	0.86350
	Q1 :	0.47576
	Q2 :	0.86350
	Q3 :	1.46216
	IQR :	0.98640
	C.V. :	0.85764

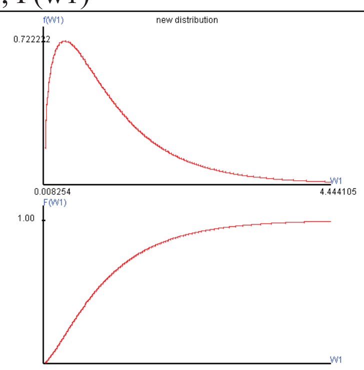
(3-1-4)k=3, $H_0 : \lambda_1 = \lambda_2 = ... = \lambda_k$=0.2, n=10,

The right tailed probability is removing 0.01,

f(w1), F(w1)	Coefficient	
	Mathematical Mean:	1.08266
	Geometrical Mean :	0.57923
	Harmonic Mean :	0.05058
	Variance :	1.43484
	S.D. :	1.19785
	Skewed Coef. :	2.77287
	Kurtosis Coef. :	17.76549
	MAD :	0.83411
	Range :	57.99537
	Mid_range :	28.99768
	Median :	0.70490
	Q1 :	0.28751
	Q2 :	0.70490
	Q3 :	1.45489
	IQR :	1.16738
	C.V. :	1.10639

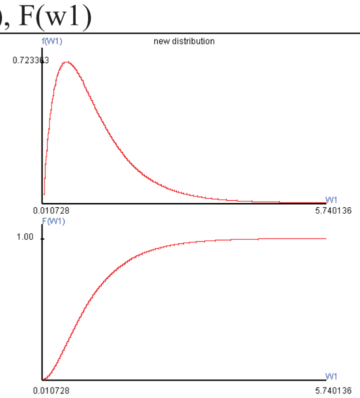
(3-1-5)k=4, $H_0 : \lambda_1 = \lambda_2 = ... = \lambda_k$=0.2, n=10,

The right tailed probability is removing 0.01,

f(w1), F(w1)	Coefficient	
	Mathematical Mean:	1.05975
	Geometrical Mean :	0.70832
	Harmonic Mean :	0.33087
	Variance :	0.89068
	S.D. :	0.94376
	Skewed Coef. :	2.14046
	Kurtosis Coef. :	11.26122
	MAD :	0.68596
	Range :	27.69103
	Mid_range :	13.84553
	Median :	0.79947
	Q1 :	0.40236
	Q2 :	0.79947
	Q3 :	1.42686
	IQR :	1.02450
	C.V. :	0.89055

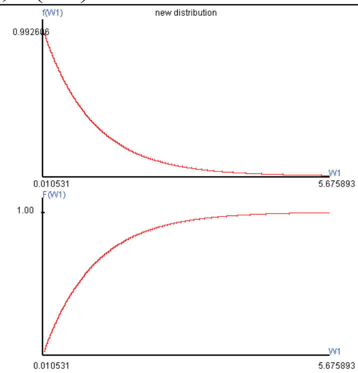
(3-1-6)k=5, $H_0 : \lambda_1 = \lambda_2 = ... = \lambda_k$=0.2, n=10,

The right tailed probability is removing 0.001,

f(w1), F(w1)	Coefficient	
	Mathematical Mean:	1.04733
	Geometrical Mean :	0.77842
	Harmonic Mean :	0.49747
	Variance :	0.64337
	S.D. :	0.80211
	Skewed Coef. :	1.81112
	Kurtosis Coef. :	8.80527
	MAD :	0.59601
	Range :	18.26582
	Mid_range :	9.13299
	Median :	0.84887
	Q1 :	0.47773
	Q2 :	0.84887
	Q3 :	1.39626
	IQR :	0.91853
	C.V. :	0.76586

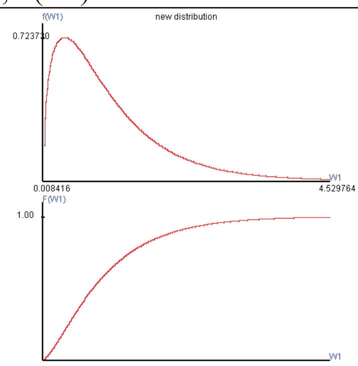
(3-1-7)k=3, $H_0 : \lambda_1 = \lambda_2 = ... = \lambda_k$=0.2, n=30,

The right tailed probability is removing 0.005,

<table>
<tr><th>f(w1), F(w1)</th><th>Coefficient</th></tr>
<tr><td>
</td>
<td>Mathematical Mean: 1.02354
Geometrical Mean : 0.56665
Harmonic Mean : 0.06268
Variance : 1.10964
S.D. : 1.05340
Skewed Coef. : 2.18523
Kurtosis Coef. : 10.60897
MAD : 0.76357
Range : 20.80437
Mid_range : 10.40219
Median : 0.69674
Q1 : 0.28746
Q2 : 0.69674
Q3 : 1.40628
IQR : 1.11882
C.V. : 1.02917</td></tr>
</table>

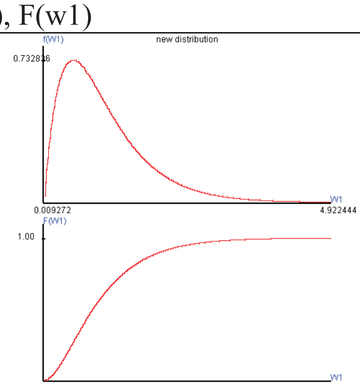
(3-1-8)k=4, $H_0 : \lambda_1 = \lambda_2 = ... = \lambda_k$=0.2, n=30,

The right tailed probability is removing 0.005,

<table>
<tr><th>f(w1), F(w1)</th><th>Coefficient</th></tr>
<tr><td>
</td>
<td>Mathematical Mean: 1.01771
Geometrical Mean : 0.69665
Harmonic Mean : 0.33411
Variance : 0.72754
S.D. : 0.85296
Skewed Coef. : 1.77455
Kurtosis Coef. : 8.01920
MAD : 0.63710
Range : 16.97153
Mid_range : 8.48578
Median : 0.79176
Q1 : 0.40378
Q2 : 0.79176
Q3 : 1.38681
IQR : 0.98304
C.V. : 0.83812</td></tr>
</table>

(3-1-9)k=5, $H_0 : \lambda_1 = \lambda_2 = ... = \lambda_k$=0.2, n=30,

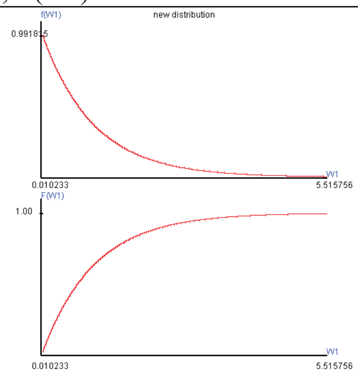
The right tailed probability is removing 0.001,

<table>
<tr><th>f(w1), F(w1)</th><th>Coefficient</th></tr>
<tr><td>
</td>
<td>Mathematical Mean: 1.01390
Geometrical Mean : 0.76767
Harmonic Mean : 0.49924
Variance : 0.53911
S.D. : 0.73424
Skewed Coef. : 1.52253
Kurtosis Coef. : 6.66105
MAD : 0.55742
Range : 12.12546
Mid_range : 6.06287
Median : 0.84192
Q1 : 0.47972
Q2 : 0.84192
Q3 : 1.36123
IQR : 0.88151
C.V. : 0.72418</td></tr>
</table>

(3-1-10)k=3, $H_0 : \lambda_1 = \lambda_2 = ... = \lambda_k$=0.2, n=50,

The right tailed probability is removing 0.005,

f(w1), F(w1)

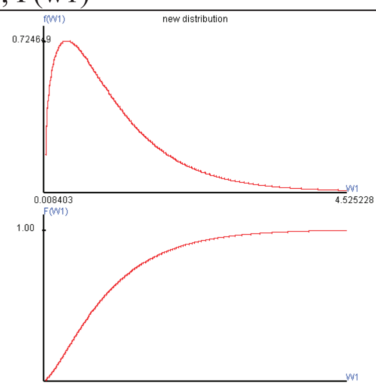


Coefficient		
Mathematical Mean	:	1.01417
Geometrical Mean	:	0.56463
Harmonic Mean	:	0.04872
Variance	:	1.06428
S.D.	:	1.03164
Skewed Coef.	:	2.10523
Kurtosis Coef.	:	9.87547
MAD	:	0.75247
Range	:	18.23656
Mid_range	:	9.11828
Median	:	0.69516
Q1	:	0.28762
Q2	:	0.69516
Q3	:	1.39855
IQR	:	1.11093
C.V.	:	1.01722

(3-1-11)k=4, $H_0 : \lambda_1 = \lambda_2 = ... = \lambda_k$=0.2, n=50,

The right tailed probability is removing 0.005,

f(w1), F(w1)

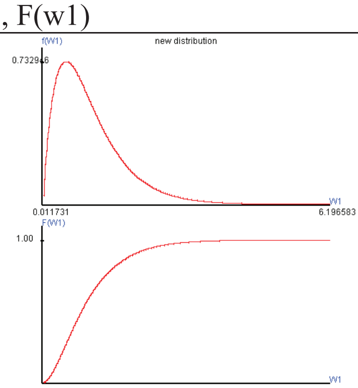


Coefficient		
Mathematical Mean	:	1.01733
Geometrical Mean	:	0.69629
Harmonic Mean	:	0.33356
Variance	:	0.72631
S.D.	:	0.85224
Skewed Coef.	:	1.76675
Kurtosis Coef.	:	7.94218
MAD	:	0.63684
Range	:	15.94130
Mid_range	:	7.97066
Median	:	0.79178
Q1	:	0.40350
Q2	:	0.79178
Q3	:	1.38662
IQR	:	0.98311
C.V.	:	0.83772

(3-1-12)k=5, $H_0 : \lambda_1 = \lambda_2 = ... = \lambda_k$=0.2, n=50,

The right tailed probability is removing 0.0001,

f(w1), F(w1)

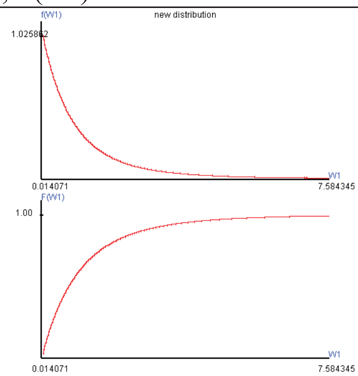


Coefficient		
Mathematical Mean	:	1.00783
Geometrical Mean	:	0.76552
Harmonic Mean	:	0.49971
Variance	:	0.52276
S.D.	:	0.72302
Skewed Coef.	:	1.48005
Kurtosis Coef.	:	6.38969
MAD	:	0.55065
Range	:	10.68217
Mid_range	:	5.34132
Median	:	0.84047
Q1	:	0.47989
Q2	:	0.84047
Q3	:	1.35457
IQR	:	0.87467
C.V.	:	0.71740

(3-2) $H_0 : \lambda_1 = \lambda_2 = ... = \lambda_k$=0.4,

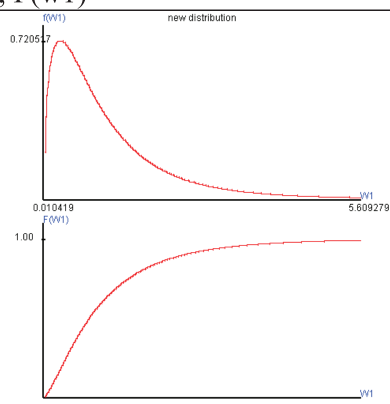
(3-2-1)k=3, $H_0 : \lambda_1 = \lambda_2 = ... = \lambda_k$=0.4, n=5,

The right tailed probability is removing 0.01,

f(w1), F(w1)	Coefficient
	Mathematical Mean: 1.22328 Geometrical Mean : 0.60119 Harmonic Mean : 0.04488 Variance : 2.68109 S.D. : 1.63740 Skewed Coef. : 5.37063 Kurtosis Coef. : 93.15497 MAD : 1.01279 Range : 180.86351 Mid_range : 90.43176 Median : 0.71578 Q1 : 0.28387 Q2 : 0.71578 Q3 : 1.54731 IQR : 1.26344 C.V. : 1.33854

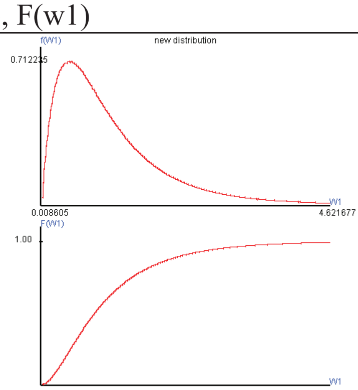
(3-2-2)k=4, $H_0 : \lambda_1 = \lambda_2 = ... = \lambda_k$=0.4, n=5,

The right tailed probability is removing 0.01,

f(w1), F(w1)	Coefficient
	Mathematical Mean: 1.15369 Geometrical Mean : 0.72925 Harmonic Mean : 0.32578 Variance : 1.39653 S.D. : 1.18175 Skewed Coef. : 3.30174 Kurtosis Coef. : 29.67488 MAD : 0.80135 Range : 73.76990 Mid_range : 36.88496 Median : 0.81059 Q1 : 0.39711 Q2 : 0.81059 Q3 : 1.50678 IQR : 1.10967 C.V. : 1.02432

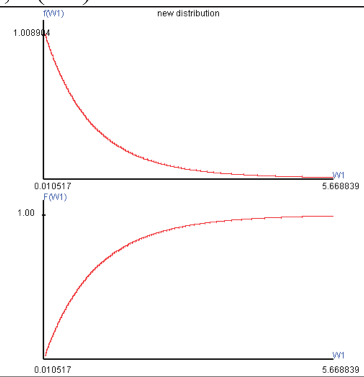
(3-2-3)k=5, $H_0 : \lambda_1 = \lambda_2 = ... = \lambda_k$=0.4, n=5,

The right tailed probability is removing 0.01,

f(w1), F(w1)	Coefficient
	Mathematical Mean: 1.11772 Geometrical Mean : 0.79747 Harmonic Mean : 0.49050 Variance : 0.92527 S.D. : 0.96191 Skewed Coef. : 2.53699 Kurtosis Coef. : 17.18906 MAD : 0.68151 Range : 49.42579 Mid_range : 24.71293 Median : 0.85943 Q1 : 0.47161 Q2 : 0.85943 Q3 : 1.46605 IQR : 0.99444 C.V. : 0.86060

(3-2-4)k=3, $H_0 : \lambda_1 = \lambda_2 = ... = \lambda_k$=0.4, n=10,

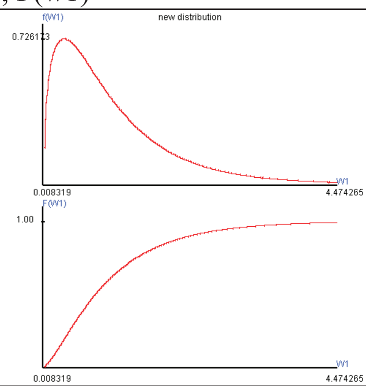
The right tailed probability is removing 0.01,



f(w1), F(w1)	Coefficient	
	Mathematical Mean:	1.08341
	Geometrical Mean :	0.57737
	Harmonic Mean :	0.04849
	Variance :	1.45638
	S.D. :	1.20681
	Skewed Coef. :	2.78902
	Kurtosis Coef. :	17.51837
	MAD :	0.83765
	Range :	52.41545
	Mid_range :	26.20773
	Median :	0.70186
	Q1 :	0.28585
	Q2 :	0.70186
	Q3 :	1.45158
	IQR :	1.16574
	C.V. :	1.11389

(3-2-5)k=4, $H_0 : \lambda_1 = \lambda_2 = ... = \lambda_k$=0.4, n=10,

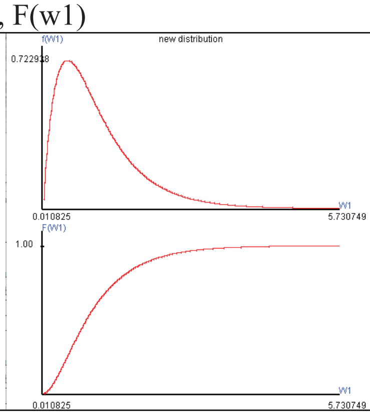
The right tailed probability is removing 0.01,



f(w1), F(w1)	Coefficient	
	Mathematical Mean:	1.06079
	Geometrical Mean :	0.70695
	Harmonic Mean :	0.32992
	Variance :	0.90130
	S.D. :	0.94937
	Skewed Coef. :	2.14696
	Kurtosis Coef. :	11.22040
	MAD :	0.68941
	Range :	29.44046
	Mid_range :	14.72025
	Median :	0.79741
	Q1 :	0.40049
	Q2 :	0.79741
	Q3 :	1.42750
	IQR :	1.02700
	C.V. :	0.89496

(3-2-6)k=5, $H_0 : \lambda_1 = \lambda_2 = ... = \lambda_k$=0.4, n=10,

The right tailed probability is removing 0.001,



f(w1), F(w1)	Coefficient	
	Mathematical Mean:	1.04800
	Geometrical Mean :	0.77764
	Harmonic Mean :	0.49610
	Variance :	0.64799
	S.D. :	0.80498
	Skewed Coef. :	1.80272
	Kurtosis Coef. :	8.62824
	MAD :	0.59821
	Range :	15.07232
	Mid_range :	7.53635
	Median :	0.84780
	Q1 :	0.47631
	Q2 :	0.84780
	Q3 :	1.39722
	IQR :	0.92092
	C.V. :	0.76811

Chapter 9, The Continuous Trinomial distribution and trial number=1,

The trinomial distribution and trial number=1,

$f_{X_1,X_2}(x_1,x_2;p_1,p_2)=p_1^{x_1}p_2^{x_2}(1-p_1-p_2)^{1-x_1-x_2}, x_1=0,1, x_2=0,1, x_1+x_2=0,1,$

$0<p_1<1, 0<p_2<1,$

analysis by Bayesian Theorem,

$P(X_1=1)=p_1$ $---P(X_2=0|X_1=1)=1,$

$---P(X_2=1|X_1=1)=0,$

$P(X_1=0)=1-p_1$ $---P(X_2=0|X_1=0)=1-\dfrac{p_2}{1-p_1},$

$---P(X_2=1|X_1=0)=\dfrac{p_2}{1-p_1},$

$P(X_1=0,X_2=0)=p_1, P(X_1=0,X_2=1)=p_2, P(X_1=0,X_2=0)=(1-p_1-p_2),$

$\Rightarrow$

$P(X_2=1)=p_2$ $---P(X_1=0|X_2=1)=1,$

$---P(X_1=1|X_2=1)=0,$

$P(X_2=0)=1-p_{21}$ $---P(X_1=0|X_2=0)=1-\dfrac{p_1}{1-p_2},$

$---P(X_1=1|X_2=0)=\dfrac{p_1}{1-p_2},$

$X_1 \sim Bernoulli(p_1), X_2 \sim Bernoulli(p_2), 1-X_1-X_2 \sim Bernoulli(1-p_1-p_2),$

$X_1+X_2 \sim Bernoulli(p_1+p_2),$

X_1 and X_2 are discrete random variables,

Let X_1 and X_2 be continuous random variables and $p_1=\lambda_1$, $p_2=\lambda_2$ to find the Continuous Trinomial distribution and trial number=1,

$f_{X_1,X_2}(x_1,x_2;\lambda_1,\lambda_2)=C(\lambda_1,\lambda_2)\lambda_1^{x_1}\lambda_2^{x_2}(1-\lambda_1-\lambda_2)^{1-x_1-x_2},$

$0<x_1<1, 0<x_2<1, 0<x_1+x_2<1, 0<\lambda_1<1, 0<\lambda_2<1, 0<\lambda_1+\lambda_2<1,$

Section 1, Setting $X_1 \sim$ Continuous Bernoulli(λ_1) , $X_2 \sim$ Continuous Bernoulli(λ_2) to find the $f_{X_1,X_2}(x_1,x_2;\lambda_1,\lambda_2)$,

$$f_{X_1}(x_1;\lambda_1) = C(\lambda_1)\lambda_1^{x_1}(1-\lambda_1)^{1-x_1}, 0 < x_1 < 1, 0 < \lambda_1 < 1,$$

$$f_{X_1}(x_1;\lambda_1) = C(\lambda_2)\lambda_2^{x_1}(1-\lambda_2)^{1-x_2}, 0 < x_2 < 1, 0 < \lambda_2 < 1,$$

$$C(\lambda_i) = \begin{cases} \dfrac{2\tanh^{-1}(1-2\lambda_i)}{1-2\lambda}, \lambda_i \neq \dfrac{1}{2} \\ 2, \lambda_i = \dfrac{1}{2} \end{cases}, i = 1,2,,$$

Getting the $f_{X_1}(x_1;\lambda_1)$ from joint probability density function of (x_1, x_2)

$$f_{X_1}(x_1;\lambda_1) = \int_0^{1-x_1} C(\lambda_1,\lambda_2)\lambda_1^{x_1}\lambda_2^{x_2}(1-\lambda_1-\lambda_2)^{1-x_1-x_2}\, dx_2,$$

$$= C(\lambda_1)\lambda_1^{x_1}(1-\lambda_1)^{1-x_1}\int_0^{1-x_1}\frac{C(\lambda_1,\lambda_2)}{C(\lambda_1)}\left(\frac{\lambda_2}{1-\lambda_1}\right)^{x_2}\left(1-\frac{\lambda_2}{1-\lambda_1}\right)^{1-x_1-x_2} dx_2,$$

$$\lambda = \frac{\lambda_2}{1-\lambda_1}, \int_0^{1-x_1}\left(\frac{\lambda_2}{1-\lambda_1}\right)^{x_2}\left(1-\frac{\lambda_2}{1-\lambda_1}\right)^{1-x_1-x_2} dx_2 = \int_0^{1-x_1}\lambda^{x_2}(1-\lambda)^{1-x_1-x_2}\, dx_2 --- (9.1)$$

$$w = \frac{x_2}{1-x_1}, \frac{dx_2}{dw} = 1-x_1, 0 < w < 1,$$

$$(9.1) \Rightarrow \int_0^1 \lambda^{(1-x_1)w}(1-\lambda)^{1-x_1-(1-x_1)w}(1-x_1)dw = (1-x_1)\int_0^1 \lambda^{(1-x_1)w}(1-\lambda)^{(1-x_1)(1-w)}(1-x_1)dw$$

$$= (1-x_1)\int_0^1 \lambda^{(1-x_1)w}(1-\lambda)^{(1-x_1)(1-w)}dw = (1-x_1)(1-\lambda)^{1-x_1}\int_0^1\left(\frac{\lambda}{1-\lambda}\right)^{(1-x_1)w} dw$$

$$= (1-x_1)(1-\lambda)^{1-x_1}\int_0^1 exp\left((1-x_1)w \times log\left(\left(\frac{\lambda}{1-\lambda}\right)\right)\right)dw$$

$$= (1-x_1)(1-\lambda)^{1-x_1} \times \frac{exp\left((1-x_1)\times log\left(\left(\frac{\lambda}{1-\lambda}\right)\right)\right)-1}{(1-x_1)\times log\left(\left(\frac{\lambda}{1-\lambda}\right)\right)}$$

$$= (1-x_1)(1-\lambda)^{1-x_1} \times \frac{\left(\frac{\lambda}{1-\lambda}\right)^{1-x_1}-1}{(1-x_1)\times(log(\lambda)-log(1-\lambda))}$$

$$= \frac{(\lambda)^{1-x_1}-(1-\lambda)^{1-x_1}}{(log(\lambda)-log(1-\lambda))}, \lambda \neq 0.5$$

$$(1)\ f_{X_1,X_2}\left(x_1,x_2;\lambda_1,\lambda_2\right)=f_{X_1}\left(x_1;\lambda_1\right)f_{X_2|x_1}\left(x_2|x_1\right)$$

$$\lambda=\frac{\lambda_2}{1-\lambda_1}\neq 0.5,$$

$$\frac{C(\lambda_1,\lambda_2)}{C(\lambda_1)}=C^*(\lambda,x_1)=\frac{log(\lambda)-log(1-\lambda)}{(\lambda)^{1-x_1}-(1-\lambda)^{1-x_1}},$$

$$f_{X_2|x_1}\left(x_2|x_1\right)=C^*(\lambda,x_1)\lambda^{x_2}(1-\lambda)^{1-x_1-x_2},$$

$$\int_0^{1-x_1}C^*(\lambda)\lambda^{x_2}(1-\lambda)^{1-x_1-x_2}dx_2=1,$$

$$\lambda=\frac{\lambda_2}{1-\lambda_1}=0.5,\frac{C(\lambda_1,\lambda_2)}{C(\lambda_1)}=\frac{1}{1-x_1}=C^*(\lambda,x_1),$$

$$f_{X_2|x_1}\left(x_2|x_1\right)=C^*(\lambda),0<x_2<1-x_1,\int_0^{1-x_1}\frac{1}{1-x_1}dx_2=1$$

$$f_{X_1,X_2}\left(x_1,x_2;\lambda_1,\lambda_2\right)=C(\lambda_1,\lambda_2)\lambda_1^{x_1}\lambda_2^{x_2}(1-\lambda_1-\lambda_2)^{1-x_1-x_2}$$
$$=f_{X_1}\left(x_1;\lambda_1\right)f_{X_2|x_1}\left(x_2|x_1\right)$$
$$=C(\lambda_1)\lambda_1^{x_1}(1-\lambda_1)^{1-x_1}\times C^*(\lambda,x_1)\lambda^{x_2}(1-\lambda)^{1-x_1-x_2}$$
$$=C(\lambda_1)\lambda_1^{x_1}(1-\lambda_1)^{1-x_1}\times C^*\left(\lambda=\frac{\lambda_2}{1-\lambda_1},x_1\right)\left(\frac{\lambda_2}{1-\lambda_1}\right)^{x_2}\left(1-\frac{\lambda_2}{1-\lambda_1}\right)^{1-x_1-x_2}$$
$$=C(\lambda_1)C^*\left(\lambda=\frac{\lambda_2}{1-\lambda_1},x_1\right)\lambda_1^{x_1}\lambda_2^{x_2}(1-\lambda_1-\lambda_2)^{1-x_1-x_2},0<x_2<x_1,0<x_1<1,$$

$$\lambda=\frac{\lambda_2}{1-\lambda_1},C(\lambda_1,\lambda_2)=C(\lambda_1)C^*\left(\lambda=\frac{\lambda_2}{1-\lambda_1},x_1\right),$$

$$C(\lambda_1)=\begin{cases}\dfrac{2tanh^{-1}(1-2\lambda_1)}{1-2\lambda_1},\lambda_1\neq\dfrac{1}{2}\\ 2,\lambda_1=\dfrac{1}{2}\end{cases},C^*\left(\lambda=\frac{\lambda_2}{1-\lambda_1},x_1\right)=\begin{cases}\dfrac{log(\lambda)-log(1-\lambda)}{(\lambda)^{1-x_1}-(1-\lambda)^{1-x_1}},\lambda\neq\dfrac{1}{2}\\ \dfrac{1}{1-x_1},\lambda=\dfrac{1}{2}\end{cases},$$

$$f_{X_1,X_2}\left(x_1,x_2;\lambda_1,\lambda_2\right)=C(\lambda_1,\lambda_2)\lambda_1^{x_1}\lambda_2^{x_2}(1-\lambda_1-\lambda_2)^{1-x_1-x_2}$$
$$=f_{X_1}\left(x_1;\lambda_1\right)f_{X_2|x_1}\left(x_2|x_1\right)$$
$$=C(\lambda_1)\lambda_1^{x_1}(1-\lambda_1)^{1-x_1}\times C^*(\lambda,x_1)\lambda^{x_2}(1-\lambda)^{1-x_1-x_2}$$
$$=C(\lambda_1)\lambda_1^{x_1}(1-\lambda_1)^{1-x_1}\times C^*\left(\lambda=\frac{\lambda_2}{1-\lambda_1},x_1\right)\left(\frac{\lambda_2}{1-\lambda_1}\right)^{x_2}\left(1-\frac{\lambda_2}{1-\lambda_1}\right)^{1-x_1-x_2}$$
$$=C(\lambda_1)C^*\left(\lambda=\frac{\lambda_2}{1-\lambda_1},x_1\right)\lambda_1^{x_1}\lambda_2^{x_2}(1-\lambda_1-\lambda_2)^{1-x_1-x_2},0<x_2<x_1,0<x_1<1,$$

$$\lambda=\frac{\lambda_2}{1-\lambda_1},C(\lambda_1,\lambda_2)=C(\lambda_1)C^*\left(\lambda=\frac{\lambda_2}{1-\lambda_1},x_1\right)$$

X_1~Continuous Bernoulli (λ_1), X_2 is not Continuous Bernoulli (λ_2),

(2) $f_{X_1,X_2}(x_1,x_2;\lambda_1,\lambda_2)=f_{X_2}(x_2;\lambda_2)f_{X_1|x_2}(x_1|x_2)\neq f_{X_1}(x_1;\lambda_1)f_{X_2|x_1}(x_2|x_1)$

$$\lambda=\frac{\lambda_1}{1-\lambda_2}\neq 0.5,$$

$$\frac{C(\lambda_1,\lambda_2)}{C(\lambda_2)}=C^{**}(\lambda,x_2)=\frac{log(\lambda)-log(1-\lambda)}{(\lambda)^{1-x_2}-(1-\lambda)^{1-x_2}},$$

$$f_{X_1|x_2}(x_1|x_2)=C^{**}(\lambda,x_2)\lambda^{x_1}(1-\lambda)^{1-x_2-x_1},$$

$$\int_0^{1-x_2}C^{**}(\lambda,x_2)\lambda^{x_1}(1-\lambda)^{1-x_2-x_1}dx_1=1,$$

$$\lambda=\frac{\lambda_1}{1-\lambda_2}=0.5,$$

$$\frac{C(\lambda_1,\lambda_2)}{C(\lambda_2)}=\frac{1}{1-x_1}=C^{**}(\lambda,x_2)f_{X_1|x_2}(x_1|x_2)=C^{**}(\lambda,x_2),0<x_1<1-x_2,$$

$$\int_0^{1-x_2}\frac{1}{1-x_2}dx_1=1$$

$$f_{X_1,X_2}(x_1,x_2;\lambda_1,\lambda_2)=C(\lambda_1,\lambda_2)\lambda_1^{x_1}\lambda_2^{x_2}(1-\lambda_1-\lambda_2)^{1-x_1-x_2}$$

$$=f_{X_2}(x_2;\lambda_2)f_{X_1|x_2}(x_1|x_2)\neq$$

$$=C(\lambda_2)\lambda_2^{x_2}(1-\lambda_2)^{1-x_2}\times C^{**}(\lambda,x_2)\lambda^{x_1}(1-\lambda)^{1-x_2-x_1}$$

$$=C(\lambda_2)\lambda_2^{x_2}(1-\lambda_2)^{1-x_2}\times C^{**}\left(\lambda=\frac{\lambda_1}{1-\lambda_2},x_2\right)\left(\frac{\lambda_1}{1-\lambda_2}\right)^{x_2}\left(1-\frac{\lambda_1}{1-\lambda_2}\right)^{1-x_2-x_1}$$

$$=C(\lambda_1)C^{*}\left(\lambda=\frac{\lambda_1}{1-\lambda_2},x_2\right)\lambda_1^{x_1}\lambda_2^{x_2}(1-\lambda_1-\lambda_2)^{1-x_1-x_2},0<x_1<x_2,0<x_2<1,$$

$$\lambda=\frac{\lambda_1}{1-\lambda_2},C(\lambda_1,\lambda_2)=C(\lambda_1)C^{**}\left(\lambda=\frac{\lambda_1}{1-\lambda_2},x_2\right),$$

$$C(\lambda_2)=\begin{cases}\dfrac{2tanh^{-1}(1-2\lambda_2)}{1-2\lambda_2},\lambda_2\neq\dfrac{1}{2}\\ 2,\lambda_2=\dfrac{1}{2}\end{cases},C^{**}\left(\lambda=\frac{\lambda_1}{1-\lambda_2},x_2\right)=\begin{cases}\dfrac{log(\lambda)-log(1-\lambda)}{(\lambda)^{1-x_2}-(1-\lambda)^{1-x_2}},\lambda\neq\dfrac{1}{2}\\ \dfrac{1}{1-x_1},\lambda=\dfrac{1}{2}\end{cases},$$

X_2 ~Continuous Bernoulli (λ_2), X_1 is not Continuous Bernoulli(λ_1),

(3)Conclusion,

$f_{X_2}(x_2;\lambda_2)f_{X_1|x_2}(x_1|x_2)\neq f_{X_1}(x_1;\lambda_1)f_{X_2|x_1}(x_2|x_1)$,

$f_{X_2}(x_2;\lambda_2)f_{X_1|x_2}(x_1|x_2)$ and $f_{X_1}(x_1;\lambda_1)f_{X_2|x_1}(x_2|x_1)$ do not have the property of joint probability density function.

The requirement of X_1 ~ Continuous Bernoulli(λ_1) and X_2 ~ Continuous Bernoulli(λ_2) cannot derive the joint probability density function $f_{X_1,X_2}(x_1,x_2;\lambda_1,\lambda_2)$.

Section 2, Following property of joint probability density function,

$$f_{X_1,X_2}(x_1,x_2;\lambda_1,\lambda_2)=C(\lambda_1,\lambda_2)\lambda_1^{x_1}\lambda_2^{x_2}(1-\lambda_1-\lambda_2)^{1-x_1-x_2},$$
$$0<x_1<1,0<x_2<1,0<x_1+x_2<1,0<\lambda_1<1,0<\lambda_2<1,0<\lambda_1+\lambda_2<1,$$

1. $C(\lambda_1,\lambda_2)$=?

$(1)\lambda_2\neq\frac{1-\lambda_1}{2}$, ($\lambda_1\neq\frac{1}{3}$ and $\lambda_2\neq\frac{1}{3}$)

$$f_{X_1}(x_1;\lambda_1,\lambda_2)=\int_0^{1-x_1}C(\lambda_1,\lambda_2)\lambda_1^{x_1}\lambda_2^{x_2}(1-\lambda_1-\lambda_2)^{1-x_1-x_2}\,dx_2,$$

$$=C(\lambda_1,\lambda_2)\lambda_1^{x_1}\int_0^{1-x_1}\lambda_2^{x_2}(1-\lambda_1-\lambda_2)^{1-x_1-x_2}\,dx_2$$

$$=C(\lambda_1,\lambda_2)\lambda_1^{x_1}(1-\lambda_1-\lambda_2)^{1-x_1}\,d\int_0^{1-x_1}\left(\frac{\lambda_2}{1-\lambda_1-\lambda_2}\right)^{x_2}dx_2\ --(9.2),$$

$$\frac{\lambda_2}{1-\lambda_1-\lambda_2}\neq 1,$$

$$(9.2)\Rightarrow C(\lambda_1,\lambda_2)\lambda_1^{x_1}(1-\lambda_1-\lambda_2)^{1-x_1}\left(\frac{\left(\frac{\lambda_2}{1-\lambda_1-\lambda_2}\right)^{x_2}}{ln\left(\frac{\lambda_2}{1-\lambda_1-\lambda_2}\right)}\Bigg|_0^{1-x_1}\right),$$

$$=C(\lambda_1,\lambda_2)\lambda_1^{x_1}(1-\lambda_1-\lambda_2)^{1-x_1}\times\frac{\left(\frac{\lambda_2}{1-\lambda_1-\lambda_2}\right)^{1-x_1}-1}{ln\left(\frac{\lambda_2}{1-\lambda_1-\lambda_2}\right)}$$

$$=C(\lambda_1,\lambda_2)\frac{\lambda_1^{x_1}(\lambda_2)^{1-x_1}-\lambda_1^{x_1}(1-\lambda_1-\lambda_2)^{1-x_1}}{ln\left(\frac{\lambda_2}{1-\lambda_1-\lambda_2}\right)}$$

$$\int_0^1 f_{X_1}(x_1;\lambda_1,\lambda_2)dx_1=\frac{C(\lambda_1,\lambda_2)}{ln\left(\frac{\lambda_2}{1-\lambda_1-\lambda_2}\right)}\left(\int_0^1\lambda_1^{x_1}(\lambda_2)^{1-x_1}dx_1-\int_0^1\lambda_1^{x_1}(1-\lambda_1-\lambda_2)^{1-x_1}dx_1\right)-(9.3)$$

$$(i)\lambda_1\neq\lambda_2,(9.3)=\frac{C(\lambda_1,\lambda_2)}{ln\left(\frac{\lambda_2}{1-\lambda_1-\lambda_2}\right)}\times\left(\frac{\lambda_1-\lambda_2}{ln\left(\frac{\lambda_1}{\lambda_2}\right)}+\frac{1-\lambda_2-2\lambda_1}{ln\left(\frac{\lambda_1}{1-\lambda_1-\lambda_2}\right)}\right)=1,$$

$$C(\lambda_1,\lambda_2)=\frac{ln\left(\frac{\lambda_2}{1-\lambda_1-\lambda_2}\right)}{\frac{\lambda_1-\lambda_2}{ln\left(\frac{\lambda_1}{\lambda_2}\right)}+\frac{1-\lambda_2-2\lambda_1}{ln\left(\frac{\lambda_1}{1-\lambda_1-\lambda_2}\right)}},$$

$$F_{X_1}(x_1;\lambda_1,\lambda_2)=\int_0^{x_1} f_{X_1}(x_1;\lambda_1,\lambda_2)dx_1$$

$$=\frac{1}{\dfrac{\lambda_1-\lambda_2}{ln\left(\dfrac{\lambda_1}{\lambda_2}\right)}+\dfrac{1-\lambda_2-2\lambda_1}{ln\left(\dfrac{\lambda_1}{1-\lambda_1-\lambda_2}\right)}}\left(\lambda_2\left(\left(\frac{\lambda_1}{\lambda_2}\right)^{x_1}-1\right)-(1-\lambda_1-\lambda_2)\left(\left(\frac{\lambda_1}{1-\lambda_1-\lambda_2}\right)^{x_1}-1\right)\right),$$

$0<x_1<1,$

$$(ii)\lambda_1=\lambda_2,(9.3)=\frac{C(\lambda_1,\lambda_2)}{ln\left(\dfrac{\lambda_2}{1-\lambda_1-\lambda_2}\right)}\times\left(\lambda_1+\frac{1-\lambda_2-2\lambda_1}{ln\left(\dfrac{\lambda_1}{1-\lambda_1-\lambda_2}\right)}\right)=1,$$

$$C(\lambda_1,\lambda_2)=\frac{ln\left(\dfrac{\lambda_2}{1-\lambda_1-\lambda_2}\right)}{\lambda_1+\dfrac{1-\lambda_2-2\lambda_1}{ln\left(\dfrac{\lambda_1}{1-\lambda_1-\lambda_2}\right)}},$$

$$F_{X_1}(x_1;\lambda_1,\lambda_2)=\int_0^{x_1} f_{X_1}(x_1;\lambda_1,\lambda_2)dx_1$$

$$=\frac{1}{\lambda_1+\dfrac{1-\lambda_2-2\lambda_1}{ln\left(\dfrac{\lambda_1}{1-\lambda_1-\lambda_2}\right)}}\left(\lambda_1x_1-(1-\lambda_1-\lambda_2)\left(\left(\frac{\lambda_1}{1-\lambda_1-\lambda_2}\right)^{x_1}-1\right)\right),0<x_1<1,$$

$$C(\lambda_1,\lambda_2)=\begin{cases}\dfrac{ln\left(\dfrac{\lambda_2}{1-\lambda_1-\lambda_2}\right)}{\dfrac{\lambda_1-\lambda_2}{ln\left(\dfrac{\lambda_1}{\lambda_2}\right)}+\dfrac{1-\lambda_2-2\lambda_1}{ln\left(\dfrac{\lambda_1}{1-\lambda_1-\lambda_2}\right)}},\lambda_1\neq\lambda_2\\[2ex]\dfrac{ln\left(\dfrac{\lambda_2}{1-\lambda_1-\lambda_2}\right)}{\lambda_1+\dfrac{1-\lambda_2-2\lambda_1}{ln\left(\dfrac{\lambda_1}{1-\lambda_1-\lambda_2}\right)}},\lambda_1=\lambda_2\end{cases},$$

$(2)\lambda_2 = \frac{1-\lambda_1}{2}$, ($\lambda_1 = \frac{1}{3}$ and $\lambda_2 = \frac{1}{3}$)

$f_{X_1,X_2}(x_1, x_2; \lambda_1, \lambda_2) = \frac{C(\lambda_1, \lambda_2)}{3}$,

$0 < x_1 < 1, 0 < x_2 < 1, 0 < x_1 + x_2 < 1, 0 < \lambda_1 < 1, 0 < \lambda_2 < 1, 0 < \lambda_1 + \lambda_2 < 1,$

$C(\lambda_1, \lambda_2)\int_0^1 \int_0^{1-x_1} \frac{1}{3} dx_2 dx_1 = C(\lambda_1, \lambda_2)\int_0^1 \frac{(1-x_1)}{3} dx_1 = \frac{C(\lambda_1, \lambda_2)}{6} = 1, C(\lambda_1, \lambda_2) = 1,$

2. The marginal probability distribution and the conditional probability distribution,

$$f_{X_1}(x_1;\lambda_1,\lambda_2)=\begin{cases}\dfrac{\left(\lambda_1^{x_1}(\lambda_2)^{1-x_1}-\lambda_1^{x_1}(1-\lambda_1-\lambda_2)^{1-x_1}\right)}{\dfrac{\lambda_1-\lambda_2}{ln\left(\dfrac{\lambda_1}{\lambda_2}\right)}+\dfrac{1-\lambda_2-2\lambda_1}{ln\left(\dfrac{\lambda_1}{1-\lambda_1-\lambda_2}\right)}},\dfrac{\lambda_2}{1-\lambda_1-\lambda_2}\neq 1,\lambda_1\neq\lambda_2\\ \dfrac{\left(\lambda_1-\lambda_1^{x_1}(1-\lambda_1-\lambda_2)^{1-x_1}\right)}{\lambda_1+\dfrac{1-\lambda_2-2\lambda_1}{ln\left(\dfrac{\lambda_1}{1-\lambda_1-\lambda_2}\right)}},\dfrac{\lambda_2}{1-\lambda_1-\lambda_2}\neq 1,\lambda_1=\lambda_2\\ 2(1-x_1),\dfrac{\lambda_2}{1-\lambda_1-\lambda_2}=1,\end{cases},0<x_1<1,$$

The marginal probability distribution parameters are λ_1,λ_2,

$f_{X_1}(x_1;\lambda_1=c_1,\lambda_2=c_2)\neq f_{X_1}(x_1;\lambda_1=c_1,\lambda_2=c_3),c_2\neq c_3$.

$$f_{X_2|X_1=x_1}(x_2|x_1)=\begin{cases}\dfrac{(1-\lambda_1-\lambda_2)^{1-x_1}\,ln\left(\dfrac{\lambda_2}{1-\lambda_1-\lambda_2}\right)}{(\lambda_2)^{1-x_1}-(1-\lambda_1-\lambda_2)^{1-x_1}}\left(\dfrac{\lambda_2}{1-\lambda_1-\lambda_2}\right)^{x_2},\dfrac{\lambda_2}{1-\lambda_1-\lambda_2}\neq 1,\\ \dfrac{1}{1-x_1},\dfrac{\lambda_2}{1-\lambda_1-\lambda_2}=1,\end{cases}$$

$0<x_2<1-x_1$,

The conditional probability distribution parameters are λ_1,λ_2.

$f_{X_2|X_1=x_1}(x_2|x_1;\lambda_1=c_1,\lambda_2=c_2)\neq f_{X_2|X_1=x_1}(x_2|x_1;\lambda_1=c_1,\lambda_2=c_3),c_2\neq c_3$.

This joint probability density function is

$f_{X_1,X_2}(x_1,x_2;\lambda_1,\lambda_2)=C(\lambda_1,\lambda_2)\lambda_1^{x_1}\lambda_2^{x_2}(1-\lambda_1-\lambda_2)^{1-x_1-x_2}$,

$\lambda_1=\lambda_2=\lambda, f_{X_1,X_2}(x_1,x_2;\lambda_1,\lambda_2)=C(\lambda,\lambda)\lambda^{x_1+x_2}(1-2\lambda)^{1-x_1-x_2}$,

$0<x_1<1, 0<x_2<1, 0<x_1+x_2<1, 0<\lambda<0.5$,

$$C(\lambda,\lambda)=\begin{cases}\dfrac{ln\left(\dfrac{\lambda}{1-2\lambda}\right)\times ln\left(\dfrac{\lambda}{1-2\lambda}\right)}{1-3\lambda},\lambda\neq\dfrac{1}{2}\\ 6 \qquad ,\lambda=\dfrac{1}{2}\end{cases}$$

$\int_0^{1-x_1}C(\lambda_1,\lambda_2)\lambda_1^{x_1}\lambda_2^{x_2}(1-\lambda_1-\lambda_2)^{1-x_1-x_2}\,dx_2\neq C(\lambda_1)\lambda_1^{x_1}(1-\lambda_1)^{1-x_1}$,

$\int_0^{1-x_2}C(\lambda_1,\lambda_2)\lambda_1^{x_1}\lambda_2^{x_2}(1-\lambda_1-\lambda_2)^{1-x_1-x_2}\,dx_1\neq C(\lambda_2)\lambda_2^{x_1}(1-\lambda_2)^{1-x_1}$,

X_i is not Continuous Bernoulli(λ_i), $i=1,2$,

X_1+X_2 is not Continuous Bernoulli($\lambda_1+\lambda_2$).

3. The simulated data is from numerical analysis

The range of $(x_1, x_2), 0 < x_1 < 1, 0 < x_2 < 1, 0 < x_1 + x_2 < 1,$

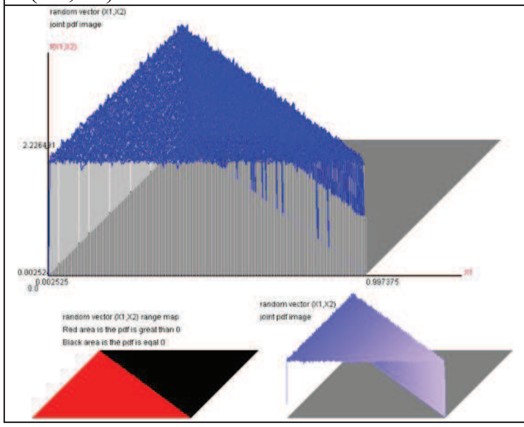
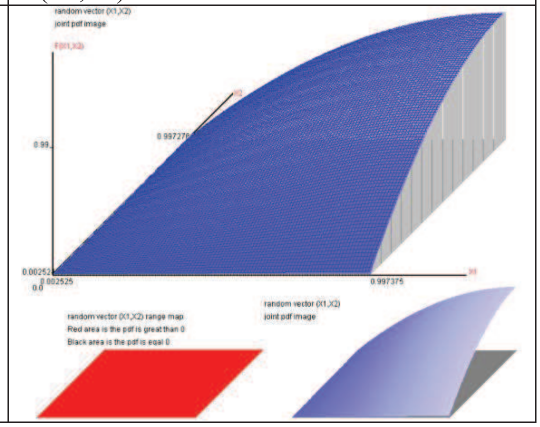
This area is cutting many very small area, the range of x_1 and x_2 many small same width segement.

$$f_{X_1,X_2}(x_1, x_2; \lambda_1, \lambda_2) \cong \sum_{x_1}^{1} \sum_{x_2}^{1-x_1} C(\lambda_1, \lambda_2) \lambda_1^{\Delta x_1} \lambda_2^{\Delta x_2} (1 - \lambda_1 - \lambda_2)^{1-\Delta x_1 - \Delta x_2} \Delta x_1 \Delta x_2,$$

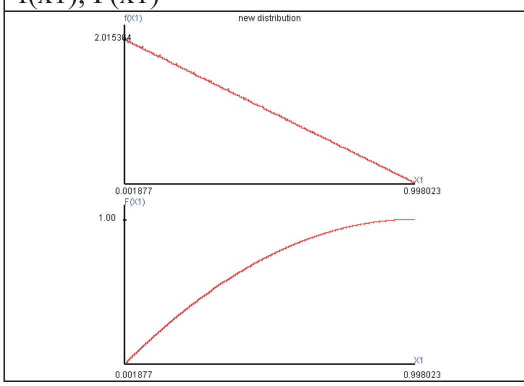
$$f_{X_1}(x_1; \lambda_1, \lambda_2) \cong \sum_{x_2}^{1-x_1} C(\lambda_1, \lambda_2) \lambda_1^{\Delta x_1} \lambda_2^{\Delta x_2} (1 - \lambda_1 - \lambda_2)^{1-\Delta x_1 - \Delta x_2} \Delta x_2,$$

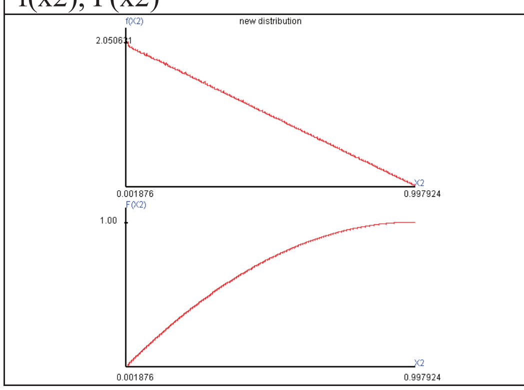
$$f_{X_2}(x_2; \lambda_1, \lambda_2) \cong \sum_{x_1}^{1-x_2} C(\lambda_1, \lambda_2) \lambda_1^{\Delta x_1} \lambda_2^{\Delta x_2} (1 - \lambda_1 - \lambda_2)^{1-\Delta x_1 - \Delta x_2} \Delta x_1$$

4.The joint probability density function and marginal probability density function,
The joint probability distribution of (x_1, x_2)',
(4-1) λ_1=0.3333, λ_2=0.3333, $C(\lambda_1,\lambda_2)$=6.0003000300,

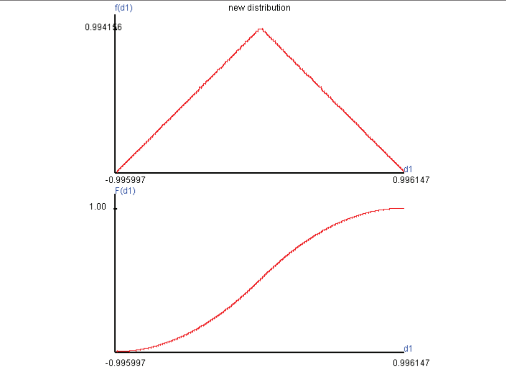
f(x1,x2)	F(x1,x2)
	

E(X1)= 0.3334, Var(X1)= 0.0556, E(X2)= 0.3332, Var(X2)= 0.0555,
Cov(X1,X2)= -0.0278, X1 and X2 correlation coefficient=-0.5002.

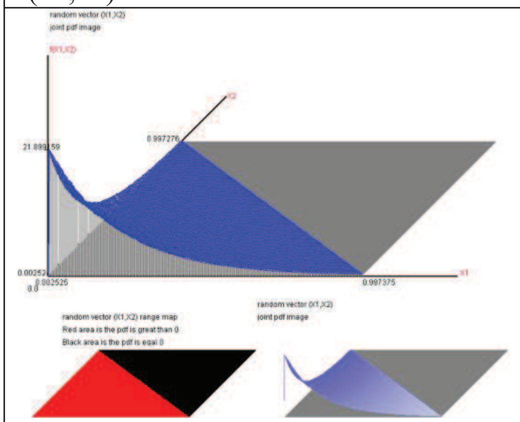
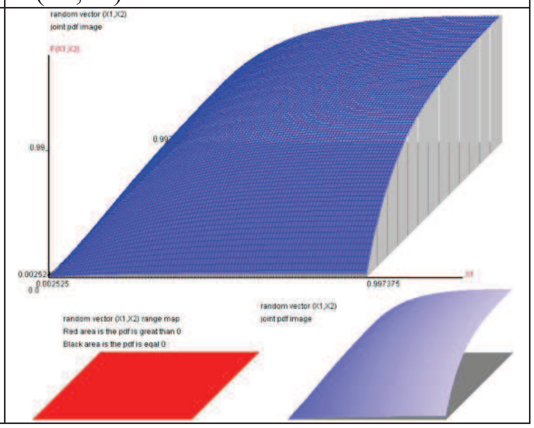
f(x1), F(x1)	Coefficient
	Mathematical Mean: 0.33339 Geometrical Mean : 0.22317 Harmonic Mean : 0.04606 Variance : 0.05560 S.D. : 0.23579 Skewed Coef. : 0.56652 Kurtosis Coef. : 2.40224 MAD : 0.19759 Range : 0.99985 Mid_range : 0.49995 Median : 0.29292 Q1 : 0.13397 Q2 : 0.29292 Q3 : 0.50002 IQR : 0.36605 C.V. : 0.70724

f(x2), F(x2)	Coefficient
	Mathematical Mean: 0.33318 Geometrical Mean : 0.22285 Harmonic Mean : 0.03883 Variance : 0.05553 S.D. : 0.23566 Skewed Coef. : 0.56645 Kurtosis Coef. : 2.40166 MAD : 0.19747 Range : 0.99975 Mid_range : 0.49990 Median : 0.29272 Q1 : 0.13387 Q2 : 0.29272 Q3 : 0.49967 IQR : 0.36580 C.V. : 0.70729

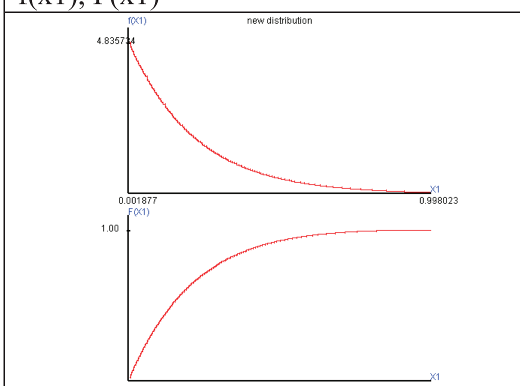
d1=X1-X2,

f(d1), F(d1)	Coefficient
	Mathematical Mean: 0.00021 Geometrical Mean : none Harmonic Mean : none Variance : 0.16671 S.D. : 0.40830 Skewed Coef. : 0.00048 Kurtosis Coef. : 2.40146 MAD : 0.33334 Range : 1.99955 Mid_range : 0.00007 Median : 0.00025 Q1 : -0.29270 Q2 : 0.00025 Q3 : 0.29300 IQR : 0.58570 C.V. : none

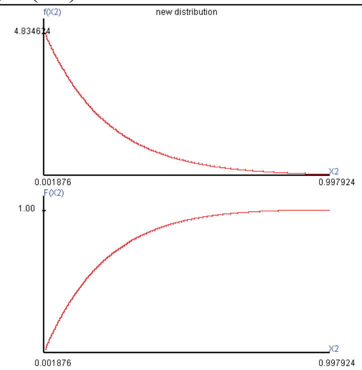
(4-2) λ_1=0.01, λ_2=0.01, $C(\lambda_1,\lambda_2)$=22.7474317294,

f(x1,x2)	F(x1,x2)
	

E(X1)= 0.1933, Var(X1)= 0.0304, E(X2)= 0.1933, Var(X2)= 0.0304, Cov(X1,X2)= -0.0035, X1 and X2 correlation coefficient=-0.1140.

f(x1), F(x1)	Coefficient
	Mathematical Mean: 0.19334 Geometrical Mean : 0.11314 Harmonic Mean : 0.02131 Variance : 0.03043 S.D. : 0.17445 Skewed Coef. : 1.33583 Kurtosis Coef. : 4.59905 MAD : 0.13624 Range : 0.99985 Mid_range : 0.49995 Median : 0.14187 Q1 : 0.05943 Q2 : 0.14187 Q3 : 0.27792 IQR : 0.21850 C.V. : 0.90230

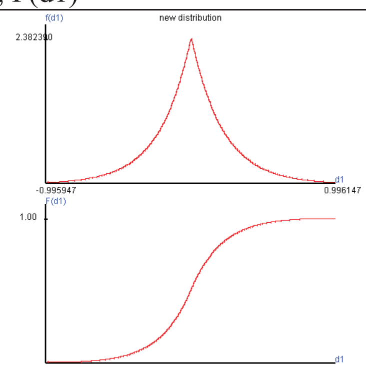
f(x2), F(x2)



Coefficient		
Mathematical Mean:		0.19331
Geometrical Mean :		0.11311
Harmonic Mean	:	0.02106
Variance	:	0.03042
S.D.	:	0.17440
Skewed Coef.	:	1.33473
Kurtosis Coef.	:	4.59486
MAD	:	0.13622
Range	:	0.99975
Mid_range	:	0.49990
Median	:	0.14187
Q1	:	0.05943
Q2	:	0.14187
Q3	:	0.27792
IQR	:	0.21850
C.V.	:	0.90219

d1=X1-X2,

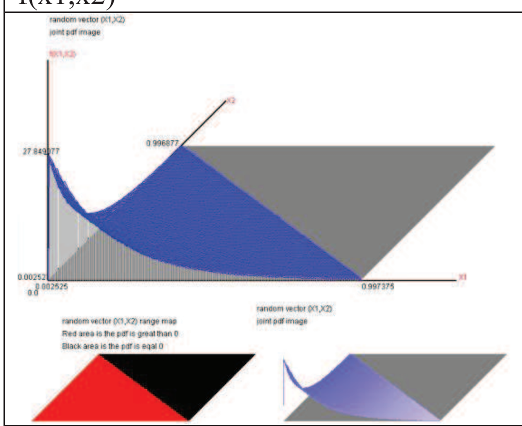
f(d1), F(d1)



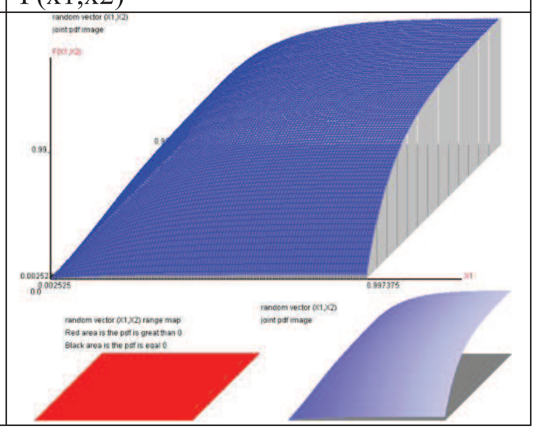
Coefficient		
Mathematical Mean:		0.00003
Geometrical Mean : none		
Harmonic Mean	: none	
Variance	:	0.06778
S.D.	:	0.26036
Skewed Coef.	:	0.00120
Kurtosis Coef.	:	3.90703
MAD	:	0.19330
Range	:	1.99950
Mid_range	:	0.00010
Median	:	0.00000
Q1	:	-0.14185
Q2	:	0.00000
Q3	:	0.14180
IQR	:	0.28365
C.V.	: none	

(4-3) λ_1=0.05, λ_2=0.05, $C(\lambda_1,\lambda_2)$=11.8420874605,

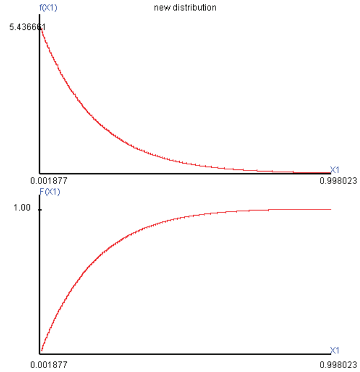
f(x1,x2)



F(x1,x2)

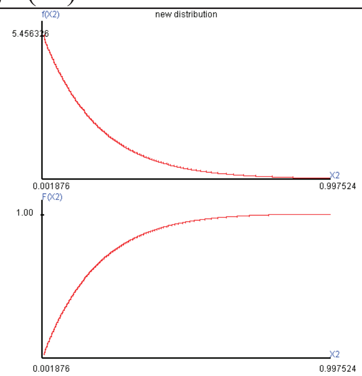


f(x1), F(x1)	Coefficient



Mathematical Mean	:	0.17531
Geometrical Mean	:	0.10133
Harmonic Mean	:	0.01915
Variance	:	0.02638
S.D.	:	0.16241
Skewed Coef.	:	1.45749
Kurtosis Coef.	:	5.15666
MAD	:	0.12523
Range	:	0.99985
Mid_range	:	0.49995
Median	:	0.12642
Q1	:	0.05278
Q2	:	0.12642
Q3	:	0.24957
IQR	:	0.19680
C.V.	:	0.92640

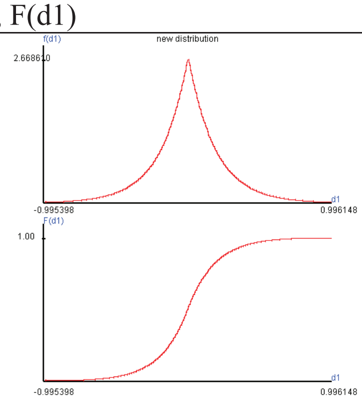
f(x2), F(x2)	Coefficient



Mathematical Mean	:	0.17531
Geometrical Mean	:	0.10129
Harmonic Mean	:	0.01910
Variance	:	0.02638
S.D.	:	0.16241
Skewed Coef.	:	1.45640
Kurtosis Coef.	:	5.15078
MAD	:	0.12526
Range	:	0.99935
Mid_range	:	0.49970
Median	:	0.12642
Q1	:	0.05273
Q2	:	0.12642
Q3	:	0.24962
IQR	:	0.19690
C.V.	:	0.92644

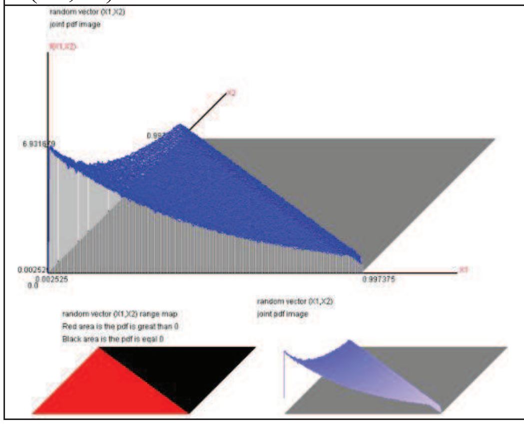
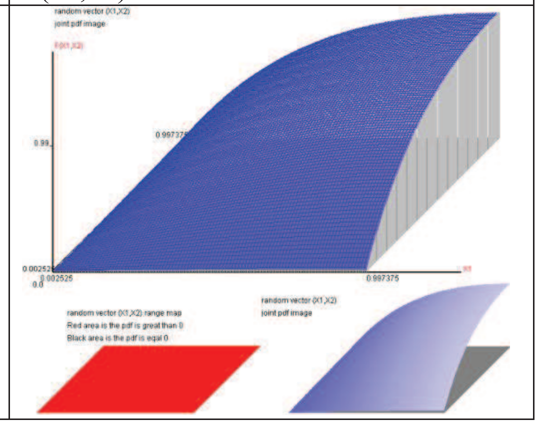
d1=X1-X2,

f(d1), F(d1)	Coefficient

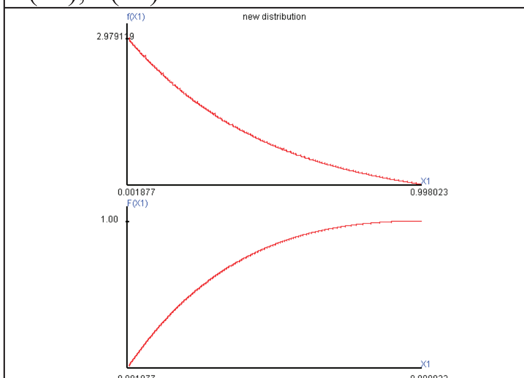


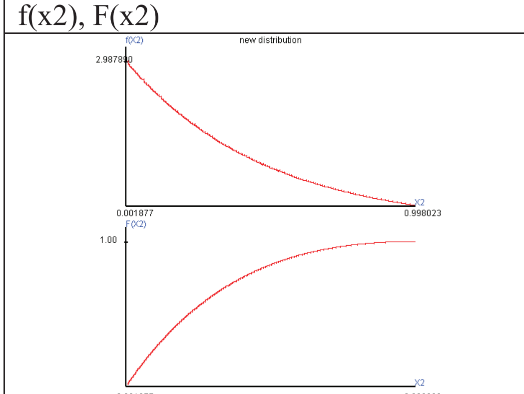
Mathematical Mean	:	-0.00000
Geometrical Mean	:	none
Harmonic Mean	:	none
Variance	:	0.05711
S.D.	:	0.23898
Skewed Coef.	:	0.00089
Kurtosis Coef.	:	4.22160
MAD	:	0.17532
Range	:	1.99895
Mid_range	:	0.00037
Median	:	0.00000
Q1	:	-0.12645
Q2	:	0.00000
Q3	:	0.12640
IQR	:	0.25285
C.V.	:	none

(4-4) λ_1=0.1, λ_2=0.1, $C(\lambda_1,\lambda_2)$=8.7879702452,

f(x1,x2)	F(x1,x2)
	

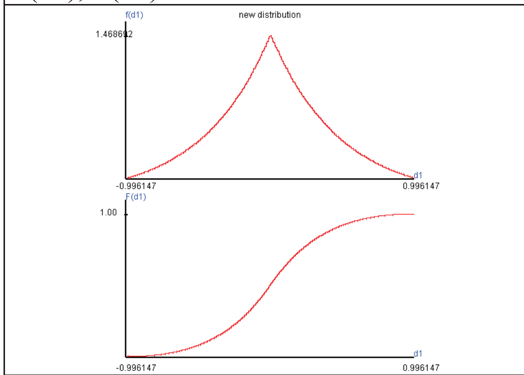
E(X1)= 0.2696, Var(X1)= 0.0458, E(X2)= 0.2696, Var(X2)= 0.0458,
Cov(X1,X2)= -0.0135, X1 and X2 correlation coefficient=-0.2947.

f(x1), F(x1)	Coefficient
	Mathematical Mean: 0.26964 Geometrical Mean : 0.16911 Harmonic Mean : 0.03274 Variance : 0.04577 S.D. : 0.21395 Skewed Coef. : 0.88087 Kurtosis Coef. : 3.05649 MAD : 0.17491 Range : 0.99985 Mid_range : 0.49995 Median : 0.21772 Q1 : 0.09447 Q2 : 0.21772 Q3 : 0.40087 IQR : 0.30640 C.V. : 0.79347

f(x2), F(x2)	Coefficient
	Mathematical Mean: 0.26959 Geometrical Mean : 0.16901 Harmonic Mean : 0.03104 Variance : 0.04576 S.D. : 0.21391 Skewed Coef. : 0.87989 Kurtosis Coef. : 3.05258 MAD : 0.17489 Range : 0.99985 Mid_range : 0.49995 Median : 0.21767 Q1 : 0.09442 Q2 : 0.21767 Q3 : 0.40092 IQR : 0.30650 C.V. : 0.79345

d1=X1-X2,

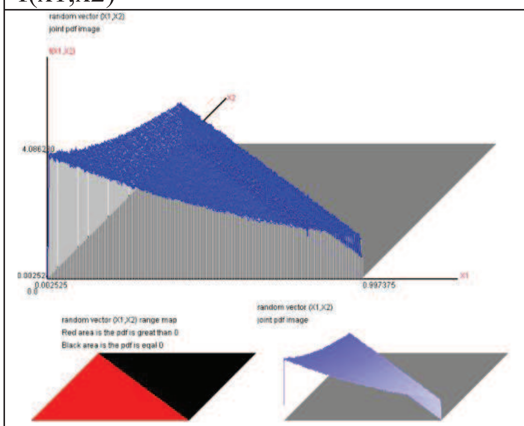
f(d1), F(d1)	Coefficient

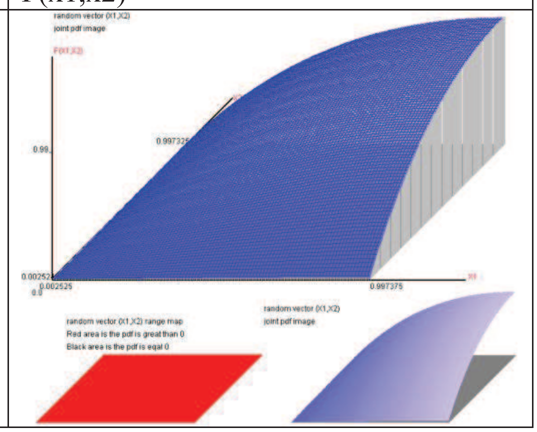


Coefficient	
Mathematical Mean:	0.00005
Geometrical Mean : none	
Harmonic Mean : none	
Variance :	0.11850
S.D. :	0.34424
Skewed Coef. :	0.00072
Kurtosis Coef. :	2.91709
MAD :	0.26968
Range :	1.99970
Mid_range :	0.00000
Median :	-0.00005
Q1 :	-0.21770
Q2 :	-0.00005
Q3 :	0.21790
IQR :	0.43560
C.V. : none	

(4-5) λ_1=0.2, λ_2=0.2, $C(\lambda_1,\lambda_2)$=6.6951731777,

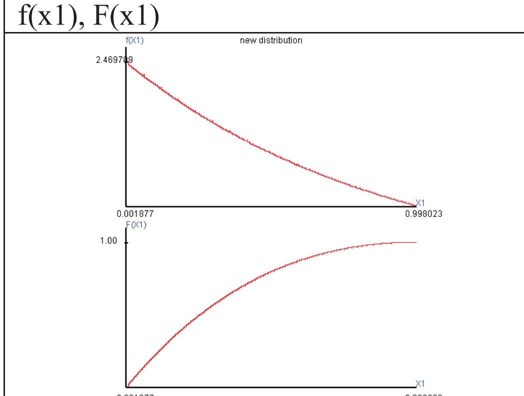
f(x1,x2)	F(x1,x2)



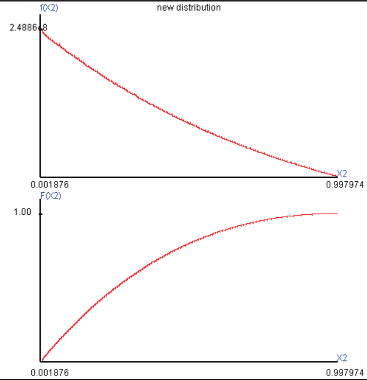


E(X1)= 0.3009, Var(X1)= 0.0509, E(X2)= 0.3007, Var(X2)= 0.0509,
Cov(X1,X2)= -0.0198, X1 and X2 correlation coefficient=-0.3894.

f(x1), F(x1)	Coefficient

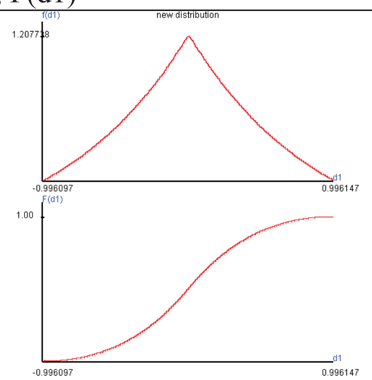


Coefficient	
Mathematical Mean:	0.30094
Geometrical Mean :	0.19485
Harmonic Mean :	0.03875
Variance :	0.05094
S.D. :	0.22571
Skewed Coef. :	0.72040
Kurtosis Coef. :	2.68484
MAD :	0.18705
Range :	0.99985
Mid_range :	0.49995
Median :	0.25352
Q1 :	0.11257
Q2 :	0.25352
Q3 :	0.45047
IQR :	0.33790
C.V. :	0.75002

f(x2), F(x2)	Coefficient
	

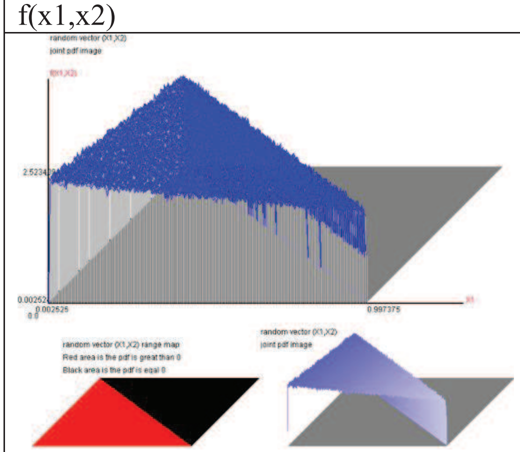
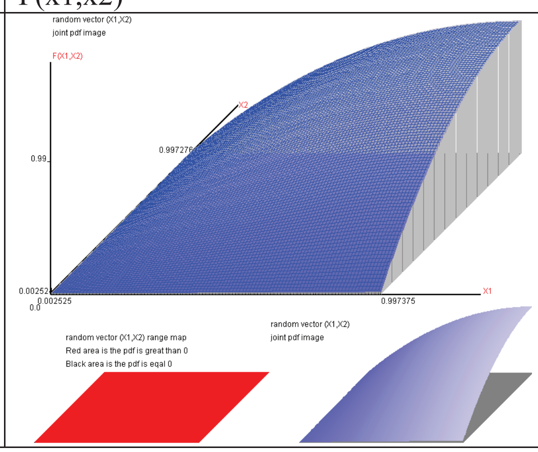
Coefficient		
Mathematical Mean:		0.30073
Geometrical Mean :		0.19462
Harmonic Mean	:	0.03507
Variance	:	0.05088
S.D.	:	0.22557
Skewed Coef.	:	0.72049
Kurtosis Coef.	:	2.68469
MAD	:	0.18693
Range	:	0.99980
Mid_range	:	0.49992
Median	:	0.25337
Q1	:	0.11257
Q2	:	0.25337
Q3	:	0.45007
IQR	:	0.33750
C.V.	:	0.75007

d1=X1-X2,

f(d1), F(d1)	Coefficient
	

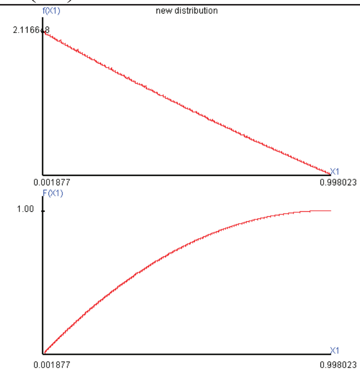
Coefficient		
Mathematical Mean:		0.00021
Geometrical Mean : none		
Harmonic Mean	: none	
Variance	:	0.14147
S.D.	:	0.37613
Skewed Coef.	:	0.00028
Kurtosis Coef.	:	2.63806
MAD	:	0.30091
Range	:	1.99965
Mid_range	:	0.00002
Median	:	0.00025
Q1	:	-0.25335
Q2	:	0.00025
Q3	:	0.25365
IQR	:	0.50700
C.V.	: none	

(4-6) λ_1=0.3, λ_2=0.3, $C(\lambda_1,\lambda_2)$=6.0432595817,

f(x1,x2)	F(x1,x2)
	

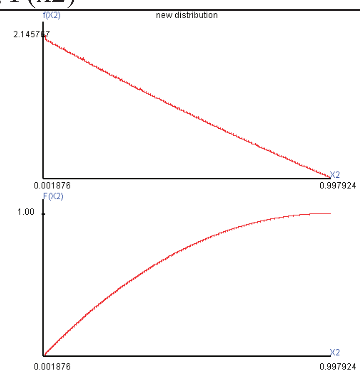
E(X1)= 0.3253, Var(X1)= 0.0545, E(X2)= 0.3252, Var(X2)= 0.0544, Cov(X1,X2)= -0.0257, X1 and X2 correlation coefficient=-0.4713.

f(x1), F(x1)



Coefficient	
Mathematical Mean:	0.32525
Geometrical Mean :	0.21595
Harmonic Mean :	0.04417
Variance :	0.05449
S.D. :	0.23343
Skewed Coef. :	0.60408
Kurtosis Coef. :	2.46500
MAD :	0.19512
Range :	0.99985
Mid_range :	0.49995
Median :	0.28287
Q1 :	0.12837
Q2 :	0.28287
Q3 :	0.48777
IQR :	0.35940
C.V. :	0.71769

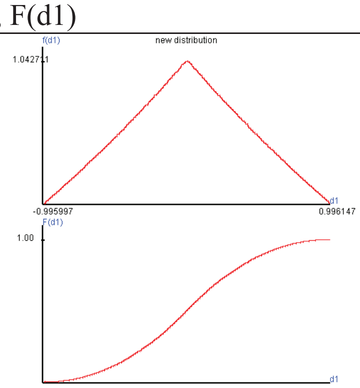
f(x2), F(x2)



Coefficient	
Mathematical Mean:	0.32523
Geometrical Mean :	0.21582
Harmonic Mean :	0.03818
Variance :	0.05445
S.D. :	0.23334
Skewed Coef. :	0.60257
Kurtosis Coef. :	2.46178
MAD :	0.19507
Range :	0.99975
Mid_range :	0.49990
Median :	0.28297
Q1 :	0.12837
Q2 :	0.28297
Q3 :	0.48782
IQR :	0.35945
C.V. :	0.71747

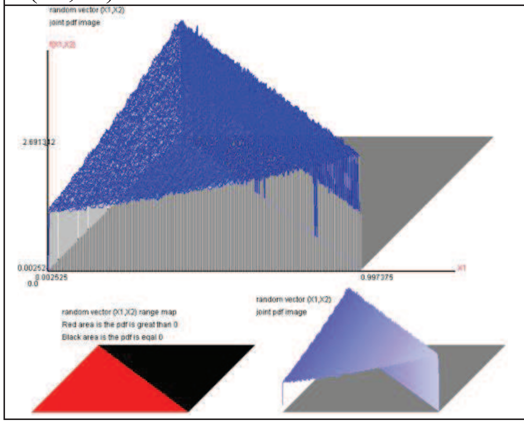
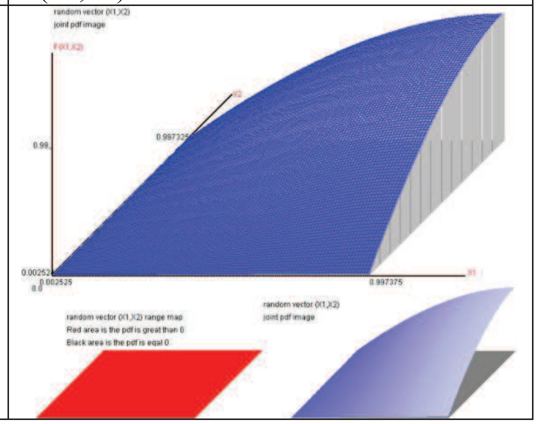
d1=X1-X2,

f(d1), F(d1)

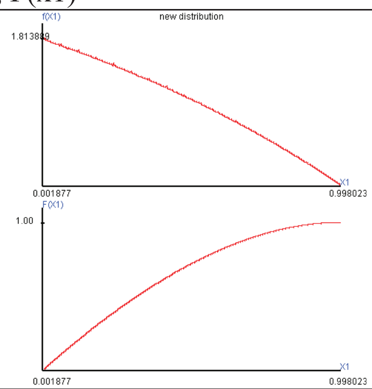


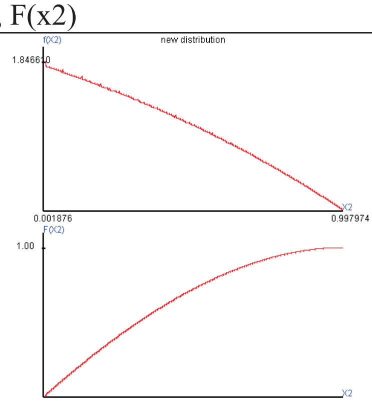
Coefficient	
Mathematical Mean:	0.00003
Geometrical Mean : none	
Harmonic Mean : none	
Variance :	0.16028
S.D. :	0.40035
Skewed Coef. :	0.00135
Kurtosis Coef. :	2.45570
MAD :	0.32526
Range :	1.99955
Mid_range :	0.00007
Median :	-0.00020
Q1 :	-0.28285
Q2 :	-0.00020
Q3 :	0.28290
IQR :	0.56575
C.V. : none	

(4-7) λ_1=0.4, λ_2=0.4, $C(\lambda_1,\lambda_2)$=6.2191290110,

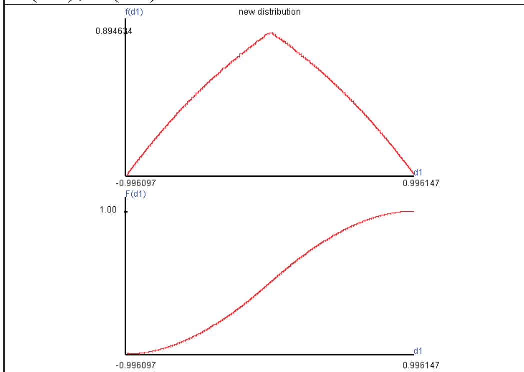
f(x1,x2)	F(x1,x2)
	

E(X1)= 0.3518, Var(X1)= 0.0580, E(X2)= 0.3516, Var(X2)= 0.0579,
Cov(X1,X2)= -0.0329, X1 and X2 correlation coefficient=-0.5680.

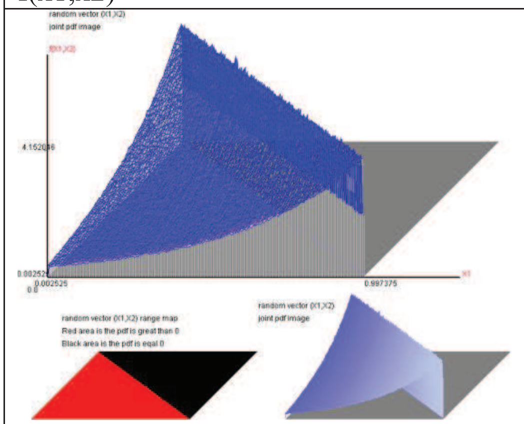
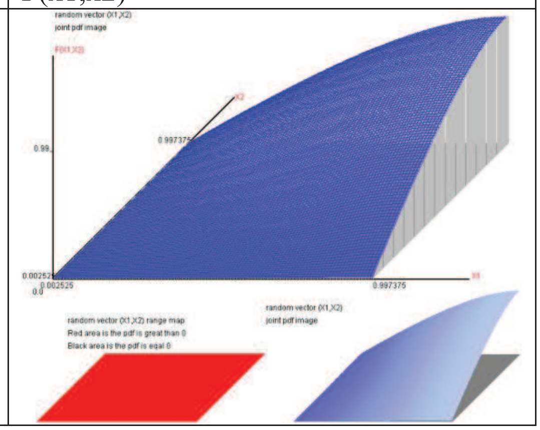
f(x1), F(x1)	Coefficient
	Mathematical Mean: 0.35176 Geometrical Mean : 0.23972 Harmonic Mean : 0.05057 Variance : 0.05796 S.D. : 0.24076 Skewed Coef. : 0.48401 Kurtosis Coef. : 2.27791 MAD : 0.20279 Range : 0.99985 Mid_range : 0.49995 Median : 0.31597 Q1 : 0.14712 Q2 : 0.31597 Q3 : 0.52732 IQR : 0.38020 C.V. : 0.68445

f(x2), F(x2)	Coefficient
	Mathematical Mean: 0.35161 Geometrical Mean : 0.23941 Harmonic Mean : 0.04041 Variance : 0.05791 S.D. : 0.24065 Skewed Coef. : 0.48326 Kurtosis Coef. : 2.27646 MAD : 0.20270 Range : 0.99980 Mid_range : 0.49992 Median : 0.31587 Q1 : 0.14707 Q2 : 0.31587 Q3 : 0.52707 IQR : 0.38000 C.V. : 0.68440

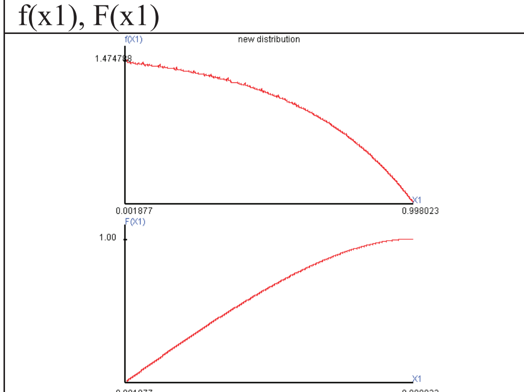
d1=X1-X2,

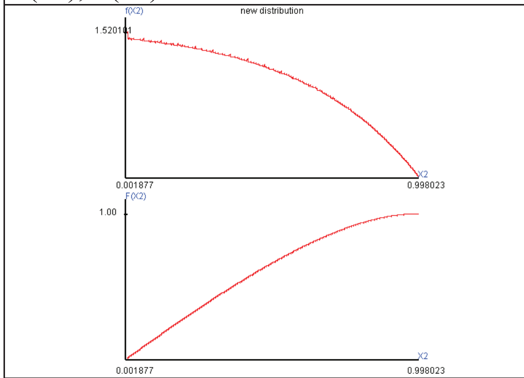
f(d1), F(d1)	Coefficient
	Mathematical Mean: 0.00014 Geometrical Mean : none Harmonic Mean : none Variance : 0.18169 S.D. : 0.42626 Skewed Coef. : 0.00088 Kurtosis Coef. : 2.28701 MAD : 0.35174 Range : 1.99965 Mid_range : 0.00002 Median : 0.00000 Q1 : -0.31590 Q2 : 0.00000 Q3 : 0.31610 IQR : 0.63200 C.V. : none

(4-8) λ_1=0.48, λ_2=0.48, $C(\lambda_1,\lambda_2)$=8.2036882336,

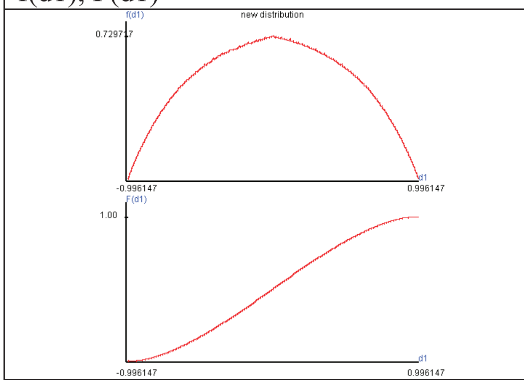
f(x1,x2)	F(x1,x2)
	

E(X1)= 0.3900, Var(X1)= 0.0625, E(X2)= 0.3897, Var(X2)= 0.0625,
Cov(X1,X2)= -0.0448, X1 and X2 correlation coefficient=-0.7169.

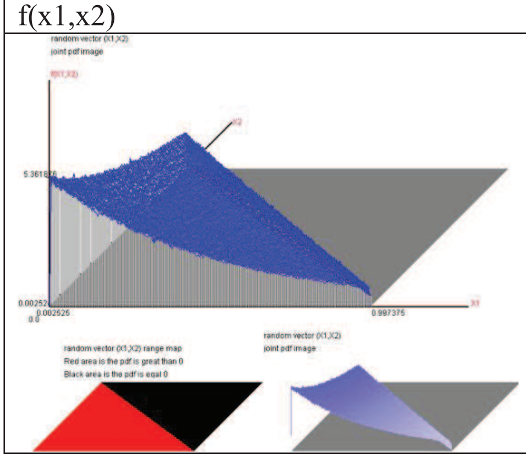
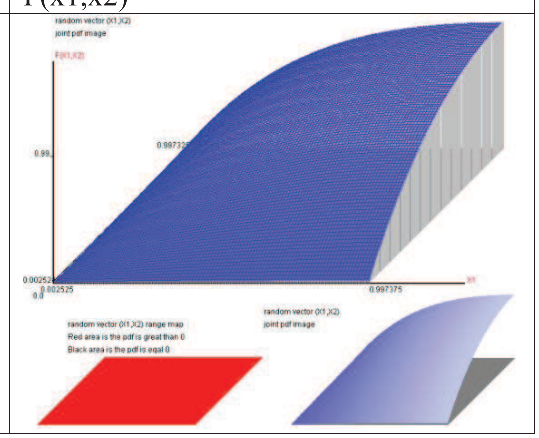
f(x1), F(x1)	Coefficient
	Mathematical Mean: 0.38996 Geometrical Mean : 0.27491 Harmonic Mean : 0.06058 Variance : 0.06251 S.D. : 0.25003 Skewed Coef. : 0.32090 Kurtosis Coef. : 2.08239 MAD : 0.21242 Range : 0.99985 Mid_range : 0.49995 Median : 0.36517 Q1 : 0.17622 Q2 : 0.36517 Q3 : 0.58252 IQR : 0.40630 C.V. : 0.64115

f(x2), F(x2)	Coefficient	
	Mathematical Mean:	0.38969
	Geometrical Mean :	0.27428
	Harmonic Mean :	0.04124
	Variance :	0.06247
	S.D. :	0.24994
	Skewed Coef. :	0.32051
	Kurtosis Coef. :	2.08063
	MAD :	0.21236
	Range :	0.99985
	Mid_range :	0.49995
	Median :	0.36492
	Q1 :	0.17602
	Q2 :	0.36492
	Q3 :	0.58227
	IQR :	0.40625
	C.V. :	0.64139

d1=X1-X2,

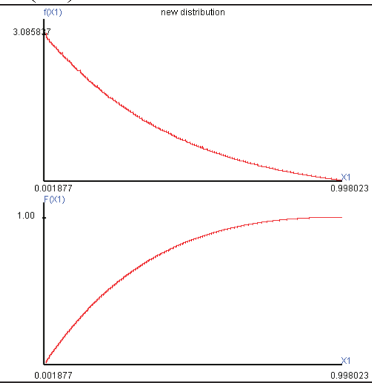
f(d1), F(d1)	Coefficient	
	Mathematical Mean:	0.00028
	Geometrical Mean : none	
	Harmonic Mean : none	
	Variance :	0.21458
	S.D. :	0.46323
	Skewed Coef. :	0.00045
	Kurtosis Coef. :	2.08761
	MAD :	0.38995
	Range :	1.99970
	Mid_range :	0.00000
	Median :	0.00030
	Q1 :	-0.36505
	Q2 :	0.00030
	Q3 :	0.36535
	IQR :	0.73040
	C.V. : none	

(4-9) λ_1=0.1, λ_2=0.2, $C(\lambda_1,\lambda_2)$=7.6357730188,

f(x1,x2)	F(x1,x2)
	

E(X1)= 0.2634, Var(X1)= 0.0443, E(X2)= 0.3082, Var(X2)= 0.0524, Cov(X1,X2)= -0.0164, X1 and X2 correlation coefficient=-0.3403.

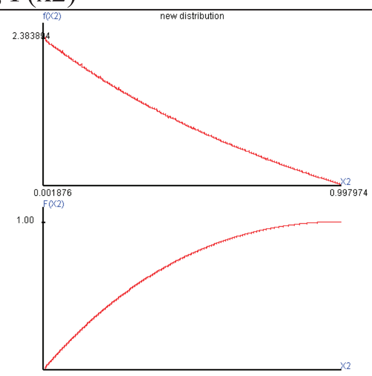
f(x1), F(x1)



Coefficient

Mathematical Mean:	0.26338
Geometrical Mean :	0.16462
Harmonic Mean :	0.03182
Variance :	0.04432
S.D. :	0.21052
Skewed Coef. :	0.90974
Kurtosis Coef. :	3.14231
MAD :	0.17159
Range :	0.99985
Mid_range :	0.49995
Median :	0.21162
Q1 :	0.09167
Q2 :	0.21162
Q3 :	0.39067
IQR :	0.29900
C.V. :	0.79931

f(x2), F(x2)

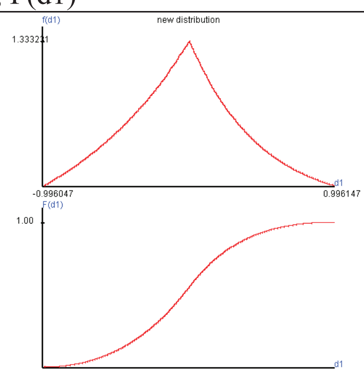


Coefficient

Mathematical Mean:	0.30818
Geometrical Mean :	0.20049
Harmonic Mean :	0.03692
Variance :	0.05235
S.D. :	0.22881
Skewed Coef. :	0.68722
Kurtosis Coef. :	2.61003
MAD :	0.19020
Range :	0.99980
Mid_range :	0.49992
Median :	0.26152
Q1 :	0.11647
Q2 :	0.26152
Q3 :	0.46217
IQR :	0.34570
C.V. :	0.74246

d1=X1-X2,

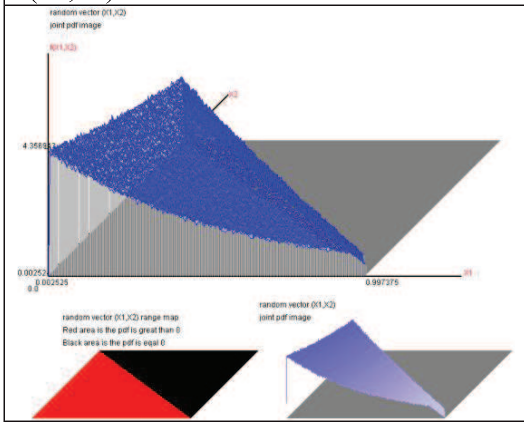
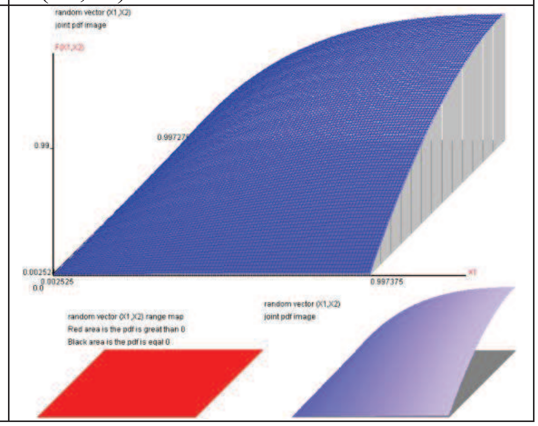
f(d1), F(d1)



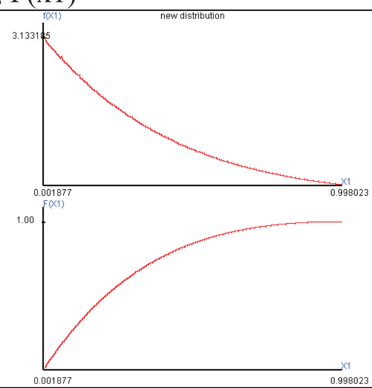
Coefficient

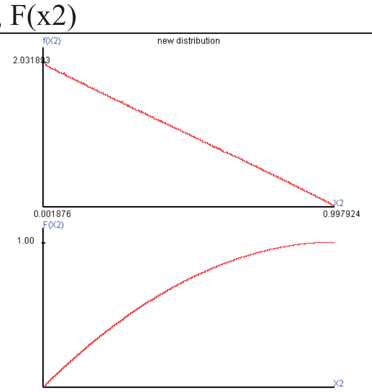
Mathematical Mean:	-0.04480
Geometrical Mean :	none
Harmonic Mean :	none
Variance :	0.12946
S.D. :	0.35980
Skewed Coef. :	0.02978
Kurtosis Coef. :	2.76650
MAD :	0.28543
Range :	1.99960
Mid_range :	0.00005
Median :	-0.03805
Q1 :	-0.28480
Q2 :	-0.03805
Q3 :	0.18695
IQR :	0.47175
C.V. :	none

(4-10) λ_1=0.1, λ_2=0.3, $C(\lambda_1,\lambda_2)$=7.1455294994,

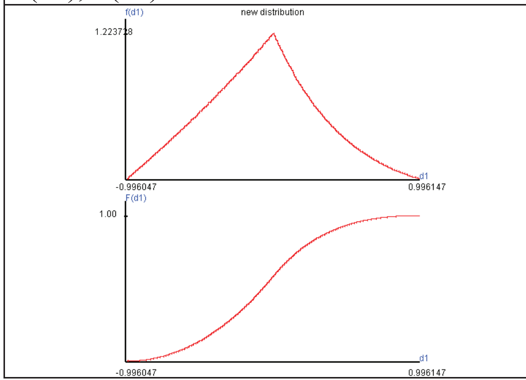
f(x1,x2)	F(x1,x2)
	

E(X1)= 0.2604, Var(X1)= 0.0436, E(X2)= 0.3359, Var(X2)= 0.0563,
Cov(X1,X2)= -0.0186, X1 and X2 correlation coefficient=-0.3753.

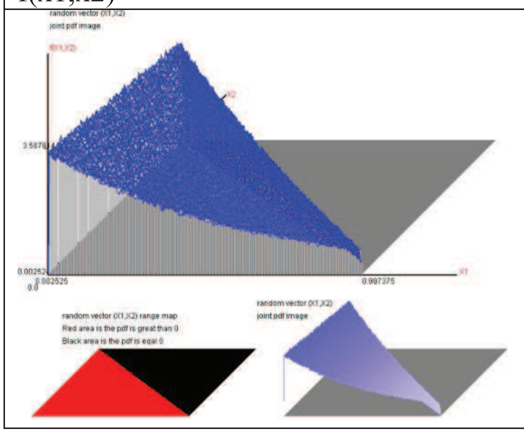
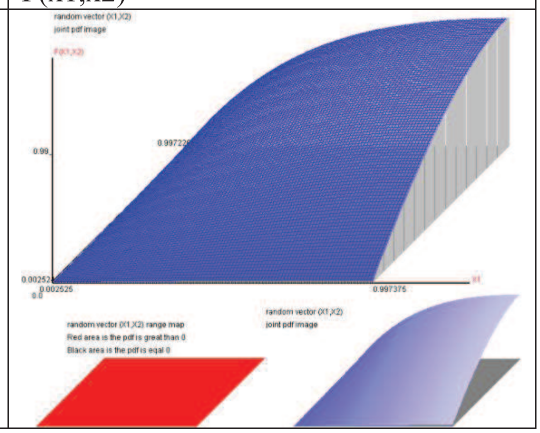
f(x1), F(x1)	Coefficient	
	Mathematical Mean:	0.26041
	Geometrical Mean :	0.16249
	Harmonic Mean :	0.03141
	Variance :	0.04365
	S.D. :	0.20892
	Skewed Coef. :	0.92426
	Kurtosis Coef. :	3.18602
	MAD :	0.17003
	Range :	0.99985
	Mid_range :	0.49995
	Median :	0.20867
	Q1 :	0.09037
	Q2 :	0.20867
	Q3 :	0.38582
	IQR :	0.29545
	C.V. :	0.80228

f(x2), F(x2)	Coefficient	
	Mathematical Mean:	0.33585
	Geometrical Mean :	0.22456
	Harmonic Mean :	0.04100
	Variance :	0.05632
	S.D. :	0.23732
	Skewed Coef. :	0.55704
	Kurtosis Coef. :	2.37828
	MAD :	0.19911
	Range :	0.99975
	Mid_range :	0.49990
	Median :	0.29532
	Q1 :	0.13472
	Q2 :	0.29532
	Q3 :	0.50447
	IQR :	0.36975
	C.V. :	0.70663

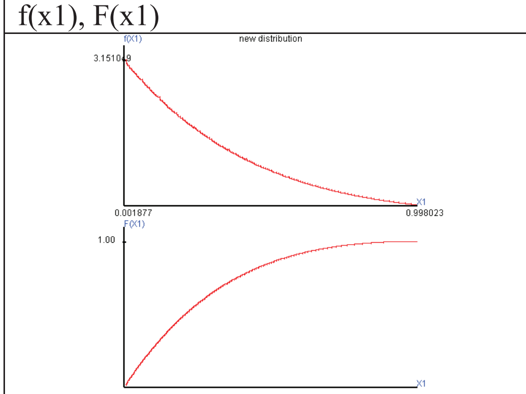
d1=X1-X2,

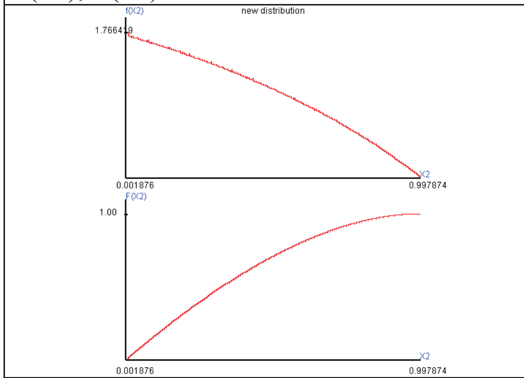
f(d1), F(d1)	Coefficient
	Mathematical Mean: -0.07545 Geometrical Mean : none Harmonic Mean : none Variance : 0.13718 S.D. : 0.37038 Skewed Coef. : 0.06820 Kurtosis Coef. : 2.66832 MAD : 0.29674 Range : 1.99960 Mid_range : 0.00005 Median : -0.06845 Q1 : -0.33250 Q2 : -0.06845 Q3 : 0.16675 IQR : 0.49925 C.V. : none

(4-11) λ_1=0.1, λ_2=0.4, $C(\lambda_1,\lambda_2)$=6.945348179,

f(x1,x2)	F(x1,x2)
	

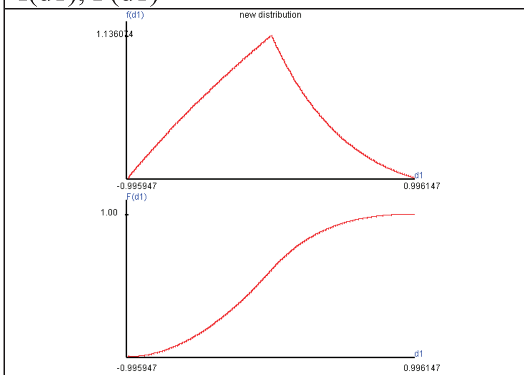
E(X1)= 0.2591, Var(X1)= 0.0433, E(X2)= 0.3594, Var(X2)= 0.0591,
Cov(X1,X2)= -0.0206, X1 and X2 correlation coefficient=-0.4077.

f(x1), F(x1)	Coefficient
	Mathematical Mean: 0.25910 Geometrical Mean : 0.16156 Harmonic Mean : 0.03123 Variance : 0.04335 S.D. : 0.20820 Skewed Coef. : 0.93048 Kurtosis Coef. : 3.20533 MAD : 0.16934 Range : 0.99985 Mid_range : 0.49995 Median : 0.20742 Q1 : 0.08982 Q2 : 0.20742 Q3 : 0.38367 IQR : 0.29385 C.V. : 0.80356

f(x2), F(x2)	Coefficient
	

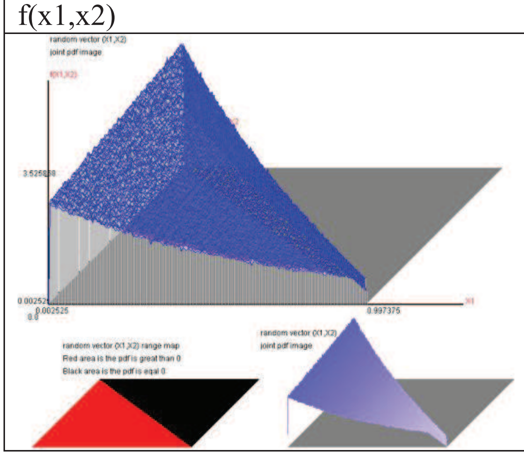
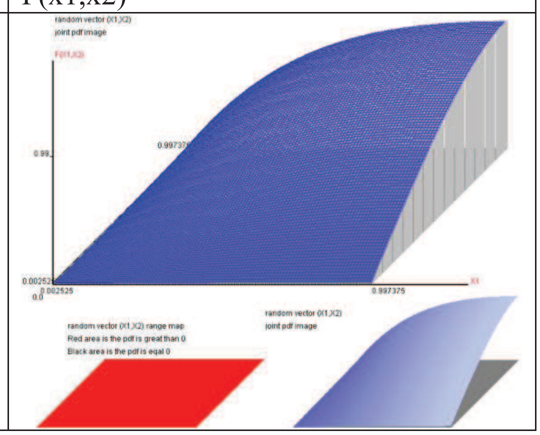
Coefficient		
Mathematical Mean:		0.35936
Geometrical Mean :		0.24622
Harmonic Mean	:	0.04503
Variance	:	0.05907
S.D.	:	0.24304
Skewed Coef.	:	0.45046
Kurtosis Coef.	:	2.22760
MAD	:	0.20517
Range	:	0.99970
Mid_range	:	0.49987
Median	:	0.32542
Q1	:	0.15232
Q2	:	0.32542
Q3	:	0.53892
IQR	:	0.38660
C.V.	:	0.67630

d1=X1-X2,

f(d1), F(d1)	Coefficient
	

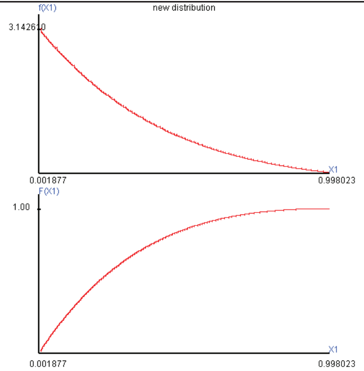
Coefficient		
Mathematical Mean:		-0.10026
Geometrical Mean	:	none
Harmonic Mean	:	none
Variance	:	0.14368
S.D.	:	0.37905
Skewed Coef.	:	0.10956
Kurtosis Coef.	:	2.59600
MAD	:	0.30605
Range	:	1.99950
Mid_range	:	0.00010
Median	:	-0.09580
Q1	:	-0.37190
Q2	:	-0.09580
Q3	:	0.15095
IQR	:	0.52285
C.V.	:	none

(4-12) λ_1=0.1, λ_2=0.5, $C(\lambda_1,\lambda_2)$=6.9453825633,

f(x1,x2)	F(x1,x2)
	

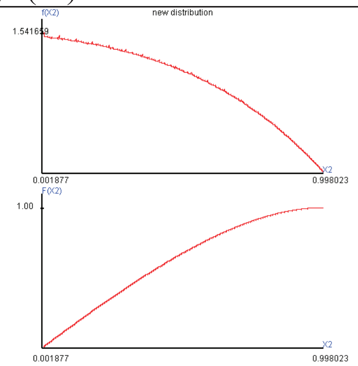
E(X1)= 0.2591, Var(X1)= 0.0434, E(X2)= 0.3814, Var(X2)= 0.0611, Cov(X1,X2)= -0.0227, X1 and X2 correlation coefficient=-0.4411.

f(x1), F(x1)	Coefficient



Mathematical Mean:		0.25909
Geometrical Mean	:	0.16156
Harmonic Mean	:	0.03126
Variance	:	0.04335
S.D.	:	0.20821
Skewed Coef.	:	0.93096
Kurtosis Coef.	:	3.20695
MAD	:	0.16933
Range	:	0.99985
Mid_range	:	0.49995
Median	:	0.20737
Q1	:	0.08982
Q2	:	0.20737
Q3	:	0.38362
IQR	:	0.29380
C.V.	:	0.80362

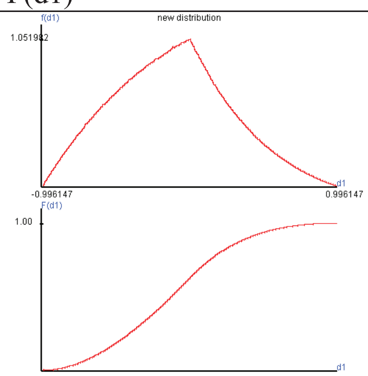
f(x2), F(x2)	Coefficient



Mathematical Mean:		0.38144
Geometrical Mean	:	0.26749
Harmonic Mean	:	0.04904
Variance	:	0.06115
S.D.	:	0.24728
Skewed Coef.	:	0.35341
Kurtosis Coef.	:	2.12112
MAD	:	0.20965
Range	:	0.99985
Mid_range	:	0.49995
Median	:	0.35457
Q1	:	0.17052
Q2	:	0.35457
Q3	:	0.56957
IQR	:	0.39905
C.V.	:	0.64827

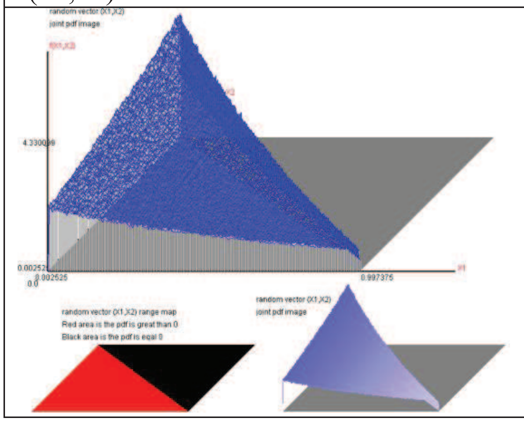
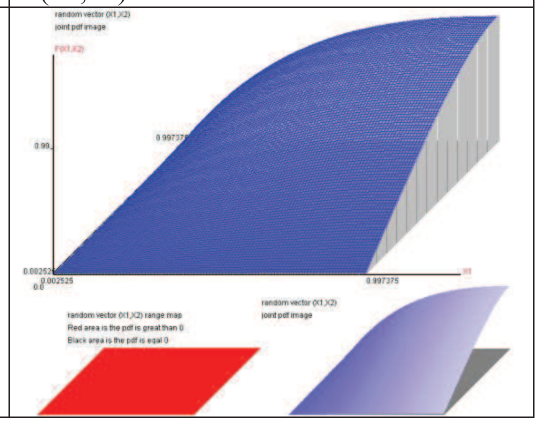
d1=X1-X2,

f(d1), F(d1)	Coefficient



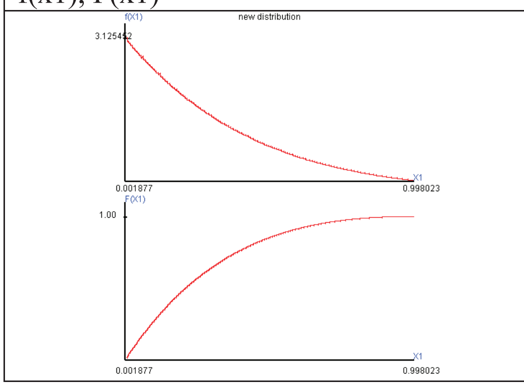
Mathematical Mean:		-0.12235
Geometrical Mean	:	none
Harmonic Mean	:	none
Variance	:	0.14992
S.D.	:	0.38719
Skewed Coef.	:	0.15481
Kurtosis Coef.	:	2.53994
MAD	:	0.31466
Range	:	1.99970
Mid_range	:	0.00000
Median	:	-0.12260
Q1	:	-0.40740
Q2	:	-0.12260
Q3	:	0.13725
IQR	:	0.54465
C.V.	:	none

(4-13) λ_1=0.1, λ_2=0.6, $C(\lambda_1,\lambda_2)$=7.1456533130,

f(x1,x2)	F(x1,x2)
	

E(X1)= 0.2604, Var(X1)= 0.0437, E(X2)= 0.4037, Var(X2)= 0.0628,
Cov(X1,X2)= -0.0251, X1 and X2 correlation coefficient=-0.4786.

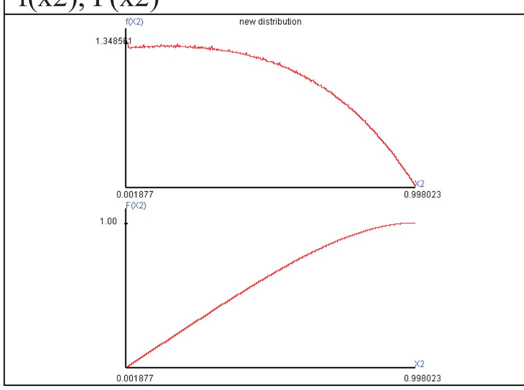
f(x1), F(x1)



Coefficient

Mathematical Mean:	0.26044
Geometrical Mean :	0.16251
Harmonic Mean :	0.03146
Variance :	0.04367
S.D. :	0.20898
Skewed Coef. :	0.92525
Kurtosis Coef. :	3.18932
MAD :	0.17006
Range :	0.99985
Mid_range :	0.49995
Median :	0.20867
Q1 :	0.09042
Q2 :	0.20867
Q3 :	0.38577
IQR :	0.29535
C.V. :	0.80242

f(x2), F(x2)

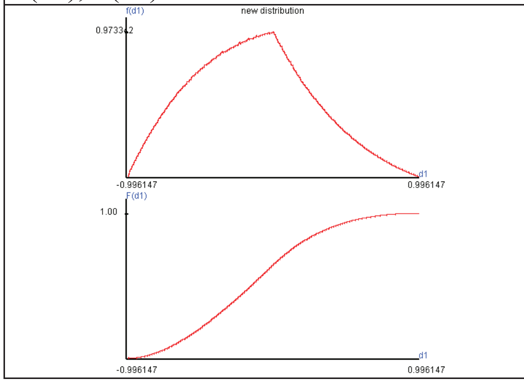


Coefficient

Mathematical Mean:	0.40367
Geometrical Mean :	0.28974
Harmonic Mean :	0.05218
Variance :	0.06276
S.D. :	0.25052
Skewed Coef. :	0.25837
Kurtosis Coef. :	2.04391
MAD :	0.21302
Range :	0.99985
Mid_range :	0.49995
Median :	0.38457
Q1 :	0.19052
Q2 :	0.38457
Q3 :	0.59882
IQR :	0.40830
C.V. :	0.62060

d1=X1-X2,

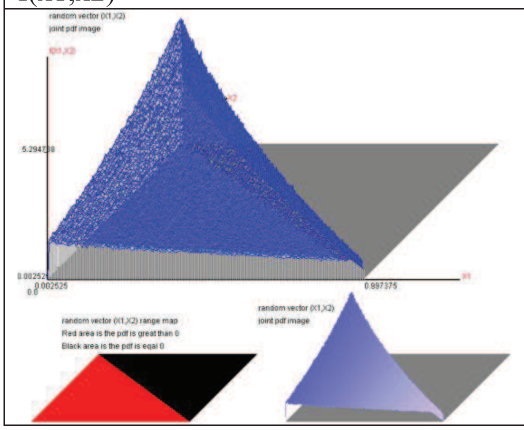
f(d1), F(d1)



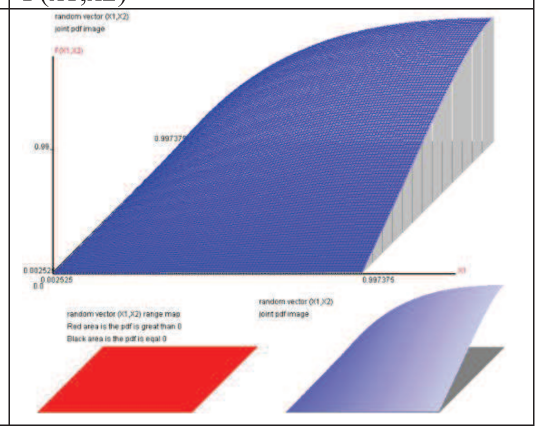
Coefficient	
Mathematical Mean:	-0.14323
Geometrical Mean : none	
Harmonic Mean : none	
Variance :	0.15654
S.D. :	0.39566
Skewed Coef. :	0.20392
Kurtosis Coef. :	2.49334
MAD :	0.32344
Range :	1.99970
Mid_range :	0.00000
Median :	-0.15020
Q1 :	-0.44170
Q2 :	-0.15020
Q3 :	0.12510
IQR :	0.56680
C.V. : none	

(4-14) λ_1=0.1, λ_2=0.7, $C(\lambda_1,\lambda_2)$=7.6360121679,

f(x1,x2)

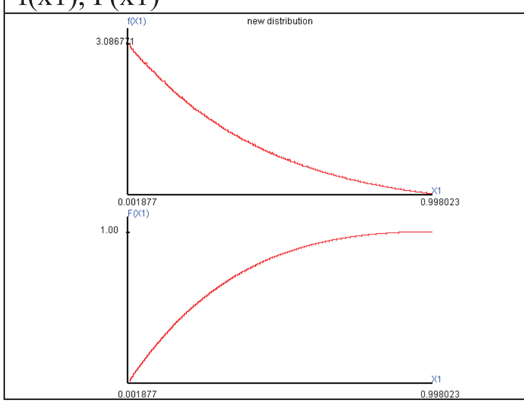


F(x1,x2)



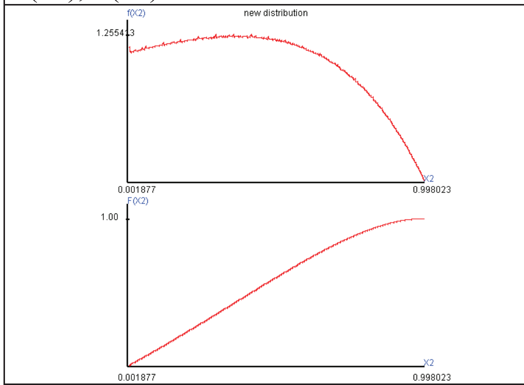
E(X1)= 0.2635, Var(X1)= 0.0444, E(X2)= 0.4283, Var(X2)= 0.0639,
Cov(X1,X2)= -0.0280, X1 and X2 correlation coefficient=-0.5252.

f(x1), F(x1)



Coefficient	
Mathematical Mean:	0.26345
Geometrical Mean :	0.16465
Harmonic Mean :	0.03183
Variance :	0.04436
S.D. :	0.21062
Skewed Coef. :	0.91087
Kurtosis Coef. :	3.14641
MAD :	0.17165
Range :	0.99985
Mid_range :	0.49995
Median :	0.21162
Q1 :	0.09172
Q2 :	0.21162
Q3 :	0.39077
IQR :	0.29905
C.V. :	0.79948

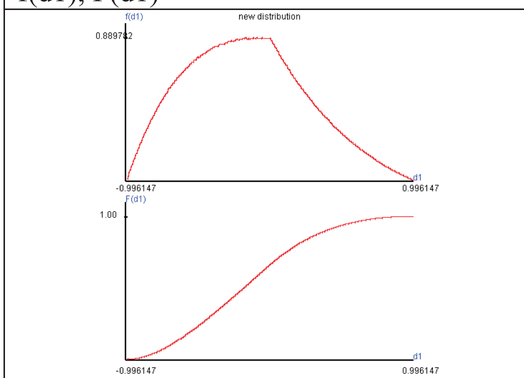
f(x2), F(x2)	Coefficient



Coefficient		
Mathematical Mean:		0.42835
Geometrical Mean :		0.31557
Harmonic Mean	:	0.05597
Variance	:	0.06393
S.D.	:	0.25285
Skewed Coef.	:	0.15523
Kurtosis Coef.	:	1.99025
MAD	:	0.21536
Range	:	0.99985
Mid_range	:	0.49995
Median	:	0.41812
Q1	:	0.21517
Q2	:	0.41812
Q3	:	0.62932
IQR	:	0.41415
C.V.	:	0.59029

d1=X1-X2,

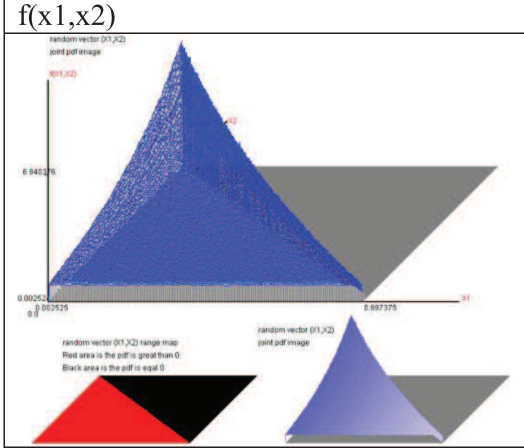
f(d1), F(d1)	Coefficient

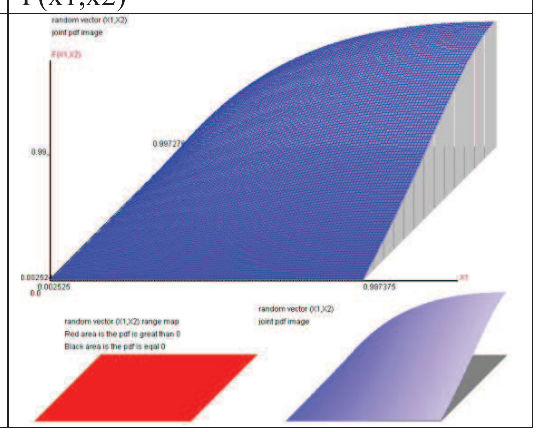


Coefficient		
Mathematical Mean:		-0.16490
Geometrical Mean	:	none
Harmonic Mean	:	none
Variance	:	0.16423
S.D.	:	0.40526
Skewed Coef.	:	0.26130
Kurtosis Coef.	:	2.45371
MAD	:	0.33313
Range	:	1.99970
Mid_range	:	0.00000
Median	:	-0.18125
Q1	:	-0.47745
Q2	:	-0.18125
Q3	:	0.11310
IQR	:	0.59055
C.V.	:	none

(4-15) λ_1=0.1, λ_2=0.8, $C(\lambda_1,\lambda_2)$=0.87884271088,

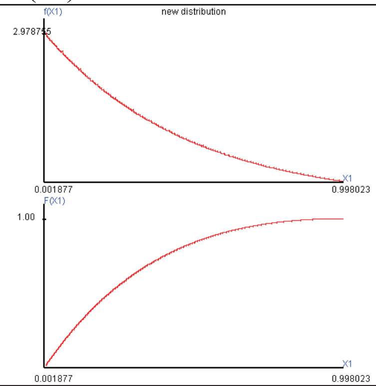
f(x1,x2)	F(x1,x2)





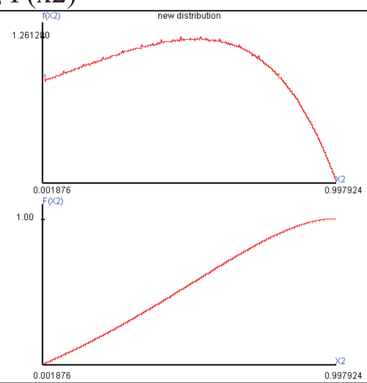
E(X1)= 0.2697, Var(X1)= 0.0458, E(X2)= 0.4606, Var(X2)= 0.0646,
Cov(X1,X2)= -0.0323, X1 and X2 correlation coefficient=-0.5940.

f(x1), F(x1)



Coefficient	
Mathematical Mean:	0.26971
Geometrical Mean :	0.16914
Harmonic Mean :	0.03275
Variance :	0.04583
S.D. :	0.21407
Skewed Coef. :	0.88238
Kurtosis Coef. :	3.06223
MAD :	0.17498
Range :	0.99985
Mid_range :	0.49995
Median :	0.21777
Q1 :	0.09447
Q2 :	0.21777
Q3 :	0.40092
IQR :	0.30645
C.V. :	0.79369

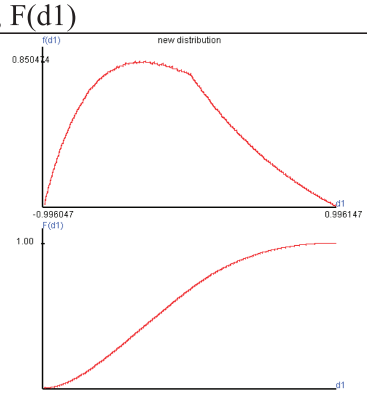
f(x2), F(x2)



Coefficient	
Mathematical Mean:	0.46061
Geometrical Mean :	0.35072
Harmonic Mean :	0.05938
Variance :	0.06459
S.D. :	0.25414
Skewed Coef. :	0.02217
Kurtosis Coef. :	1.96643
MAD :	0.21644
Range :	0.99975
Mid_range :	0.49990
Median :	0.46217
Q1 :	0.25067
Q2 :	0.46217
Q3 :	0.66607
IQR :	0.41540
C.V. :	0.55175

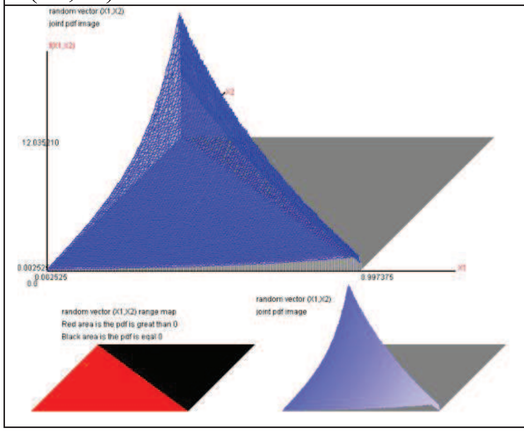
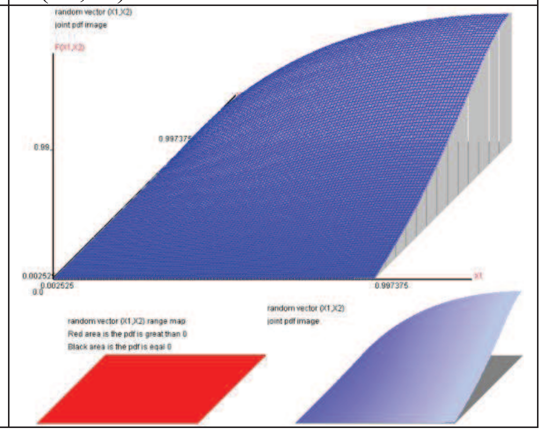
d1=X1-X2,

f(d1), F(d1)

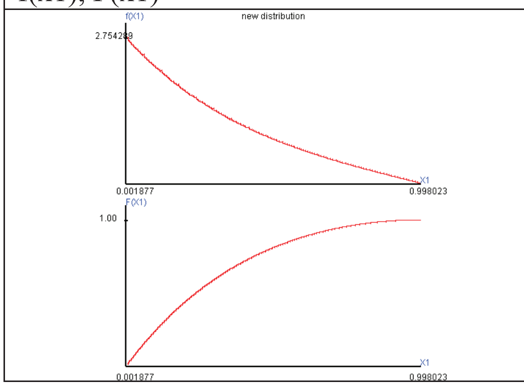


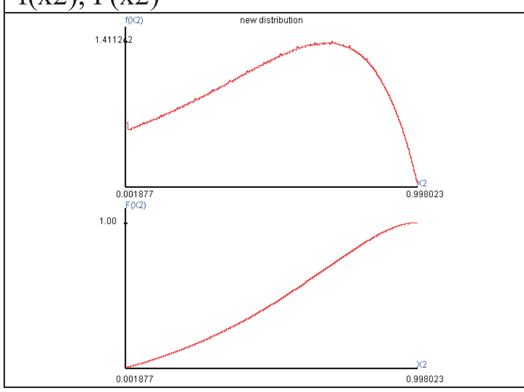
Coefficient	
Mathematical Mean:	-0.19090
Geometrical Mean :	none
Harmonic Mean :	none
Variance :	0.17505
S.D. :	0.41839
Skewed Coef. :	0.33967
Kurtosis Coef. :	2.42189
MAD :	0.34583
Range :	1.99960
Mid_range :	0.00005
Median :	-0.22195
Q1 :	-0.52110
Q2 :	-0.22195
Q3 :	0.09930
IQR :	0.62040
C.V. :	none

(4-16) λ_1=0.1, λ_2=0.89, $C(\lambda_1, \lambda_2)$=13.9288280159,

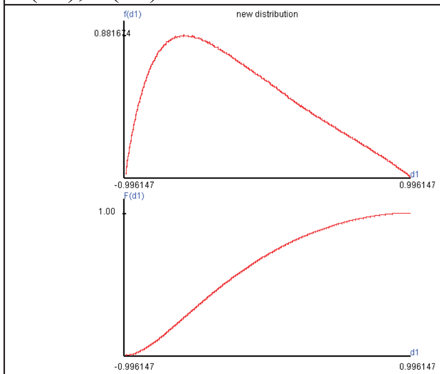
f(x1,x2)	F(x1,x2)
	

E(X1)= 0.2891, Var(X1)= 0.0507, E(X2)= 0.5273, Var(X2)= 0.0636,
Cov(X1,X2)= -0.0434, X1 and X2 correlation coefficient=-0.7639.

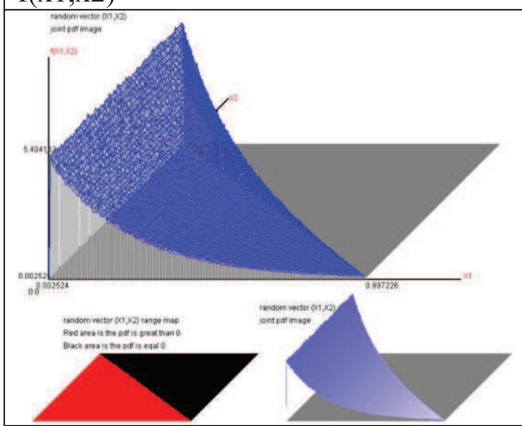
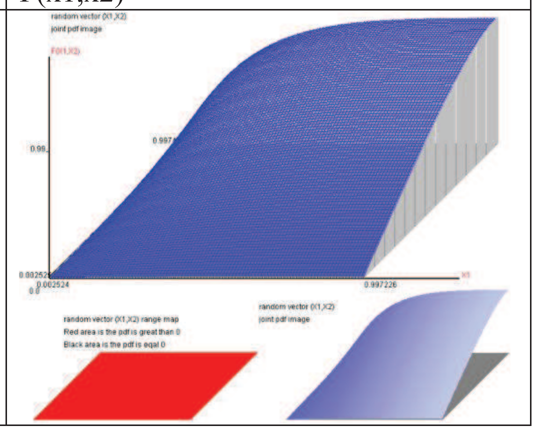
f(x1), F(x1)	Coefficient	
	Mathematical Mean: Geometrical Mean : Harmonic Mean : Variance : S.D. : Skewed Coef. : Kurtosis Coef. : MAD : Range : Mid_range : Median : Q1 : Q2 : Q3 : IQR : C.V. :	0.28912 0.18262 0.03519 0.05068 0.22513 0.80025 2.82457 0.18567 0.99985 0.49995 0.23647 0.10252 0.23647 0.43327 0.33075 0.77868

f(x2), F(x2)	Coefficient	
	Mathematical Mean: Geometrical Mean : Harmonic Mean : Variance : S.D. : Skewed Coef. : Kurtosis Coef. : MAD : Range : Mid_range : Median : Q1 : Q2 : Q3 : IQR : C.V. :	0.52728 0.42671 0.05734 0.06355 0.25209 -0.24858 2.06515 0.21319 0.99985 0.49995 0.55107 0.33292 0.55107 0.73382 0.40090 0.47810

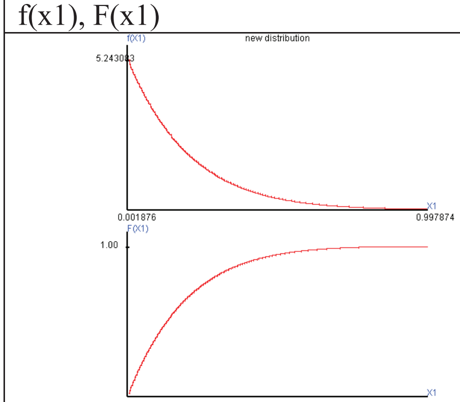
d1=X1-X2,

f(d1), F(d1)	Coefficient
	Mathematical Mean: -0.23816 Geometrical Mean : none Harmonic Mean : none Variance : 0.20094 S.D. : 0.44826 Skewed Coef. : 0.49886 Kurtosis Coef. : 2.41031 MAD : 0.37308 Range : 1.99970 Mid_range : 0.00000 Median : -0.30470 Q1 : -0.60250 Q2 : -0.30470 Q3 : 0.07485 IQR : 0.67735 C.V. : none

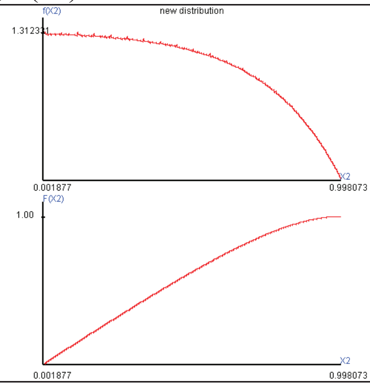
(4-17) λ_1=0.01, λ_2=0.5, $C(\lambda_1, \lambda_2)$=10.5265104948,

f(x1,x2)	F(x1,x2)
	

E(X1)= 0.1773, Var(X1)= 0.0259, E(X2)= 0.4125, Var(X2)= 0.0651,
Cov(X1,X2)= -0.0130, X1 and X2 correlation coefficient=-0.3166.

f(x1), F(x1)	Coefficient
	Mathematical Mean: 0.17732 Geometrical Mean : 0.10387 Harmonic Mean : 0.01982 Variance : 0.02586 S.D. : 0.16082 Skewed Coef. : 1.39661 Kurtosis Coef. : 4.94160 MAD : 0.12481 Range : 0.99970 Mid_range : 0.49987 Median : 0.13017 Q1 : 0.05473 Q2 : 0.13017 Q3 : 0.25382 IQR : 0.19910 C.V. : 0.90698

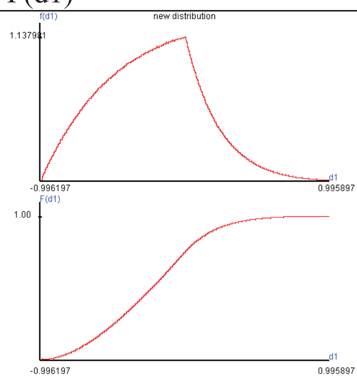
f(x2), F(x2)	Coefficient



Mathematical Mean:		0.41251
Geometrical Mean :		0.29594
Harmonic Mean	:	0.06718
Variance	:	0.06506
S.D.	:	0.25507
Skewed Coef.	:	0.22824
Kurtosis Coef.	:	1.99682
MAD	:	0.21763
Range	:	0.99990
Mid_range	:	0.49997
Median	:	0.39487
Q1	:	0.19417
Q2	:	0.39487
Q3	:	0.61457
IQR	:	0.42040
C.V.	:	0.61833

d1=X1-X2,

f(d1), F(d1)	Coefficient



Mathematical Mean:		-0.23520
Geometrical Mean : none		
Harmonic Mean	: none	
Variance	:	0.11690
S.D.	:	0.34191
Skewed Coef.	:	0.14965
Kurtosis Coef.	:	2.65729
MAD	:	0.27932
Range	:	1.99950
Mid_range	:	-0.00015
Median	:	-0.22775
Q1	:	-0.49110
Q2	:	-0.22775
Q3	:	0.00085
IQR	:	0.49195
C.V.	: none	

6.The conditional probability $f_{X_2|x_1}(x_2|x_1)$,

$$f_{X_2|x_1}(x_2|x_1)=\frac{\lambda_1^{x_1}\lambda_2^{x_2}(1-\lambda_1-\lambda_2)^{1-x_1-x_2}}{\int_0^{1-x_1}\lambda_2^{x_2}(1-\lambda_1-\lambda_2)^{1-x_1-x_2}dx_2},0\le x_2\le 1-x_1,$$

$$\int_0^{1-x_1}C(\lambda_1,\lambda_2)\lambda_1^{x_1}\lambda_2^{x_2}(1-\lambda_1-\lambda_2)^{1-x_1-x_2}dx_2\ne C(\lambda_1)\lambda_1^{x_1}(1-\lambda_1)^{1-x_1},$$

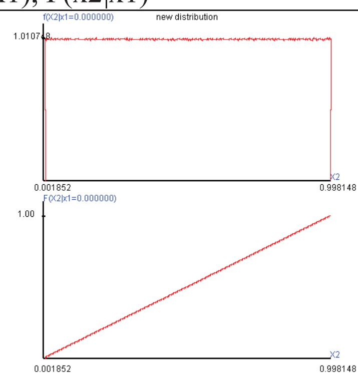
The numerical analysis,

$$f_{X_1}(x_1;\lambda_1,\lambda_2)\cong\sum_{x_2}^{1-x_1}C(\lambda_1,\lambda_2)\lambda_1^{\Delta x_1}\lambda_2^{\Delta x_2}(1-\lambda_1-\lambda_2)^{1-\Delta x_1-\Delta x_2}\Delta x_2,$$

$$f_{X_2|x_1}(x_2|x_1)\cong\frac{\lambda_2^{x_2}(1-\lambda_1-\lambda_2)^{1-x_1-x_2}}{\sum_{x_2}^{1-x_1}\lambda_2^{\Delta x_2}(1-\lambda_1-\lambda_2)^{1-\Delta x_1-\Delta x_2}\Delta x_2}$$

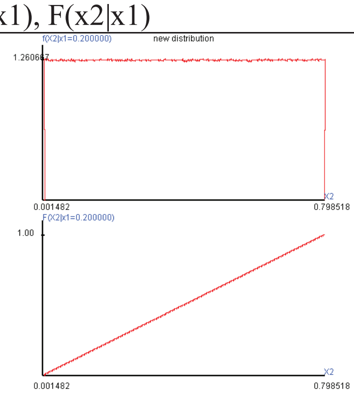
(1) λ_1=0.2, λ_2=0.4,

(1-1) x1=0,

f(x2|x1), F(x2|x1)	Coefficient	
	Mathematical Mean:	0.50000
	Geometrical Mean :	0.36789
	Harmonic Mean :	0.05082
	Variance :	0.08334
	S.D. :	0.28868
	Skewed Coef. :	-0.00003
	Kurtosis Coef. :	1.79981
	MAD :	0.25002
	Range :	1.00000
	Mid_range :	0.50000
	Median :	0.50001
	Q1 :	0.24997
	Q2 :	0.50001
	Q3 :	0.74997
	IQR :	0.50000
	C.V. :	0.57737

$x_1=0,\int_0^{1-x_1}\lambda_2^{x_2}(1-\lambda_2)^{1-x_1-x_2}dx_2=\int_0^1 0.4^{x_2}0.4^{1-x_2}dx_2\cong 1/2.5$ (numerical analysis),

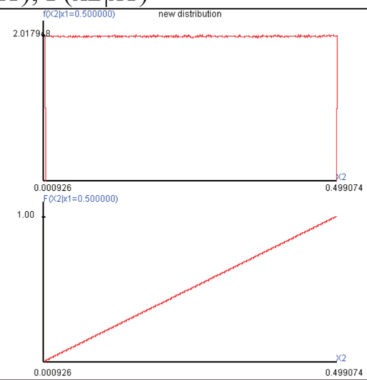
(1-2) x1=0.2,

f(x2|x1), F(x2|x1)	Coefficient	
	Mathematical Mean:	0.40008
	Geometrical Mean :	0.29432
	Harmonic Mean :	0.04496
	Variance :	0.05335
	S.D. :	0.23097
	Skewed Coef. :	-0.00033
	Kurtosis Coef. :	1.79987
	MAD :	0.20004
	Range :	0.80000
	Mid_range :	0.40000
	Median :	0.40009
	Q1 :	0.20002
	Q2 :	0.40009
	Q3 :	0.60017
	IQR :	0.40015
	C.V. :	0.57732

$x_1=0.2,\int_0^{1-x_1}\lambda_2^{x_2}(1-\lambda_2)^{1-x_1-x_2}dx_2$

$=\int_0^{0.8}0.4^{x_2}0.4^{0.8-x_2}dx_2\cong 1/2.6017288003$(numerical analysis),

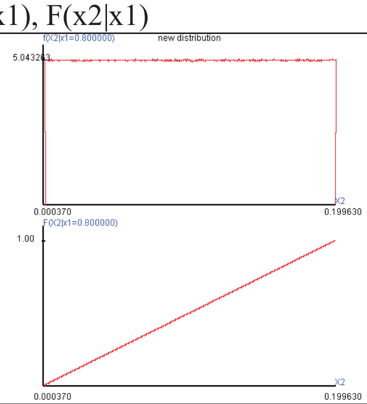
(1-3) x1=0.5,

<table>
<tr><th>f(x2|x1), F(x2|x1)</th><th>Coefficient</th></tr>
<tr><td>
</td><td>Mathematical Mean: 0.25005
Geometrical Mean : 0.18396
Harmonic Mean : 0.02864
Variance : 0.02084
S.D. : 0.14436
Skewed Coef. : -0.00020
Kurtosis Coef. : 1.80002
MAD : 0.12502
Range : 0.50000
Mid_range : 0.25000
Median : 0.25003
Q1 : 0.12504
Q2 : 0.25003
Q3 : 0.37508
IQR : 0.25004
C.V. : 0.57731</td></tr>
</table>

$$x_1 = 0.5, \int_0^{1-x_1} \lambda_2^{x_2}\left(1-\lambda_2\right)^{1-x_1-x_2} dx_2$$

$$= \int_0^{0.5} 0.4^{x_2} 0.4^{0.5-x_2} dx_2 \cong 1/3.1622777168\text{(numerical analysis)},$$

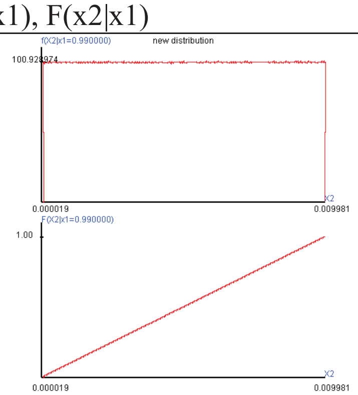
(1-4) x1=0.8,

<table>
<tr><th>f(x2|x1), F(x2|x1)</th><th>Coefficient</th></tr>
<tr><td>
</td><td>Mathematical Mean: 0.10001
Geometrical Mean : 0.07359
Harmonic Mean : 0.01116
Variance : 0.00333
S.D. : 0.05774
Skewed Coef. : -0.00013
Kurtosis Coef. : 1.80000
MAD : 0.05001
Range : 0.20000
Mid_range : 0.10000
Median : 0.10001
Q1 : 0.05001
Q2 : 0.10001
Q3 : 0.15002
IQR : 0.10001
C.V. : 0.57732</td></tr>
</table>

$$x_1 = 0.8, \int_0^{1-x_1} \lambda_2^{x_2}\left(1-\lambda_2\right)^{1-x_1-x_2} dx_2$$

$$= \int_0^{0.2} 0.4^{x_2} 0.4^{0.2-x_2} dx_2 \cong 1/6.0056222271\text{(numerical analysis)},$$

(1-5) x1=0.99,

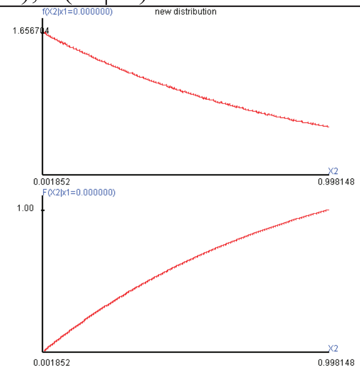
<table>
<tr><th>f(x2|x1), F(x2|x1)</th><th>Coefficient</th></tr>
<tr><td>
</td><td>Mathematical Mean: 0.00500
Geometrical Mean : 0.00368
Harmonic Mean : 0.00051
Variance : 0.00001
S.D. : 0.00289
Skewed Coef. : -0.00072
Kurtosis Coef. : 1.79982
MAD : 0.00250
Range : 0.01000
Mid_range : 0.00500
Median : 0.00500
Q1 : 0.00250
Q2 : 0.00500
Q3 : 0.00750
IQR : 0.00500
C.V. : none</td></tr>
</table>

$$x_1 = 0.99, \int_0^{1-x_1} \lambda_2^{x_2}\left(1-\lambda_2\right)^{1-x_1-x_2} dx_2$$

$$= \int_0^{0.01} 0.4^{x_2} 0.4^{0.01-x_2} dx_2 \cong 1/100.9255571552\text{(numerical analysis)},$$

(2) λ_1=0.2, λ_2=0.2, ,

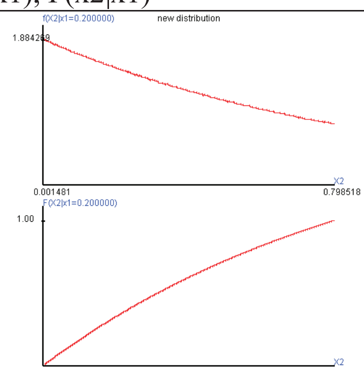
(2-1) x1=0,

<table>
<tr><td>f(x2|x1), F(x2|x1)</td><td>Coefficient</td></tr>
<tr><td>
</td><td>Mathematical Mean: 0.41032
Geometrical Mean : 0.27624
Harmonic Mean : 0.03384
Variance : 0.07856
S.D. : 0.28029
Skewed Coef. : 0.37814
Kurtosis Coef. : 1.99835
MAD : 0.24002
Range : 1.00000
Mid_range : 0.50000
Median : 0.36917
Q1 : 0.16594
Q2 : 0.36917
Q3 : 0.63115
IQR : 0.46520
C.V. : 0.68309</td></tr>
</table>

$x_1 = 0, \int_0^{1-x_1} \lambda_2^{x_2} (1-\lambda_2)^{1-x_1-x_2} dx_2$

$= \int_0^1 0.2^{x_2} 0.6^{1-x_2} dx_2 \cong 1/2.7465307527$(numerical analysis),

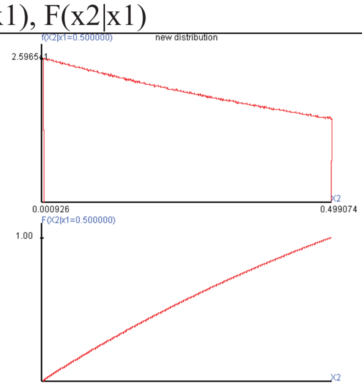
(2-2) x1=0.2,

<table>
<tr><td>f(x2|x1), F(x2|x1)</td><td>Coefficient</td></tr>
<tr><td>
</td><td>Mathematical Mean: 0.34223
Geometrical Mean : 0.23435
Harmonic Mean : 0.03093
Variance : 0.05136
S.D. : 0.22662
Skewed Coef. : 0.30325
Kurtosis Coef. : 1.92736
MAD : 0.19483
Range : 0.80000
Mid_range : 0.40000
Median : 0.31485
Q1 : 0.14391
Q2 : 0.31485
Q3 : 0.52560
IQR : 0.38170
C.V. : 0.66217</td></tr>
</table>

$x_1 = 0.2, \int_0^{1-x_1} \lambda_2^{x_2} (1-\lambda_2)^{1-x_1-x_2} dx_2$

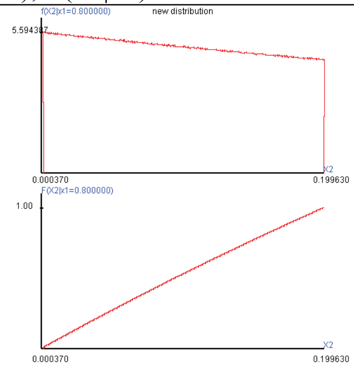
$= \int_0^{0.8} 0.2^{x_2} 0.6^{0.8-x_2} dx_2 \cong 1/2.8271477494$(numerical analysis),

(2-3) x1=0.5,

<table>
<tr><td>f(x2|x1), F(x2|x1)</td><td>Coefficient</td></tr>
<tr><td>
</td><td>Mathematical Mean: 0.22723
Geometrical Mean : 0.15976
Harmonic Mean : 0.01920
Variance : 0.02052
S.D. : 0.14326
Skewed Coef. : 0.18990
Kurtosis Coef. : 1.84970
MAD : 0.12371
Range : 0.50000
Mid_range : 0.25000
Median : 0.21611
Q1 : 0.10162
Q2 : 0.21611
Q3 : 0.34702
IQR : 0.24540
C.V. : 0.63048</td></tr>
</table>

$x_1 = 0.5, \int_0^{1-x_1} \lambda_2^{x_2} (1-\lambda_2)^{1-x_1-x_2} dx_2 = \int_0^{0.5} 0.2^{x_2} 0.6^{0.5-x_2} dx_2 \cong 1/3.3557494792$(numerical analysis),

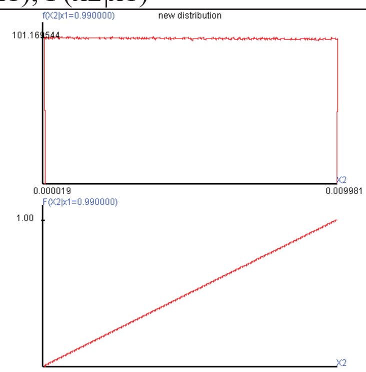
(2-4) x1=0.8,

f(x2|x1), F(x2|x1)	Coefficient		
	Mathematical Mean	:	0.09636
	Geometrical Mean	:	0.06960
	Harmonic Mean	:	0.00979
	Variance	:	0.00333
	S.D.	:	0.05767
	Skewed Coef.	:	0.07584
	Kurtosis Coef.	:	1.80779
	MAD	:	0.04992
	Range	:	0.20000
	Mid_range	:	0.10000
	Median	:	0.09454
	Q1	:	0.04604
	Q2	:	0.09454
	Q3	:	0.14576
	IQR	:	0.09973
	C.V.	:	0.59854

$$x_1 = 0.8, \int_0^{1-x_1} \lambda_2^{x_2} (1-\lambda_2)^{1-x_1-x_2} dx_2$$

$$= \int_0^{0.2} 0.2^{x_2} 0.6^{0.2-x_2} dx_2 \cong 1/6.1684864632 \text{(numerical analysis)},$$

(2-5) x1=0.99,

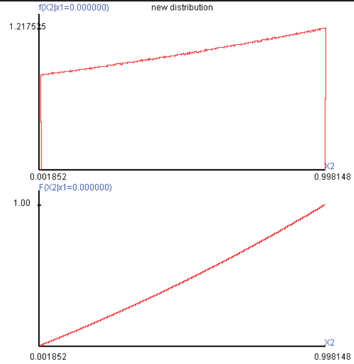
f(x2|x1), F(x2|x1)	Coefficient		
	Mathematical Mean	:	0.00499
	Geometrical Mean	:	0.00367
	Harmonic Mean	:	0.00054
	Variance	:	0.00001
	S.D.	:	0.00289
	Skewed Coef.	:	0.00376
	Kurtosis Coef.	:	1.80001
	MAD	:	0.00250
	Range	:	0.01000
	Mid_range	:	0.00500
	Median	:	0.00499
	Q1	:	0.00249
	Q2	:	0.00499
	Q3	:	0.00749
	IQR	:	0.00500
	C.V.	:	none

$$x_1 = 0.99, \int_0^{1-x_1} \lambda_2^{x_2} (1-\lambda_2)^{1-x_1-x_2} dx_2$$

$$= \int_0^{0.01} 0.2^{x_2} 0.6^{0.01-x_2} dx_2 \cong 1/101.0652638264 \text{(numerical analysis)},$$

(3) λ_1=0.8, λ_2=0.12,

(3-1) x1=0,

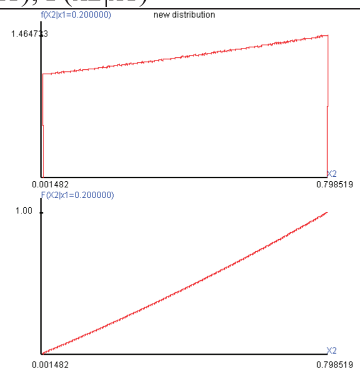


f(x2|x1), F(x2|x1)	Coefficient	
(see figure above)	Mathematical Mean:	0.53374
	Geometrical Mean :	0.40612
	Harmonic Mean :	0.06393
	Variance :	0.08267
	S.D. :	0.28752
	Skewed Coef. :	-0.14051
	Kurtosis Coef. :	1.82713
	MAD :	0.24861
	Range :	1.00000
	Mid_range :	0.50000
	Median :	0.55040
	Q1 :	0.29048
	Q2 :	0.55040
	Q3 :	0.78549
	IQR :	0.49500
	C.V. :	0.53868

$$x_1 = 0, \int_0^{1-x_1} \lambda_2^{x_2} (1-\lambda_2)^{1-x_1-x_2} dx_2$$

$$= \int_0^1 0.12^{x_2} 0.08^{1-x_2} dx_2 \cong 1/10.1366279471\text{(numerical analysis)},$$

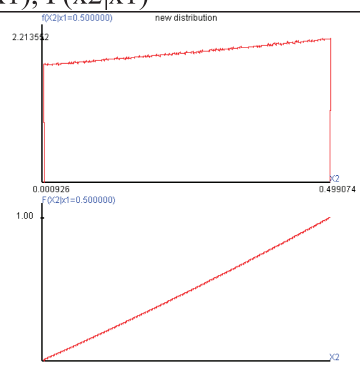
(3-2) x1=0.2,



f(x2|x1), F(x2|x1)	Coefficient	
(see figure above)	Mathematical Mean:	0.42164
	Geometrical Mean :	0.31868
	Harmonic Mean :	0.05109
	Variance :	0.05306
	S.D. :	0.23036
	Skewed Coef. :	-0.11250
	Kurtosis Coef. :	1.81743
	MAD :	0.19929
	Range :	0.80000
	Mid_range :	0.40000
	Median :	0.43235
	Q1 :	0.22564
	Q2 :	0.43235
	Q3 :	0.62309
	IQR :	0.39745
	C.V. :	0.54633

$$x_1 = 0.2, \int_0^{1-x_1} \lambda_2^{x_2} (1-\lambda_2)^{1-x_1-x_2} dx_2 = \int_0^{0.8} 0.12^{x_2} 0.08^{0.8-x_2} dx_2 \cong 1/7.9817702346\text{(numerical analysis)},$$

(3-3) x1=0.5,

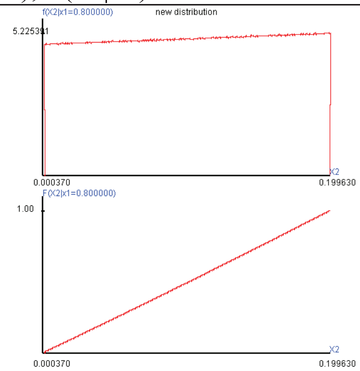


f(x2|x1), F(x2|x1)	Coefficient	
(see figure above)	Mathematical Mean:	0.25849
	Geometrical Mean :	0.19339
	Harmonic Mean :	0.03086
	Variance :	0.02080
	S.D. :	0.14421
	Skewed Coef. :	-0.07054
	Kurtosis Coef. :	1.80676
	MAD :	0.12484
	Range :	0.50000
	Mid_range :	0.25000
	Median :	0.26271
	Q1 :	0.13483
	Q2 :	0.26271
	Q3 :	0.38428
	IQR :	0.24945
	C.V. :	0.55789

$$x_1 = 0.5, \int_0^{1-x_1} \lambda_2^{x_2} (1-\lambda_2)^{1-x_1-x_2} dx_2$$

$$= \int_0^{0.5} 0.12^{x_2} 0.08^{0.5-x_2} dx_2 \cong 1/6.3785022548\text{(numerical analysis)},$$

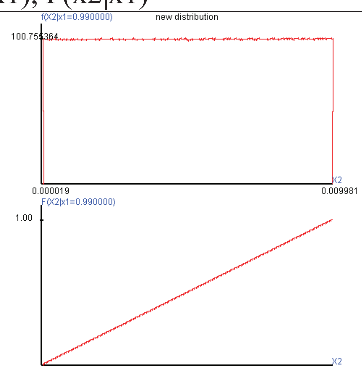
(3-4) x1=0.8,

f(x2|x1), F(x2|x1)	Coefficient	
	Mathematical Mean:	0.10136
	Geometrical Mean :	0.07508
	Harmonic Mean :	0.01078
	Variance :	0.00333
	S.D. :	0.05773
	Skewed Coef. :	-0.02819
	Kurtosis Coef. :	1.80092
	MAD :	0.04999
	Range :	0.20000
	Mid_range :	0.10000
	Median :	0.10203
	Q1 :	0.05154
	Q2 :	0.10203
	Q3 :	0.15151
	IQR :	0.09997
	C.V. :	0.56956

$$x_1 = 0.8, \int_0^{1-x_1} \lambda_2^{x_2} \left(1-\lambda_2\right)^{1-x_1-x_2} dx_2$$

$$= \int_0^{0.2} 0.12^{x_2} 0.08^{0.2-x_2} dx_2 \cong 1/7.9547016206 \text{(numerical analysis)},$$

(3-5) x1=0.99,

f(x2|x1), F(x2|x1)	Coefficient	
	Mathematical Mean:	0.00500
	Geometrical Mean :	0.00368
	Harmonic Mean :	0.00055
	Variance :	0.00001
	S.D. :	0.00289
	Skewed Coef. :	-0.00166
	Kurtosis Coef. :	1.79986
	MAD :	0.00250
	Range :	0.01000
	Mid_range :	0.00500
	Median :	0.00501
	Q1 :	0.00250
	Q2 :	0.00501
	Q3 :	0.00751
	IQR :	0.00500
	C.V. :	none

$$x_1 = 0.99, \int_0^{1-x_1} \lambda_2^{x_2} \left(1-\lambda_2\right)^{1-x_1-x_2} dx_2$$

$$= \int_0^{0.01} 0.12^{x_2} 0.08^{0.01-x_2} dx_2 \cong 1/102.3501187054 \text{(numerical analysis)},$$

Chapter 10, The Continuous Trinomial distribution and trial number=n,

Section 1, The joint probability density function,
The function setting,

$$f_{X_1,X_2}(x_1,x_2;\lambda_1,\lambda_2)=C(n,\lambda_1,\lambda_2)\lambda_1^{x_1}\lambda_2^{x_2}(1-\lambda_1-\lambda_2)^{n-x_1-x_2},$$

$$0<x_1<n,0<x_2<n,0<x_1+x_2<n,0<\lambda_1<1,0<\lambda_2<1,0<\lambda_1+\lambda_2<1,$$

$$f_{X_1}(x_1;n,\lambda_1,\lambda_2)=\int_0^{n-x_1}C(n,\lambda_1,\lambda_2)\lambda_1^{x_1}\lambda_2^{x_2}(1-\lambda_1-\lambda_2)^{n-x_1-x_2}\,dx_2,$$

$$f_{X_2}(x_2;n,\lambda_1,\lambda_2)=\int_0^{n-x_2}C(n,\lambda_1,\lambda_2)\lambda_1^{x_1}\lambda_2^{x_2}(1-\lambda_1-\lambda_2)^{n-x_1-x_2}\,dx_1,$$

$$f_{X_2|x_1}(x_2|x_1)=\frac{\lambda_1^{x_1}\lambda_2^{x_2}(1-\lambda_1-\lambda_2)^{n-x_1-x_2}}{\int_0^{n-x_1}\lambda_2^{x_2}(1-\lambda_1-\lambda_2)^{n-x_1-x_2}\,dx_2},0\le x_2\le n-x_1,$$

$$f_{X_1|x_2}(x_1|x_2)=\frac{\lambda_1^{x_1}(1-\lambda_1-\lambda_2)^{n-x_1-x_2}}{\int_0^{n-x_2}\lambda_1^{x_1}(1-\lambda_1-\lambda_2)^{n-x_1-x_2}\,dx_1},0\le x_1\le n-x_2,$$

$C(n,\lambda_1,\lambda_2)$ could be computed using numerical analysis only.

The marginal probability distributions of X_1 and X_2 are not the continuous binomial distribution.

Section 2, The simulation method,

(1)The simulator,

The joint probability density function can not be found using transformation, but the probability distribution simulator can compute this function.

The method is

$(X_{1,1}, X_{2,1}),(X_{1,2}, X_{2,2}),....,(X_{1,n}, X_{2,n})$ are independent paired random variables,

$(X_{1,i}, X_{2,i}) \sim$ Continuous trinomial distribution (λ_1, λ_2) and trial number=1,

$i = 1,2,..., n$.

Let $X_1 = \sum_{i=1}^{n} X_{1,i}, X_2 = \sum_{i=1}^{n} X_{2,i}$,

$(X_1, X_2) \sim$ Continuous trinomial distribution (λ_1, λ_2) and trial number=n.

The simulated process,

(i) Getting the database of $(X_{1,1}, X_{2,1})$ using the numerical analysis and random number simulator. [$(X_{1,1}, X_{2,1}),(X_{1,2}, X_{2,2}),....,(X_{1,n}, X_{2,n})$ are same distribution]

(ii) Repeat n times using the random number and taking the paired data of $(X_{1,1}, X_{2,1})$, the summation of the 1st part ($X_{1,1}$) is the sample data of X_1 and the summation of the 2nd part ($X_{2,1}$) is the sample data of X_2 .

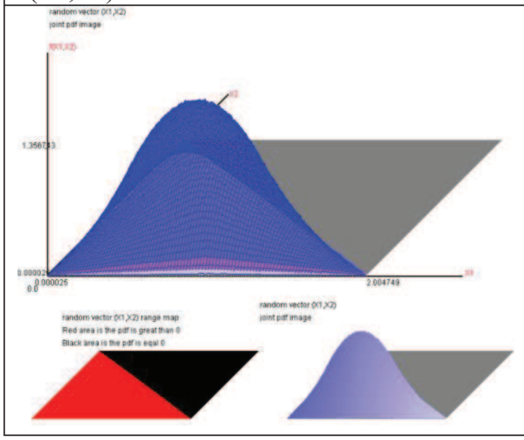
(iii)Finished 100,000,000 times of process (ii), the new database of (X_1, X_2) can represent the Continuous trinomial distribution (λ_1, λ_2) and trial number=n.

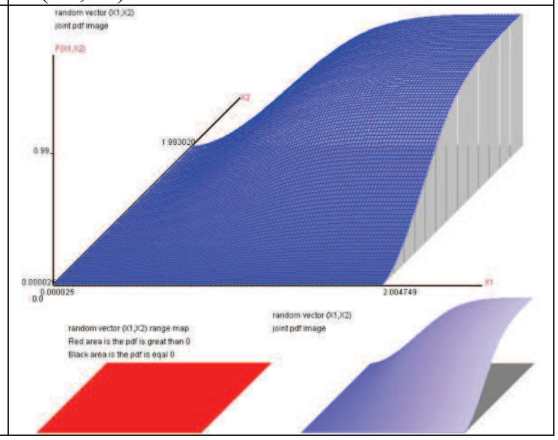
(2) The joint probability distribution and marginal probability distribution,

(1)The joint probability distribution of (x_1, x_2)’,n=2,

(1-1) λ_1=0.3333, λ_2=0.3333,

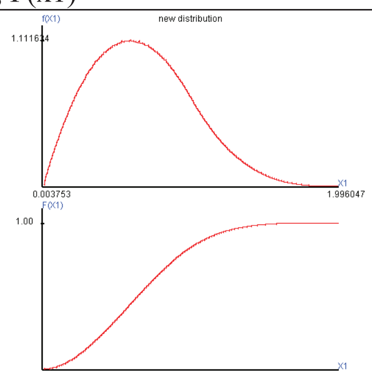
f(x1,x2)	F(x1,x2)





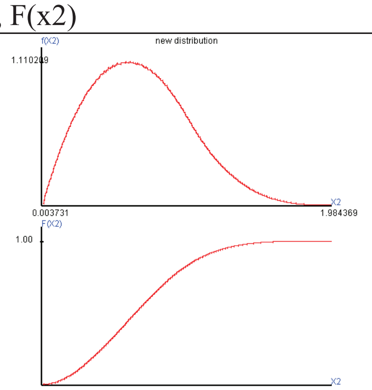
E(X1)= 0.6670, Var(X1)= 0.1112, E(X2)= 0.6663, Var(X2)= 0.1111,
Cov(X1,X2)= -0.0556, X1 and X2 correlation coefficient=-0.5001.

f(x1), F(x1)



Coefficient		
Mathematical Mean:		0.66696
Geometrical Mean :		0.56672
Harmonic Mean	:	0.42307
Variance	:	0.11123
S.D.	:	0.33351
Skewed Coef.	:	0.40035
Kurtosis Coef.	:	2.70107
MAD	:	0.27229
Range	:	1.99970
Mid_range	:	0.99990
Median	:	0.64245
Q1	:	0.41110
Q2	:	0.64245
Q3	:	0.89175
IQR	:	0.48065
C.V.	:	0.50005

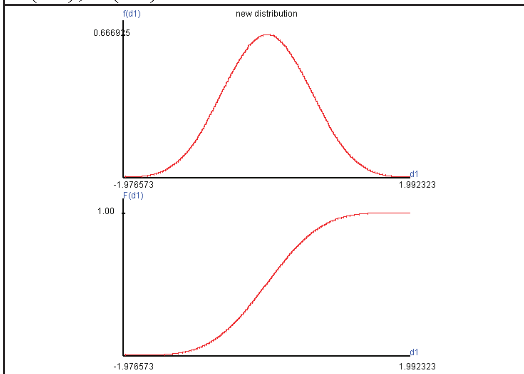
f(x2), F(x2)



Coefficient		
Mathematical Mean:		0.66634
Geometrical Mean :		0.56614
Harmonic Mean	:	0.42208
Variance	:	0.11109
S.D.	:	0.33330
Skewed Coef.	:	0.40074
Kurtosis Coef.	:	2.70103
MAD	:	0.27212
Range	:	1.98800
Mid_range	:	0.99405
Median	:	0.64180
Q1	:	0.41065
Q2	:	0.64180
Q3	:	0.89100
IQR	:	0.48035
C.V.	:	0.50020

d1=X1-X2,

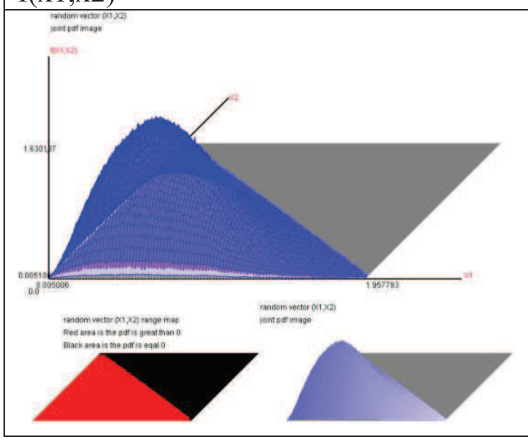
f(d1), F(d1)



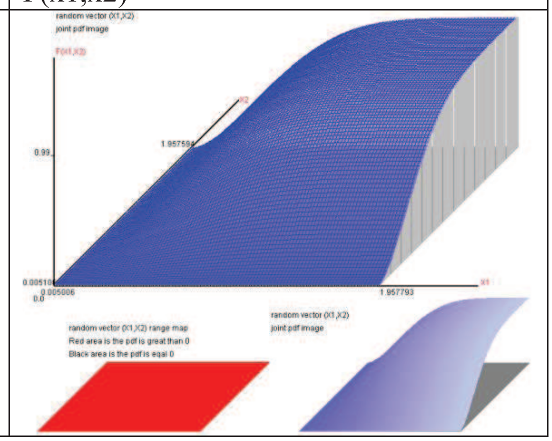
Coefficient	
Mathematical Mean:	0.00062
Geometrical Mean : none	
Harmonic Mean : none	
Variance :	0.33351
S.D. :	0.57750
Skewed Coef. :	0.00010
Kurtosis Coef. :	2.70089
MAD :	0.46677
Range :	3.98365
Mid_range :	0.00787
Median :	0.00070
Q1 :	-0.40215
Q2 :	0.00070
Q3 :	0.40340
IQR :	0.80555
C.V. : none	

(1-2) λ_1=0.1, λ_2=0.1,

f(x1,x2)

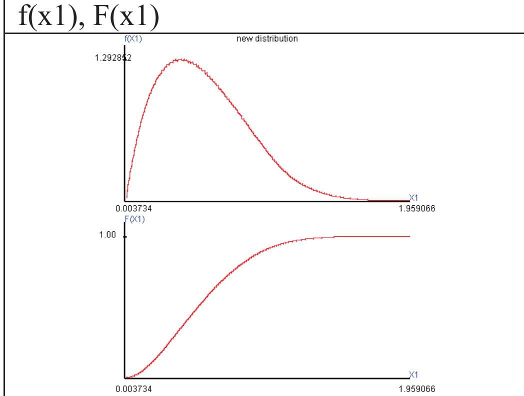


F(x1,x2)

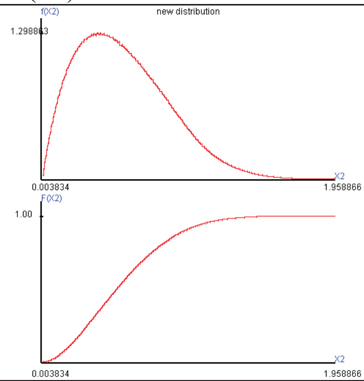


E(X1)= 0.5394, Var(X1)= 0.0915, E(X2)= 0.5392, Var(X2)= 0.0915,
Cov(X1,X2)= -0.0270, X1 and X2 correlation coefficient=-0.2945.

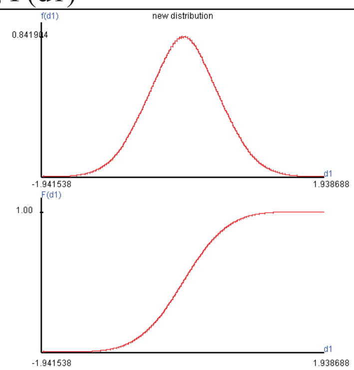
f(x1), F(x1)



Coefficient	
Mathematical Mean:	0.53939
Geometrical Mean :	0.44218
Harmonic Mean :	0.31248
Variance :	0.09155
S.D. :	0.30257
Skewed Coef. :	0.62224
Kurtosis Coef. :	3.02668
MAD :	0.24585
Range :	1.96260
Mid_range :	0.98140
Median :	0.50055
Q1 :	0.30365
Q2 :	0.50055
Q3 :	0.73585
IQR :	0.43220
C.V. :	0.56094

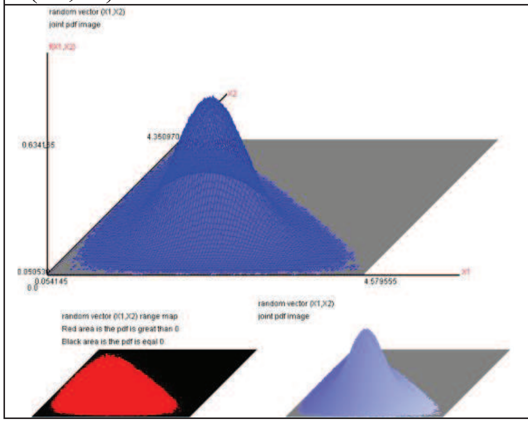
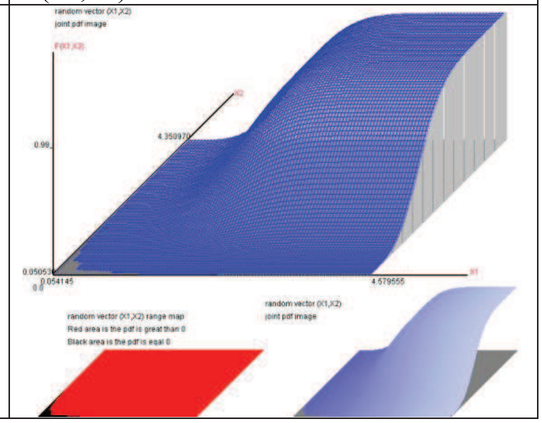
f(x2), F(x2)	Coefficient	
	Mathematical Mean:	0.53916
	Geometrical Mean :	0.44199
	Harmonic Mean :	0.31267
	Variance :	0.09147
	S.D. :	0.30244
	Skewed Coef. :	0.62212
	Kurtosis Coef. :	3.02647
	MAD :	0.24575
	Range :	1.96230
	Mid_range :	0.98135
	Median :	0.50035
	Q1 :	0.30350
	Q2 :	0.50035
	Q3 :	0.73545
	IQR :	0.43195
	C.V. :	0.56095

d1=X1-X2,

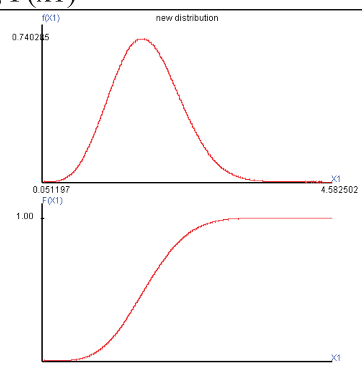
f(d1), F(d1)	Coefficient	
	Mathematical Mean:	0.00023
	Geometrical Mean : none	
	Harmonic Mean : none	
	Variance :	0.23692
	S.D. :	0.48675
	Skewed Coef. :	0.00074
	Kurtosis Coef. :	2.95772
	MAD :	0.38771
	Range :	3.89465
	Mid_range :	-0.00143
	Median :	0.00020
	Q1 :	-0.32585
	Q2 :	0.00020
	Q3 :	0.32620
	IQR :	0.65205
	C.V. : none	

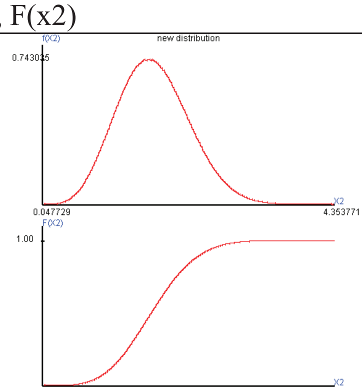
(2)The joint probability distribution of (x_1, x_2)',n=5,

(2-1) λ_1=0.3333, λ_2=0.3333,

f(x1,x2)	F(x1,x2)
	

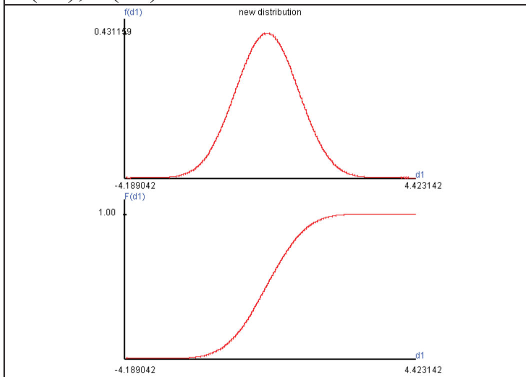
E(X1)= 1.6670, Var(X1)= 0.2779, E(X2)= 1.6668, Var(X2)= 0.2776,
Cov(X1,X2)= -0.1389, X1 and X2 correlation coefficient=-0.5000.

f(x1), F(x1)	Coefficient		
	Mathematical Mean:		1.66705
	Geometrical Mean	:	1.57712
	Harmonic Mean	:	1.47559
	Variance	:	0.27791
	S.D.	:	0.52717
	Skewed Coef.	:	0.25286
	Kurtosis Coef.	:	2.87998
	MAD	:	0.42401
	Range	:	4.54815
	Mid_range	:	2.31685
	Median	:	1.64367
	Q1	:	1.29127
	Q2	:	1.64367
	Q3	:	2.01747
	IQR	:	0.72620
	C.V.	:	0.31623

f(x2), F(x2)	Coefficient		
	Mathematical Mean:		1.66678
	Geometrical Mean	:	1.57695
	Harmonic Mean	:	1.47549
	Variance	:	0.27756
	S.D.	:	0.52684
	Skewed Coef.	:	0.25274
	Kurtosis Coef.	:	2.88158
	MAD	:	0.42367
	Range	:	4.32205
	Mid_range	:	2.20075
	Median	:	1.64352
	Q1	:	1.29112
	Q2	:	1.64352
	Q3	:	2.01682
	IQR	:	0.72570
	C.V.	:	0.31608

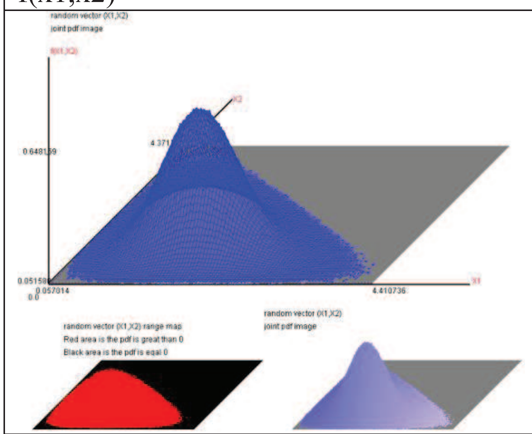
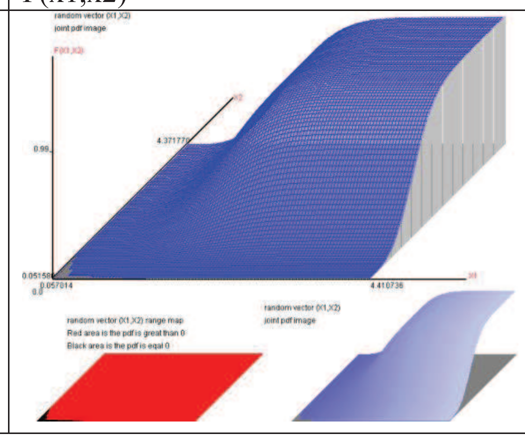
d1=X1-X2,

f(d1), F(d1)



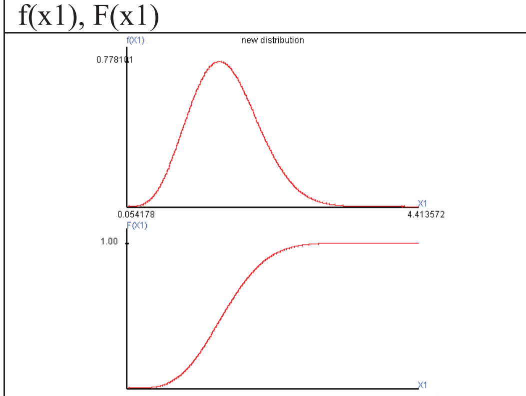
Coefficient		
Mathematical Mean:		0.00027
Geometrical Mean	:	none
Harmonic Mean	:	none
Variance	:	0.83322
S.D.	:	0.91281
Skewed Coef.	:	0.00036
Kurtosis Coef.	:	2.88168
MAD	:	0.73193
Range	:	8.64420
Mid_range	:	0.11705
Median	:	0.00015
Q1	:	-0.62330
Q2	:	0.00015
Q3	:	0.62375
IQR	:	1.24705
C.V.	:	none

(2-2) λ_1=0.2, λ_2=0.2,

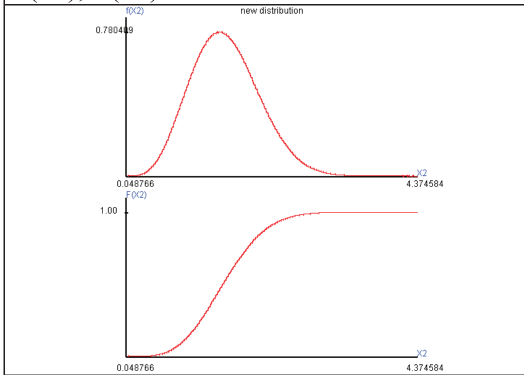
f(x1,x2)	F(x1,x2)
	

E(X1)=.5045, Var(X1)= 0.2547, E(X2)= 1.5043, Var(X2)= 0.2547,
Cov(X1,X2)= -0.0991, X1 and X2 correlation coefficient=-0.3890.

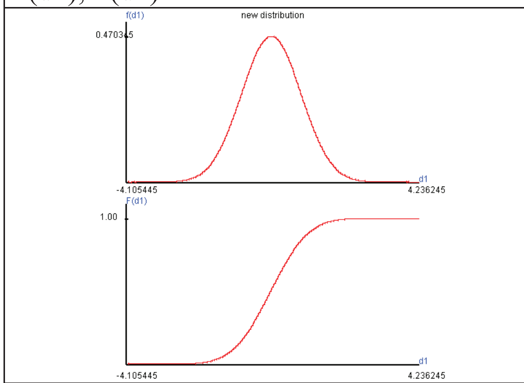
f(x1), F(x1)



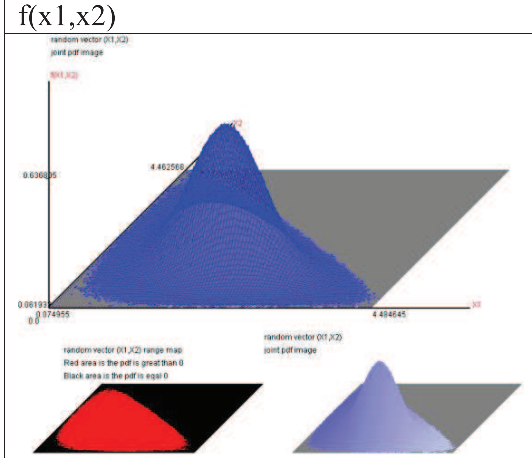
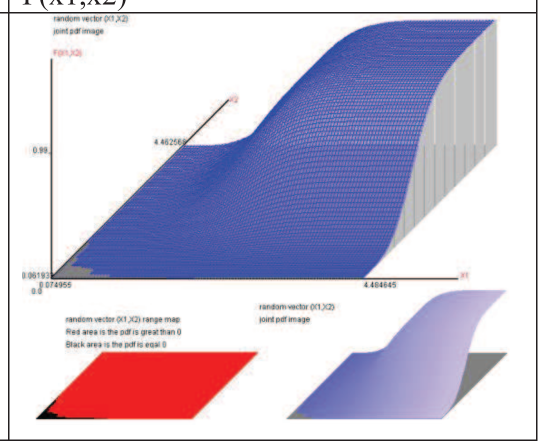
Coefficient		
Mathematical Mean:		1.50455
Geometrical Mean	:	1.41367
Harmonic Mean	:	1.31136
Variance	:	0.25473
S.D.	:	0.50471
Skewed Coef.	:	0.32150
Kurtosis Coef.	:	2.93495
MAD	:	0.40565
Range	:	4.37560
Mid_range	:	2.23387
Median	:	1.47622
Q1	:	1.14162
Q2	:	1.47622
Q3	:	1.83612
IQR	:	0.69450
C.V.	:	0.33545

f(x2), F(x2)	Coefficient
	Mathematical Mean: 1.50432 Geometrical Mean : 1.41346 Harmonic Mean : 1.31120 Variance : 0.25466 S.D. : 0.50464 Skewed Coef. : 0.32207 Kurtosis Coef. : 2.93587 MAD : 0.40559 Range : 4.34190 Mid_range : 2.21167 Median : 1.47567 Q1 : 1.14157 Q2 : 1.47567 Q3 : 1.83587 IQR : 0.69430 C.V. : 0.33546

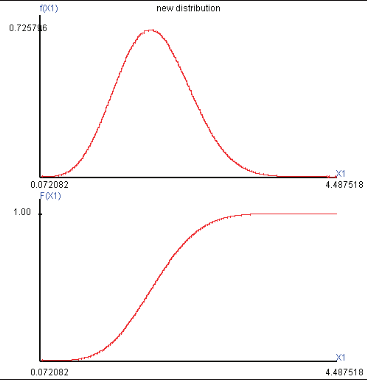
d1=X1-X2,

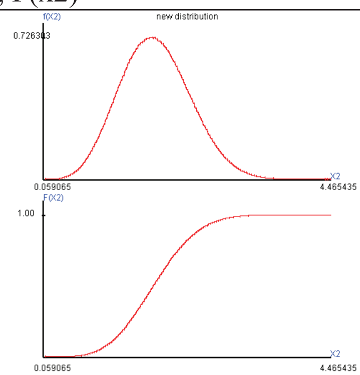
f(d1), F(d1)	Coefficient
	Mathematical Mean: 0.00023 Geometrical Mean : none Harmonic Mean : none Variance : 0.70757 S.D. : 0.84117 Skewed Coef. : -0.00050 Kurtosis Coef. : 2.92547 MAD : 0.67319 Range : 8.37270 Mid_range : 0.06540 Median : 0.00045 Q1 : -0.57145 Q2 : 0.00045 Q3 : 0.57195 IQR : 1.14340 C.V. : none

(2-3) λ_1=0.4, λ_2=0.4,

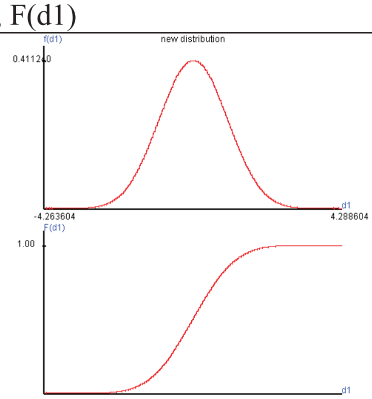
f(x1,x2)	F(x1,x2)
	

E(X1)= 1.7587, Var(X1)= 0.2896, E(X2)= 1.7583, Var(X2)= 0.2896, Cov(X1,X2)= -0.1644, X1 and X2 correlation coefficient=-0.5678.

f(x1), F(x1)	Coefficient
	Mathematical Mean: 1.75866 Geometrical Mean : 1.66962 Harmonic Mean : 1.56896 Variance : 0.28961 S.D. : 0.53815 Skewed Coef. : 0.21500 Kurtosis Coef. : 2.85198 MAD : 0.43301 Range : 4.43185 Mid_range : 2.27980 Median : 1.73837 Q1 : 1.37637 Q2 : 1.73837 Q3 : 2.11862 IQR : 0.74225 C.V. : 0.30600

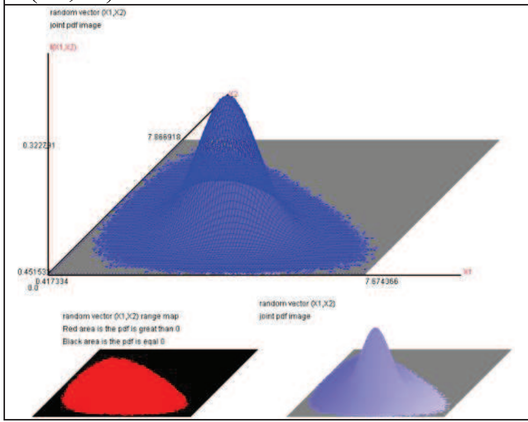
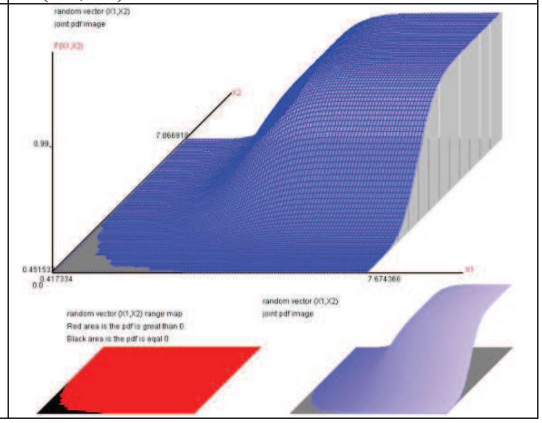
f(x2), F(x2)	Coefficient
	Mathematical Mean: 1.75826 Geometrical Mean : 1.66921 Harmonic Mean : 1.56855 Variance : 0.28959 S.D. : 0.53814 Skewed Coef. : 0.21572 Kurtosis Coef. : 2.85394 MAD : 0.43296 Range : 4.42275 Mid_range : 2.26225 Median : 1.73797 Q1 : 1.37602 Q2 : 1.73797 Q3 : 2.11797 IQR : 0.74195 C.V. : 0.30606

d1=X1-X2,

f(d1), F(d1)	Coefficient
	Mathematical Mean: 0.00040 Geometrical Mean : none Harmonic Mean : none Variance : 0.90807 S.D. : 0.95293 Skewed Coef. : -0.00035 Kurtosis Coef. : 2.85514 MAD : 0.76509 Range : 8.58400 Mid_range : 0.01250 Median : 0.00030 Q1 : -0.65260 Q2 : 0.00030 Q3 : 0.65355 IQR : 1.30615 C.V. : none

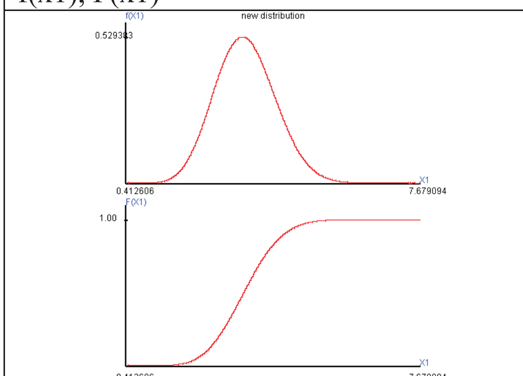
(3)The joint probability distribution of (x_1, x_2)',n=10,

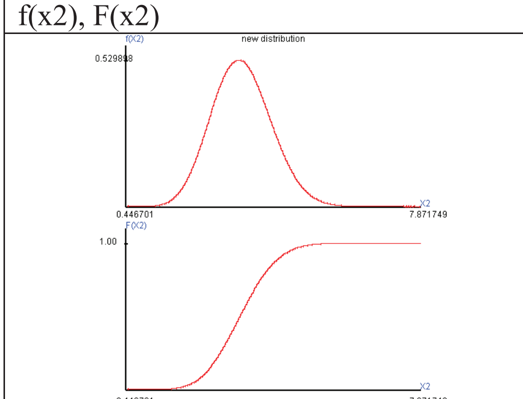
(3-1) λ_1=0.3333, λ_2=0.3333,

f(x1,x2)	F(x1,x2)
	

E(X1)= 3.3344, Var(X1)= 0.5561, E(X2)= 3.3322, Var(X2)= 0.5554,

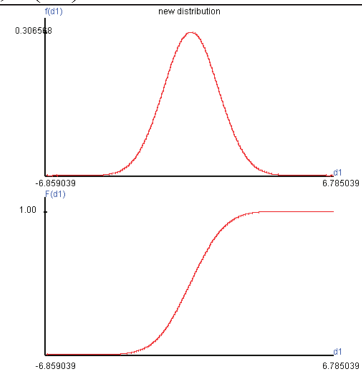
Cov(X1,X2)= -0.2780, X1 and X2 correlation coefficient=-0.5003.

f(x1), F(x1)	Coefficient
	Mathematical Mean: 3.33442 Geometrical Mean : 3.24789 Harmonic Mean : 3.15654 Variance : 0.55608 S.D. : 0.74570 Skewed Coef. : 0.17912 Kurtosis Coef. : 2.94022 MAD : 0.59732 Range : 7.29350 Mid_range : 4.04585 Median : 3.31160 Q1 : 2.81380 Q2 : 3.31160 Q3 : 3.83015 IQR : 1.01635 C.V. : 0.22364

f(x2), F(x2)	Coefficient
	Mathematical Mean: 3.33218 Geometrical Mean : 3.24569 Harmonic Mean : 3.15438 Variance : 0.55536 S.D. : 0.74523 Skewed Coef. : 0.17876 Kurtosis Coef. : 2.93995 MAD : 0.59693 Range : 7.45265 Mid_range : 4.15922 Median : 3.30945 Q1 : 2.81185 Q2 : 3.30945 Q3 : 3.82755 IQR : 1.01570 C.V. : 0.22365

d1=X1-X2,

f(d1), F(d1)

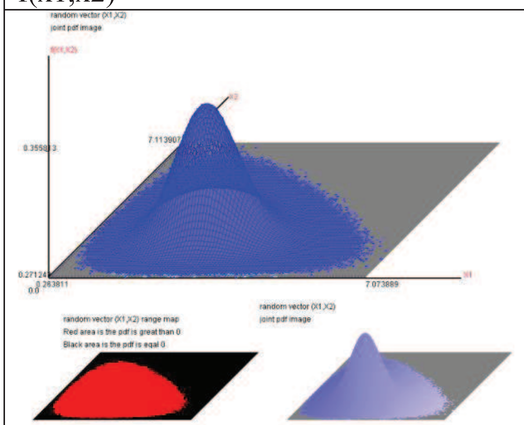


Coefficient

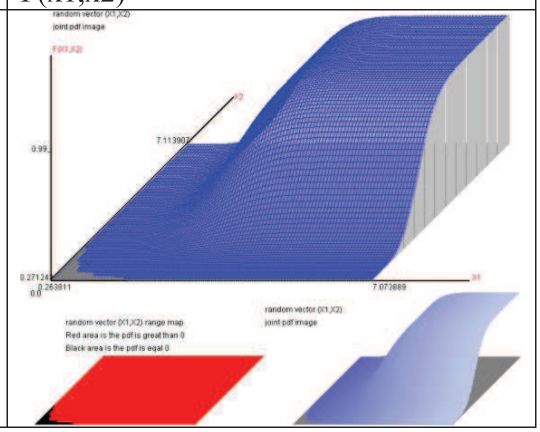
Mathematical Mean:		0.00225
Geometrical Mean	: none	
Harmonic Mean	: none	
Variance	:	1.66751
S.D.	:	1.29132
Skewed Coef.	:	0.00024
Kurtosis Coef.	:	2.93950
MAD	:	1.03296
Range	:	13.69480
Mid_range	:	-0.03700
Median	:	0.00230
Q1	:	-0.87455
Q2	:	0.00230
Q3	:	0.87890
IQR	:	1.75345
C.V.	: none	

(3-2) λ_1=0.1, λ_2=0.1,

f(x1,x2)

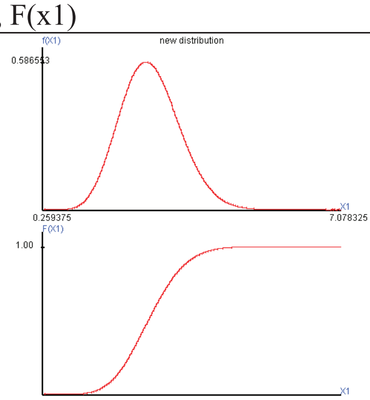


F(x1,x2)



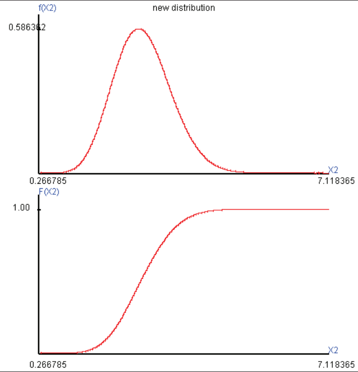
E(X1)= 2.6961, Var(X1)= 0.4576, E(X2)= 2.6959, Var(X2)= 0.4576, Cov(X1,X2)= -0.1348, X1 and X2 correlation coefficient=-0.2946.

f(x1), F(x1)

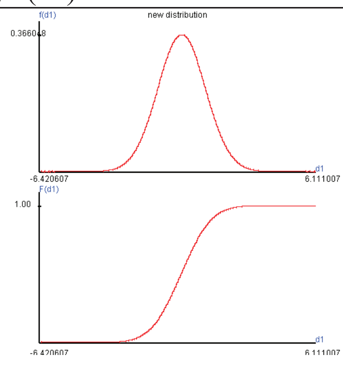


Coefficient

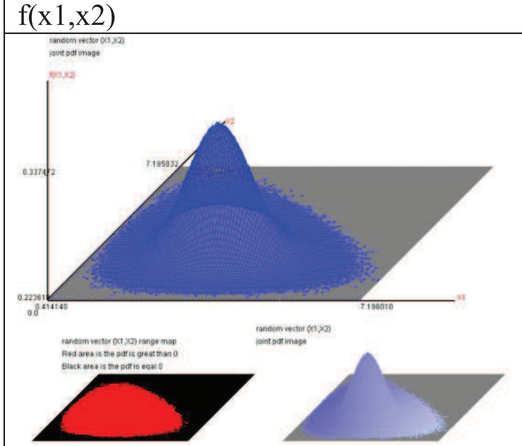
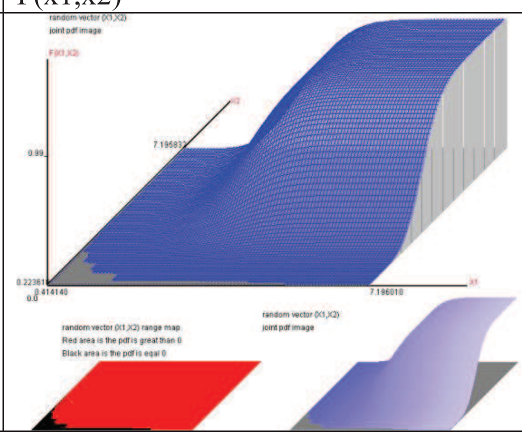
Mathematical Mean:		2.69606
Geometrical Mean :		2.60840
Harmonic Mean	:	2.51601
Variance	:	0.45759
S.D.	:	0.67645
Skewed Coef.	:	0.27868
Kurtosis Coef.	:	3.00637
MAD	:	0.54137
Range	:	6.84430
Mid_range	:	3.66885
Median	:	2.66380
Q1	:	2.21830
Q2	:	2.66380
Q3	:	3.13860
IQR	:	0.92030
C.V.	:	0.25090

f(x2), F(x2)	Coefficient
 	Mathematical Mean: 2.69588 Geometrical Mean : 2.60822 Harmonic Mean : 2.51582 Variance : 0.45756 S.D. : 0.67643 Skewed Coef. : 0.27836 Kurtosis Coef. : 3.00542 MAD : 0.54138 Range : 6.87705 Mid_range : 3.69257 Median : 2.66360 Q1 : 2.21815 Q2 : 2.66360 Q3 : 3.13845 IQR : 0.92030 C.V. : 0.25091

d1=X1-X2,

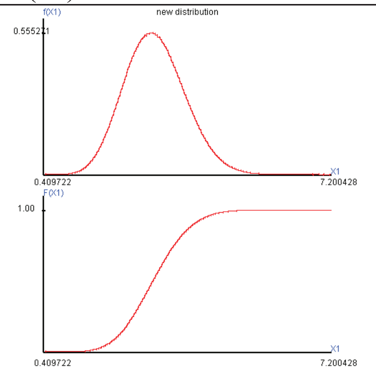
f(d1), F(d1)	Coefficient
 	Mathematical Mean: 0.00018 Geometrical Mean : none Harmonic Mean : none Variance : 1.18476 S.D. : 1.08847 Skewed Coef. : 0.00021 Kurtosis Coef. : 2.99187 MAD : 0.86870 Range : 12.57820 Mid_range : -0.15480 Median : 0.00020 Q1 : -0.73445 Q2 : 0.00020 Q3 : 0.73460 IQR : 1.46905 C.V. : none

(3-3) λ_1=0.2, λ_2=0.2,

f(x1,x2)	F(x1,x2)
	

E(X1)= 3.0089, Var(X1)= 0.5094, E(X2)= 3.0069, Var(X2)= 0.5089,
Cov(X1,X2)= -0.1980, X1 and X2 correlation coefficient=-0.3890.

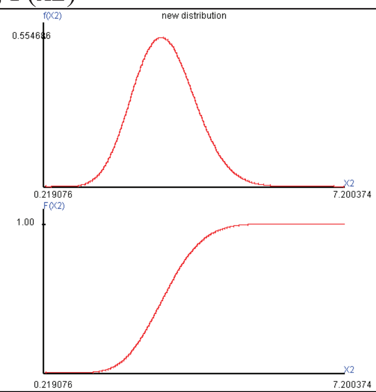
f(x1), F(x1)



Coefficient

Mathematical Mean	:	3.00886
Geometrical Mean	:	2.92119
Harmonic Mean	:	2.82866
Variance	:	0.50941
S.D.	:	0.71373
Skewed Coef.	:	0.22809
Kurtosis Coef.	:	2.96787
MAD	:	0.57153
Range	:	6.81595
Mid_range	:	3.80507
Median	:	2.98095
Q1	:	2.50745
Q2	:	2.98095
Q3	:	3.47975
IQR	:	0.97230
C.V.	:	0.23721

f(x2), F(x2)

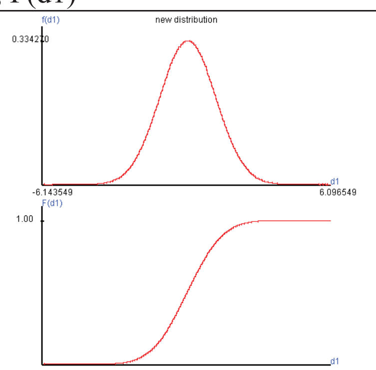


Coefficient

Mathematical Mean	:	3.00691
Geometrical Mean	:	2.91925
Harmonic Mean	:	2.82672
Variance	:	0.50886
S.D.	:	0.71334
Skewed Coef.	:	0.22690
Kurtosis Coef.	:	2.96726
MAD	:	0.57125
Range	:	7.00725
Mid_range	:	3.70972
Median	:	2.97920
Q1	:	2.50590
Q2	:	2.97920
Q3	:	3.47770
IQR	:	0.97180
C.V.	:	0.23723

d1=X1-X2,

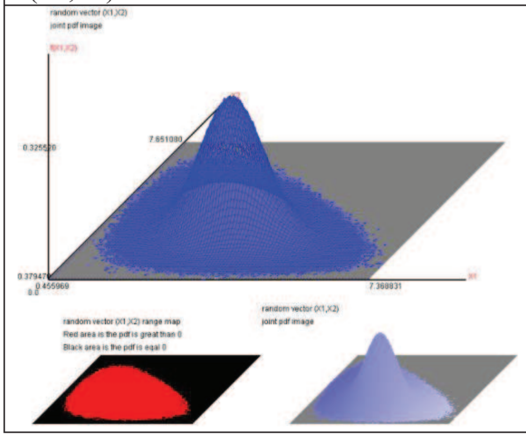
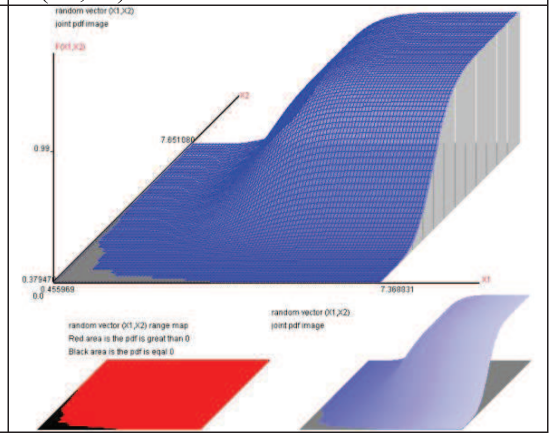
f(d1), F(d1)



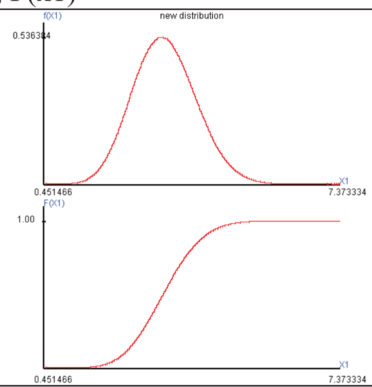
Coefficient

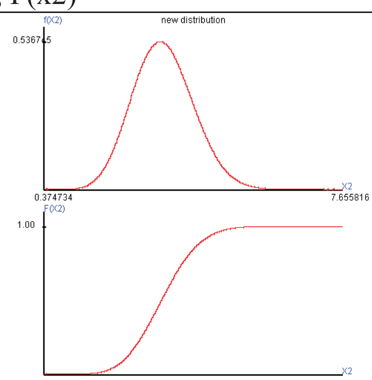
Mathematical Mean	:	0.00195
Geometrical Mean	:	none
Harmonic Mean	:	none
Variance	:	1.41436
S.D.	:	1.18927
Skewed Coef.	:	0.00103
Kurtosis Coef.	:	2.96168
MAD	:	0.95044
Range	:	12.28560
Mid_range	:	-0.02350
Median	:	0.00185
Q1	:	-0.80370
Q2	:	0.00185
Q3	:	0.80710
IQR	:	1.61080
C.V.	:	none

(3-4) λ_1=0.3, λ_2=0.3,

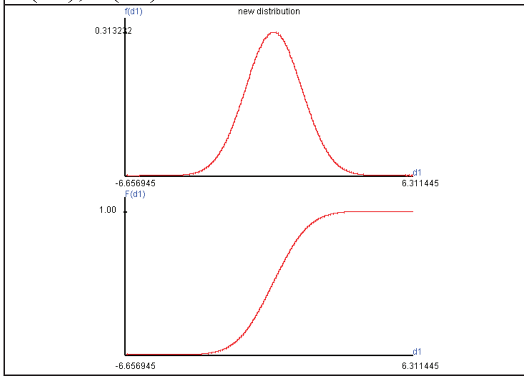
f(x1,x2)	F(x1,x2)
	

E(X1)= 3.2534, Var(X1)= 0.5454, E(X2)= 3.2520, Var(X2)= 0.5443, Cov(X1,X2)= -0.2571, X1 and X2 correlation coefficient=-0.4720.

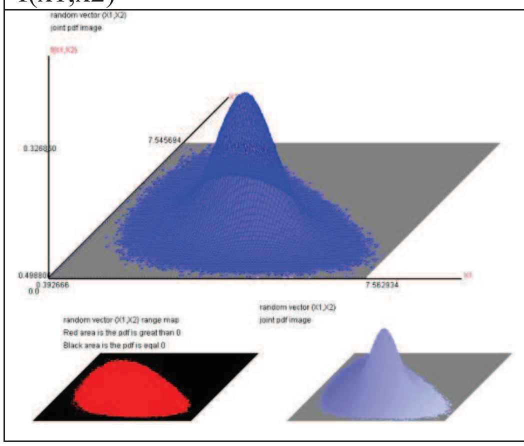
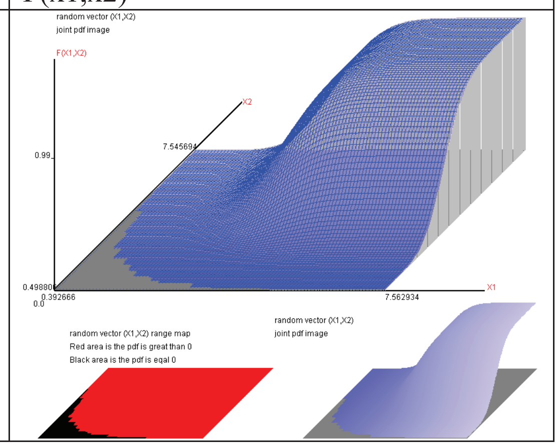
f(x1), F(x1)	Coefficient
	Mathematical Mean: 3.25336 Geometrical Mean : 3.16642 Harmonic Mean : 3.07463 Variance : 0.54539 S.D. : 0.73850 Skewed Coef. : 0.19132 Kurtosis Coef. : 2.94683 MAD : 0.59147 Range : 6.94760 Mid_range : 3.91240 Median : 3.22905 Q1 : 2.73705 Q2 : 3.22905 Q3 : 3.74330 IQR : 1.00625 C.V. : 0.22700

f(x2), F(x2)	Coefficient
	Mathematical Mean: 3.25200 Geometrical Mean : 3.16519 Harmonic Mean : 3.07353 Variance : 0.54431 S.D. : 0.73777 Skewed Coef. : 0.19064 Kurtosis Coef. : 2.94644 MAD : 0.59088 Range : 7.30815 Mid_range : 4.01527 Median : 3.22800 Q1 : 2.73625 Q2 : 3.22800 Q3 : 3.74145 IQR : 1.00520 C.V. : 0.22687

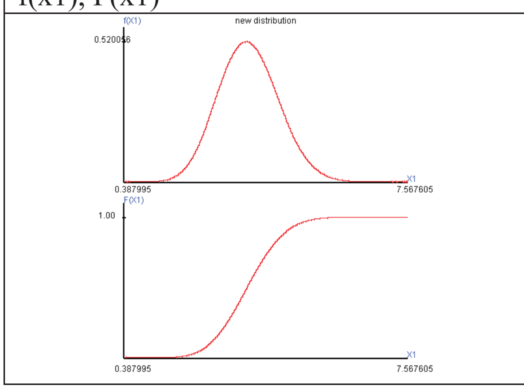
d1=X1-X2,

f(d1), F(d1)	Coefficient
	Mathematical Mean: 0.00136 Geometrical Mean : none Harmonic Mean : none Variance : 1.60398 S.D. : 1.26648 Skewed Coef. : 0.00083 Kurtosis Coef. : 2.94676 MAD : 1.01271 Range : 13.01660 Mid_range : -0.17275 Median : 0.00105 Q1 : -0.85760 Q2 : 0.00105 Q3 : 0.86025 IQR : 1.71785 C.V. : none

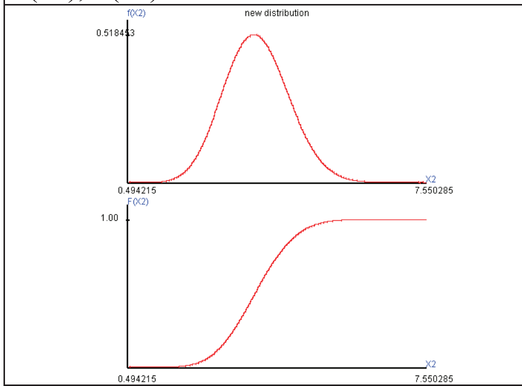
(3-5) λ_1=0.4, λ_2=0.4,

f(x1,x2)	F(x1,x2)
	

E(X1)= 3.5176, Var(X1)= 0.5791, E(X2)= 3.5161, Var(X2)= 0.5790,
Cov(X1,X2)= -0.3287, X1 and X2 correlation coefficient=-0.5677.

f(x1), F(x1)	Coefficient
	Mathematical Mean: 3.51759 Geometrical Mean : 3.43210 Harmonic Mean : 3.34184 Variance : 0.57910 S.D. : 0.76098 Skewed Coef. : 0.15275 Kurtosis Coef. : 2.92882 MAD : 0.60965 Range : 7.20630 Mid_range : 3.97780 Median : 3.49780 Q1 : 2.98795 Q2 : 3.49780 Q3 : 4.02560 IQR : 1.03765 C.V. : 0.21634

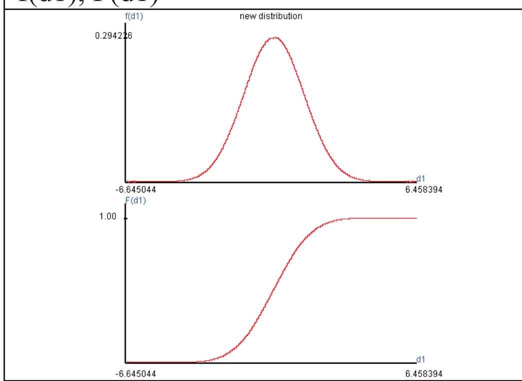
f(x2), F(x2)	Coefficient



Coefficient	
Mathematical Mean:	3.51609
Geometrical Mean :	3.43058
Harmonic Mean :	3.34031
Variance :	0.57901
S.D. :	0.76093
Skewed Coef. :	0.15294
Kurtosis Coef. :	2.92712
MAD :	0.60963
Range :	7.08230
Mid_range :	4.02225
Median :	3.49615
Q1 :	2.98645
Q2 :	3.49615
Q3 :	4.02385
IQR :	1.03740
C.V. :	0.21641

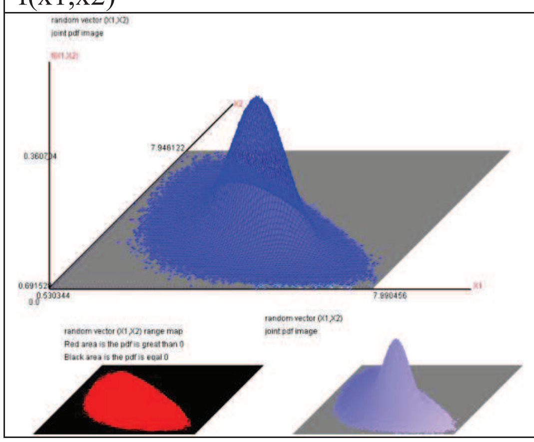
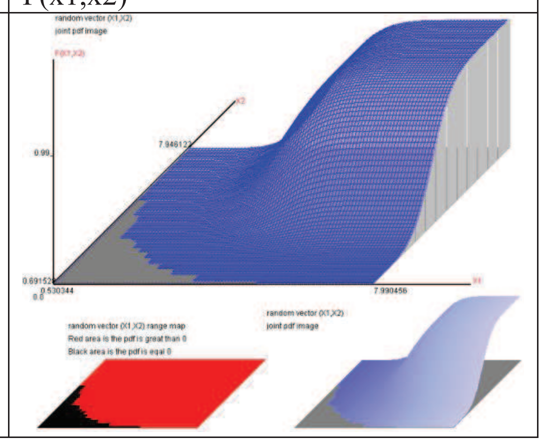
d1=X1-X2,

f(d1), F(d1)	Coefficient

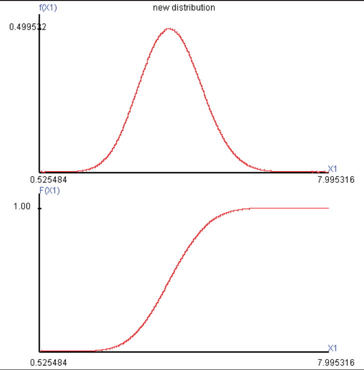


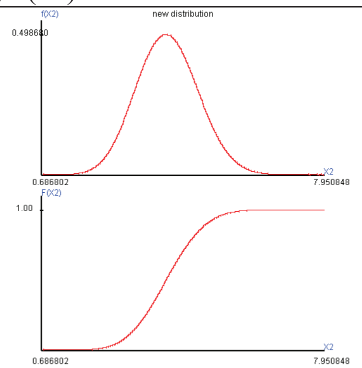
Coefficient	
Mathematical Mean:	0.00150
Geometrical Mean :	none
Harmonic Mean :	none
Variance :	1.81558
S.D. :	1.34743
Skewed Coef. :	-0.00007
Kurtosis Coef. :	2.92918
MAD :	1.07831
Range :	13.15215
Mid_range :	-0.09332
Median :	0.00185
Q1 :	-0.91405
Q2 :	0.00185
Q3 :	0.91735
IQR :	1.83140
C.V. :	none

(3-6) λ_1=0.48, λ_2=0.48,

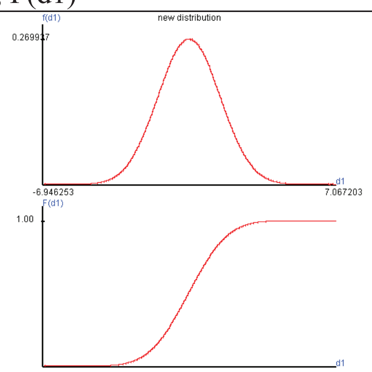
f(x1,x2)	F(x1,x2)
	

E(X1)= 3.8992, Var(X1)= 0.6245, E(X2)= 3.8966, Var(X2)= 0.6246, Cov(X1,X2)= -0.4476, X1 and X2 correlation coefficient=-0.7167.

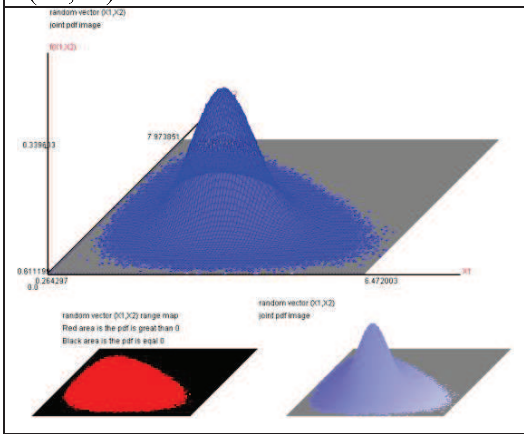
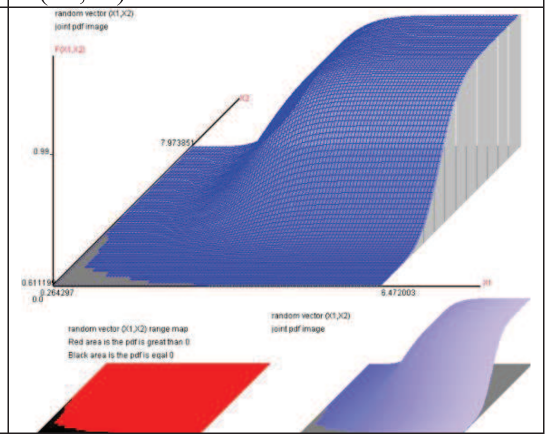
f(x1), F(x1)	Coefficient	
 	Mathematical Mean:	3.89918
	Geometrical Mean :	3.81589
	Harmonic Mean :	3.72798
	Variance :	0.62455
	S.D. :	0.79028
	Skewed Coef. :	0.10165
	Kurtosis Coef. :	2.90851
	MAD :	0.63328
	Range :	7.49760
	Mid_range :	4.26040
	Median :	3.88555
	Q1 :	3.35285
	Q2 :	3.88555
	Q3 :	4.43055
	IQR :	1.07770
	C.V. :	0.20268

f(x2), F(x2)	Coefficient	
 	Mathematical Mean:	3.89656
	Geometrical Mean :	3.81320
	Harmonic Mean :	3.72521
	Variance :	0.62461
	S.D. :	0.79032
	Skewed Coef. :	0.10178
	Kurtosis Coef. :	2.90877
	MAD :	0.63332
	Range :	7.29105
	Mid_range :	4.31882
	Median :	3.88285
	Q1 :	3.35005
	Q2 :	3.88285
	Q3 :	4.42810
	IQR :	1.07805
	C.V. :	0.20283

d1=X1-X2,

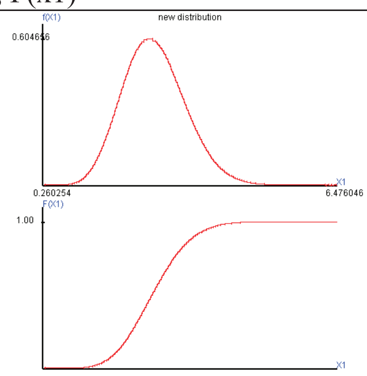
f(d1), F(d1)	Coefficient	
 	Mathematical Mean:	0.00262
	Geometrical Mean : none	
	Harmonic Mean : none	
	Variance :	2.14439
	S.D. :	1.46437
	Skewed Coef. :	-0.00029
	Kurtosis Coef. :	2.90923
	MAD :	1.17290
	Range :	14.06555
	Mid_range :	0.06048
	Median :	0.00250
	Q1 :	-0.99475
	Q2 :	0.00250
	Q3 :	1.00000
	IQR :	1.99475
	C.V. : none	

(3-7) λ_1=0.1, λ_2=0.5,

f(x1,x2)	F(x1,x2)
	

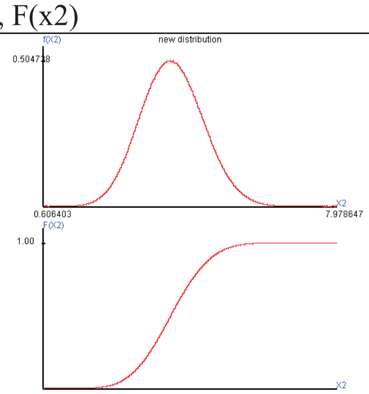
E(X1)= 2.5899, Var(X1)= 0.4332, E(X2)= 3.8154, Var(X2)= 0.6118,
Cov(X1,X2)= -0.2269, X1 and X2 correlation coefficient=-0.4408.

f(x1), F(x1)



Coefficient		
Mathematical Mean	:	2.58988
Geometrical Mean	:	2.50362
Harmonic Mean	:	2.41281
Variance	:	0.43316
S.D.	:	0.65815
Skewed Coef.	:	0.29429
Kurtosis Coef.	:	3.01957
MAD	:	0.52660
Range	:	6.23890
Mid_range	:	3.36815
Median	:	2.55645
Q1	:	2.12450
Q2	:	2.55645
Q3	:	3.01905
IQR	:	0.89455
C.V.	:	0.25412

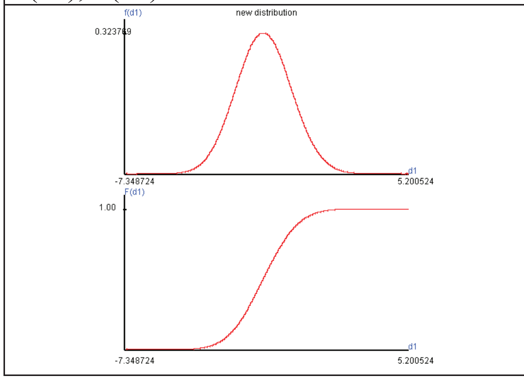
f(x2), F(x2)



Coefficient		
Mathematical Mean	:	3.81536
Geometrical Mean	:	3.73203
Harmonic Mean	:	3.64409
Variance	:	0.61176
S.D.	:	0.78215
Skewed Coef.	:	0.11222
Kurtosis Coef.	:	2.91218
MAD	:	0.62674
Range	:	7.39965
Mid_range	:	4.29252
Median	:	3.80035
Q1	:	3.27385
Q2	:	3.80035
Q3	:	4.34060
IQR	:	1.06675
C.V.	:	0.20500

d1=X1-X2,

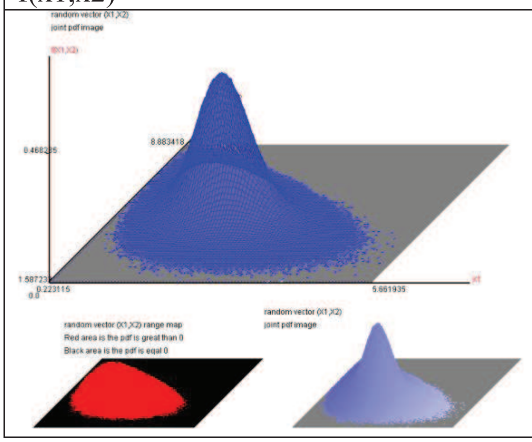
f(d1), F(d1)	Coefficient

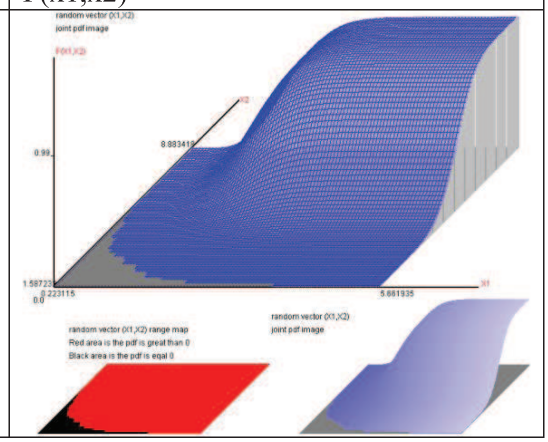


Mathematical Mean	:	-1.22549
Geometrical Mean	:	none
Harmonic Mean	:	none
Variance	:	1.49870
S.D.	:	1.22421
Skewed Coef.	:	0.04863
Kurtosis Coef.	:	2.95315
MAD	:	0.97882
Range	:	12.59590
Mid_range	:	-1.07410
Median	:	-1.23550
Q1	:	-2.06105
Q2	:	-1.23550
Q3	:	-0.40100
IQR	:	1.66005
C.V.	:	none

(3-8) $\lambda_1=0.01$, $\lambda_2=0.95$, X1 and X2 two tailed probability removing 0.002,

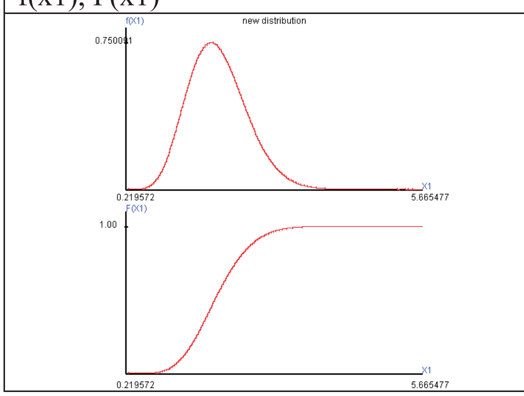
f(x1,x2)	F(x1,x2)





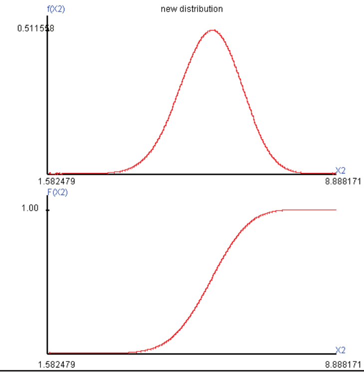
E(X1)= 1.8807, Var(X1)= 0.2880, E(X2)= 5.6784, Var(X2)= 0.5968,
Cov(X1,X2)= -0.2334, X1 and X2 correlation coefficient=-0.5631.

f(x1), F(x1)	Coefficient



Mathematical Mean	:	1.88071
Geometrical Mean	:	1.80270
Harmonic Mean	:	1.72138
Variance	:	0.28797
S.D.	:	0.53663
Skewed Coef.	:	0.42748
Kurtosis Coef.	:	3.17011
MAD	:	0.42840
Range	:	5.46615
Mid_range	:	2.94252
Median	:	1.84120
Q1	:	1.49640
Q2	:	1.84120
Q3	:	2.22240
IQR	:	0.72600
C.V.	:	0.28533

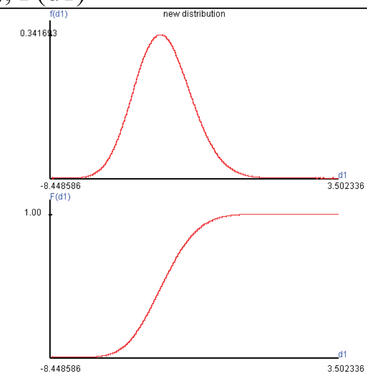
f(x2), F(x2)	Coefficient



Coefficient		
Mathematical Mean:		5.67838
Geometrical Mean :		5.62389
Harmonic Mean	:	5.56699
Variance	:	0.59679
S.D.	:	0.77252
Skewed Coef.	:	-0.12922
Kurtosis Coef.	:	2.92499
MAD	:	0.61878
Range	:	7.33285
Mid_range	:	5.23532
Median	:	5.69560
Q1	:	5.16115
Q2	:	5.69560
Q3	:	6.21400
IQR	:	1.05285
C.V.	:	0.13605

d1=X1-X2,

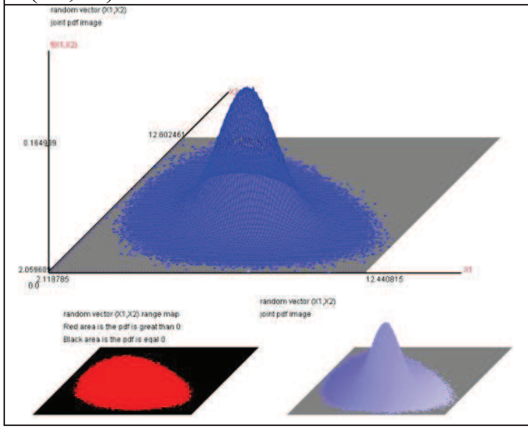
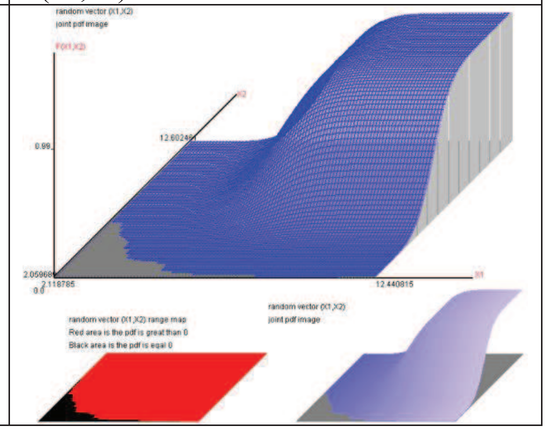
f(d1), F(d1)	Coefficient



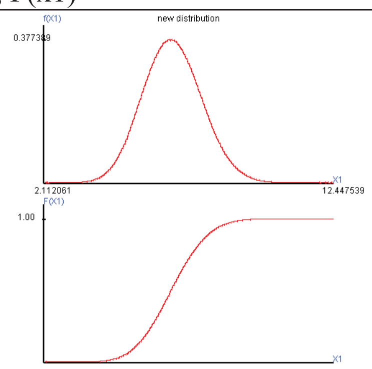
Coefficient		
Mathematical Mean:		-3.79767
Geometrical Mean	:	none
Harmonic Mean	:	none
Variance	:	1.35166
S.D.	:	1.16261
Skewed Coef.	:	0.21311
Kurtosis Coef.	:	2.99849
MAD	:	0.92944
Range	:	11.99535
Mid_range	:	-2.47312
Median	:	-3.83975
Q1	:	-4.60870
Q2	:	-3.83975
Q3	:	-3.03215
IQR	:	1.57655
C.V.	:	none

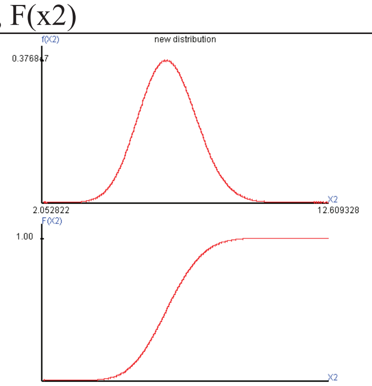
(4)The joint probability distribution of (x_1, x_2)’,n=20,

(4-1) λ_1=0.3333, λ_2=0.3333,

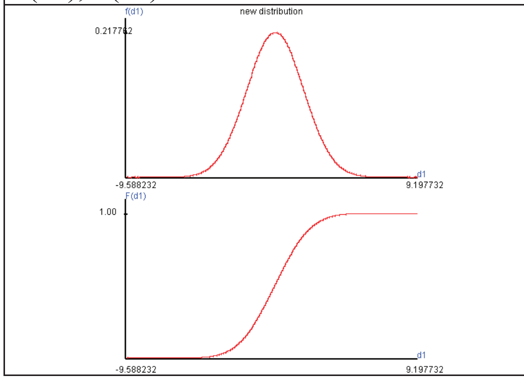
f(x1,x2)	F(x1,x2)
	

E(X1)= 6.6682, Var(X1)= 1.1113, E(X2)= 6.6669, Var(X2)= 1.1107,
Cov(X1,X2)= -0.5557, X1 and X2 correlation coefficient=-0.5002.

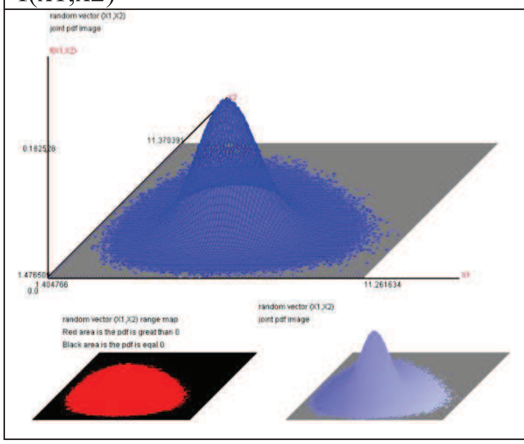
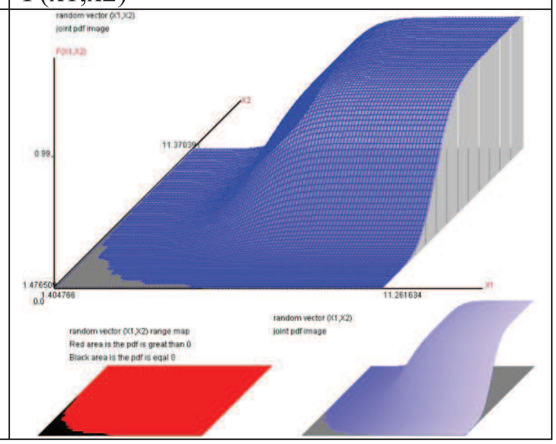
f(x1), F(x1)	Coefficient	
	Mathematical Mean:	6.66816
	Geometrical Mean :	6.58329
	Harmonic Mean :	6.49623
	Variance :	1.11129
	S.D. :	1.05418
	Skewed Coef. :	0.12570
	Kurtosis Coef. :	2.96752
	MAD :	0.84281
	Range :	10.37390
	Mid_range :	7.27980
	Median :	6.64590
	Q1 :	5.94110
	Q2 :	6.64590
	Q3 :	7.37085
	IQR :	1.42975
	C.V. :	0.15809

f(x2), F(x2)	Coefficient	
	Mathematical Mean:	6.66686
	Geometrical Mean :	6.58202
	Harmonic Mean :	6.49498
	Variance :	1.11074
	S.D. :	1.05392
	Skewed Coef. :	0.12604
	Kurtosis Coef. :	2.96848
	MAD :	0.84260
	Range :	10.59575
	Mid_range :	7.33107
	Median :	6.64410
	Q1 :	5.94030
	Q2 :	6.64410
	Q3 :	7.36935
	IQR :	1.42905
	C.V. :	0.15808

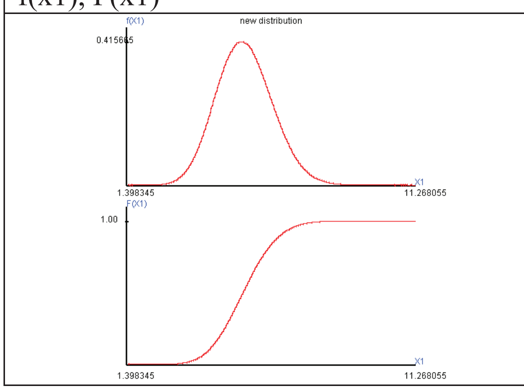
d1=X1-X2,

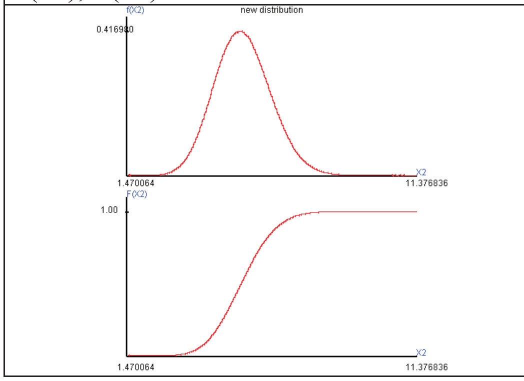
f(d1), F(d1)	Coefficient
	Mathematical Mean: 0.00130 Geometrical Mean : none Harmonic Mean : none Variance : 3.33344 S.D. : 1.82577 Skewed Coef. : -0.00002 Kurtosis Coef. : 2.96763 MAD : 1.45871 Range : 18.85580 Mid_range : -0.19525 Median : 0.00135 Q1 : -1.23410 Q2 : 0.00135 Q3 : 1.23705 IQR : 2.47115 C.V. : none

(4-2) λ_1=0.1, λ_2=0.1,

f(x1,x2)	F(x1,x2)
	

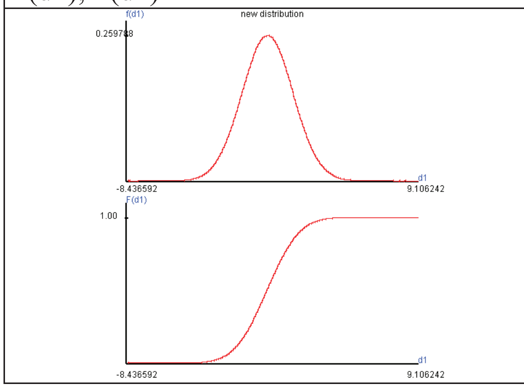
E(X1)= 5.3929, Var(X1)= 0.9155, E(X2)= 5.3924, Var(X2)= 0.9156, Cov(X1,X2)= -0.2700, X1 and X2 correlation coefficient=-0.2949.

f(x1), F(x1)	Coefficient
	Mathematical Mean: 5.39290 Geometrical Mean : 5.30663 Harmonic Mean : 5.21818 Variance : 0.91554 S.D. : 0.95684 Skewed Coef. : 0.19732 Kurtosis Coef. : 3.00183 MAD : 0.76469 Range : 9.90640 Mid_range : 6.33320 Median : 5.36085 Q1 : 4.72725 Q2 : 5.36085 Q3 : 6.02380 IQR : 1.29655 C.V. : 0.17743

f(x2), F(x2)	Coefficient
	

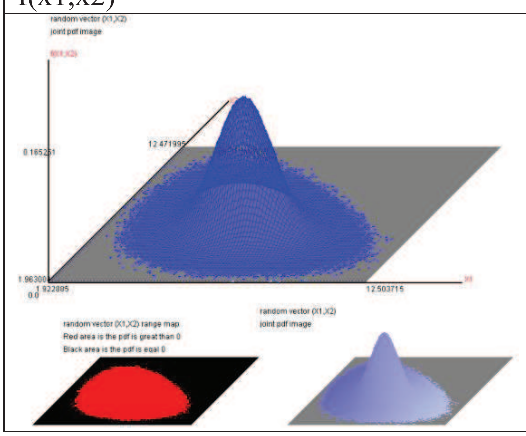
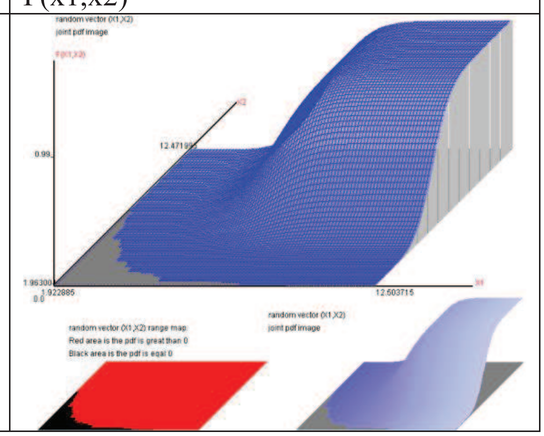
Coefficient		
Mathematical Mean:		5.39238
Geometrical Mean :		5.30611
Harmonic Mean	:	5.21765
Variance	:	0.91561
S.D.	:	0.95687
Skewed Coef.	:	0.19758
Kurtosis Coef.	:	3.00306
MAD	:	0.76460
Range	:	9.94360
Mid_range	:	6.42345
Median	:	5.36050
Q1	:	4.72710
Q2	:	5.36050
Q3	:	6.02305
IQR	:	1.29595
C.V.	:	0.17745

d1=X1-X2,

f(d1), F(d1)	Coefficient
	

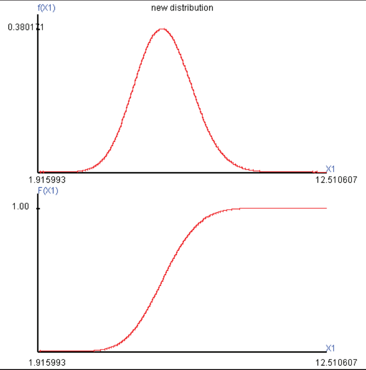
Coefficient		
Mathematical Mean:		0.00051
Geometrical Mean	:	none
Harmonic Mean	:	none
Variance	:	2.37112
S.D.	:	1.53985
Skewed Coef.	:	0.00034
Kurtosis Coef.	:	2.99497
MAD	:	1.22876
Range	:	17.60805
Mid_range	:	0.33482
Median	:	0.00065
Q1	:	-1.03885
Q2	:	0.00065
Q3	:	1.03945
IQR	:	2.07830
C.V.	:	none

(4-3) $\lambda_1=0.3$, $\lambda_2=0.3$,

f(x1,x2)	F(x1,x2)
	

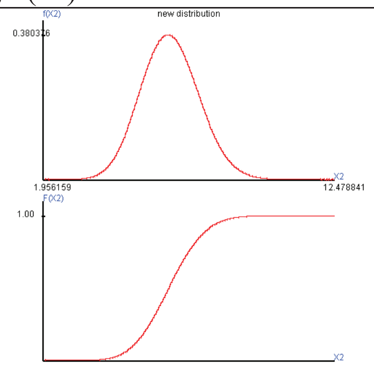
E(X1)= 6.5055, Var(X1)= 1.0903, E(X2)= 6.5020, Var(X2)= 1.0895,
Cov(X1,X2)= -0.5142, X1 and X2 correlation coefficient=-0.4717.

f(x1), F(x1)	Coefficient



Mathematical Mean:		6.50553
Geometrical Mean	:	6.42020
Harmonic Mean	:	6.33267
Variance	:	1.09033
S.D.	:	1.04419
Skewed Coef.	:	0.13534
Kurtosis Coef.	:	2.97246
MAD	:	0.83483
Range	:	10.63400
Mid_range	:	7.21330
Median	:	6.48165
Q1	:	5.78460
Q2	:	6.48165
Q3	:	7.20050
IQR	:	1.41590
C.V.	:	0.16051

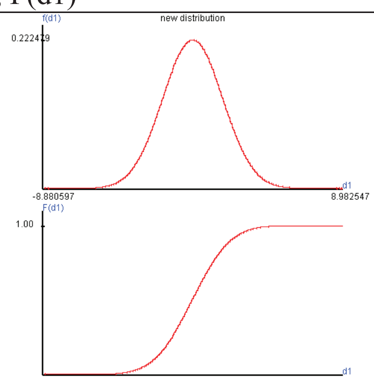
f(x2), F(x2)	Coefficient



Mathematical Mean:		6.50195
Geometrical Mean	:	6.41665
Harmonic Mean	:	6.32914
Variance	:	1.08953
S.D.	:	1.04381
Skewed Coef.	:	0.13518
Kurtosis Coef.	:	2.97065
MAD	:	0.83455
Range	:	10.56180
Mid_range	:	7.21750
Median	:	6.47820
Q1	:	5.78110
Q2	:	6.47820
Q3	:	7.19675
IQR	:	1.41565
C.V.	:	0.16054

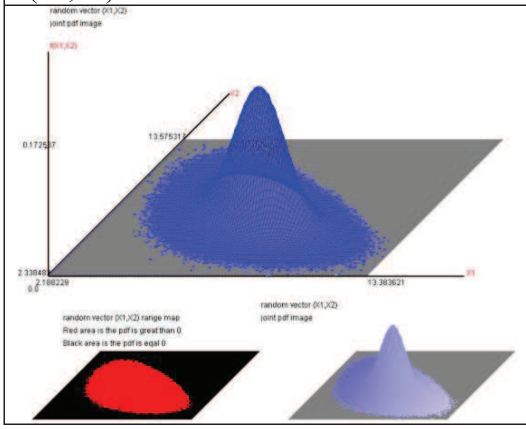
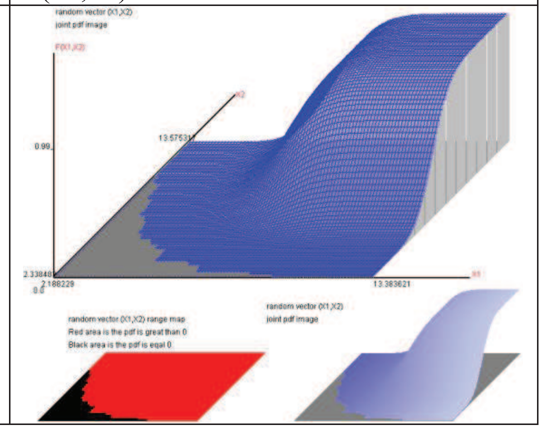
d1=X1-X2,

f(d1), F(d1)	Coefficient

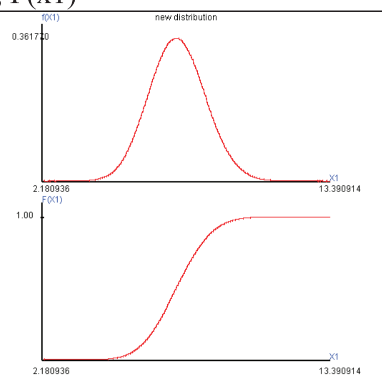


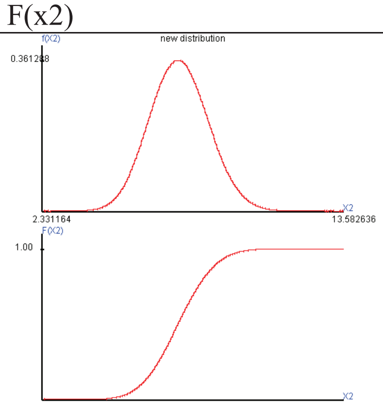
Mathematical Mean:		0.00357
Geometrical Mean	:	none
Harmonic Mean	:	none
Variance	:	3.20817
S.D.	:	1.79114
Skewed Coef.	:	0.00035
Kurtosis Coef.	:	2.97127
MAD	:	1.43090
Range	:	17.92955
Mid_range	:	0.05097
Median	:	0.00370
Q1	:	-1.20865
Q2	:	0.00370
Q3	:	1.21550
IQR	:	2.42415
C.V.	:	none

(4-4) λ_1=0.45, λ_2=0.45,

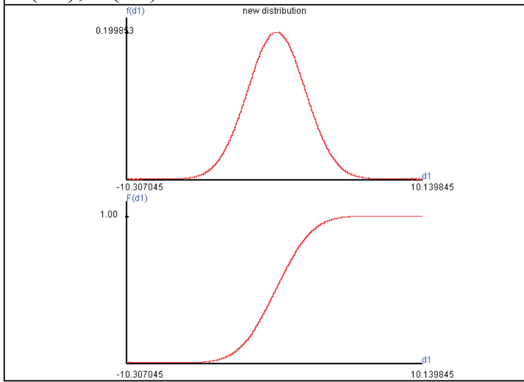
f(x1,x2)	F(x1,x2)
	

E(X1)= 7.4115, Var(X1)= 1.2037, E(X2)= 7.4098, Var(X2)= 1.2046,
Cov(X1,X2)= -0.7712, X1 and X2 correlation coefficient=-0.6405.

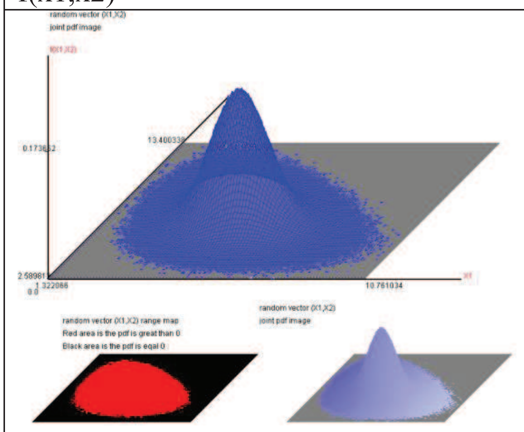
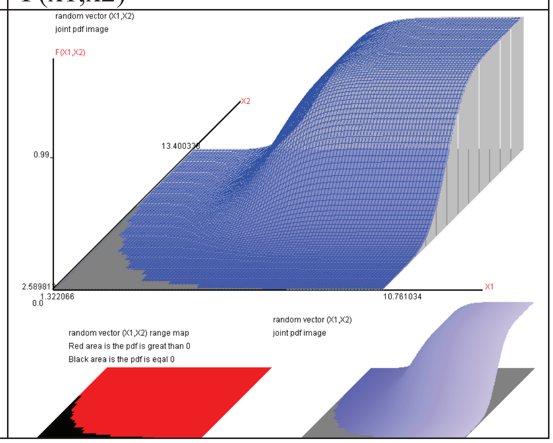
f(x1), F(x1)	Coefficient
	Mathematical Mean: 7.41146 Geometrical Mean : 7.32870 Harmonic Mean : 7.24381 Variance : 1.20370 S.D. : 1.09713 Skewed Coef. : 0.08943 Kurtosis Coef. : 2.95914 MAD : 0.87724 Range : 11.25165 Mid_range : 7.78592 Median : 7.39510 Q1 : 6.65835 Q2 : 7.39510 Q3 : 8.14665 IQR : 1.48830 C.V. : 0.14803

f(x2), F(x2)	Coefficient
	Mathematical Mean: 7.40983 Geometrical Mean : 7.32699 Harmonic Mean : 7.24202 Variance : 1.20458 S.D. : 1.09753 Skewed Coef. : 0.08996 Kurtosis Coef. : 2.96035 MAD : 0.87745 Range : 11.29330 Mid_range : 7.95690 Median : 7.39340 Q1 : 6.65660 Q2 : 7.39340 Q3 : 8.14470 IQR : 1.48810 C.V. : 0.14812

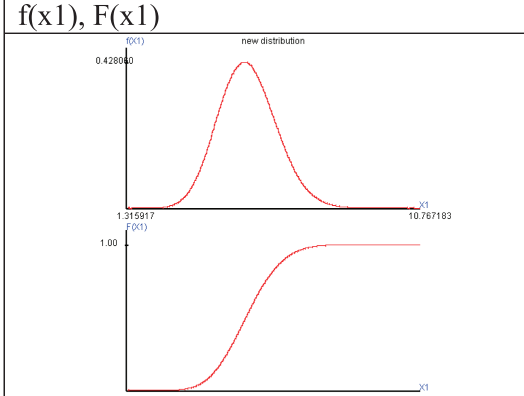
d1=X1-X2,

f(d1), F(d1)	Coefficient		
	Mathematical Mean:		0.00164
	Geometrical Mean	:	none
	Harmonic Mean	:	none
	Variance	:	3.95075
	S.D.	:	1.98765
	Skewed Coef.	:	-0.00063
	Kurtosis Coef.	:	2.96032
	MAD	:	1.58860
	Range	:	20.52290
	Mid_range	:	-0.08360
	Median	:	0.00160
	Q1	:	-1.34490
	Q2	:	0.00160
	Q3	:	1.34860
	IQR	:	2.69350
	C.V.	:	none

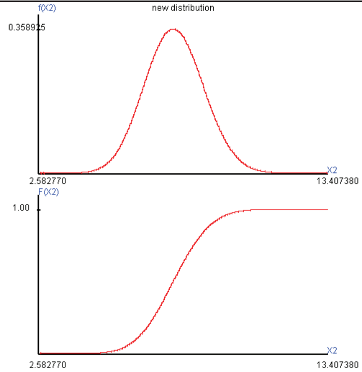
(4-5) λ_1=0.1, λ_2=0.5,

f(x1,x2)	F(x1,x2)
	

E(X1)= 5.1797, Var(X1)= 0.8659, E(X2)= 7.6312, Var(X2)= 1.2239,
Cov(X1,X2)= -0.4536, X1 and X2 correlation coefficient=-0.4407.

f(x1), F(x1)	Coefficient		
	Mathematical Mean:		5.17968
	Geometrical Mean	:	5.09479
	Harmonic Mean	:	5.00778
	Variance	:	0.86587
	S.D.	:	0.93052
	Skewed Coef.	:	0.20776
	Kurtosis Coef.	:	3.01049
	MAD	:	0.74344
	Range	:	9.48640
	Mid_range	:	6.04155
	Median	:	5.14705
	Q1	:	4.53190
	Q2	:	5.14705
	Q3	:	5.79210
	IQR	:	1.26020
	C.V.	:	0.17965

f(x2), F(x2)

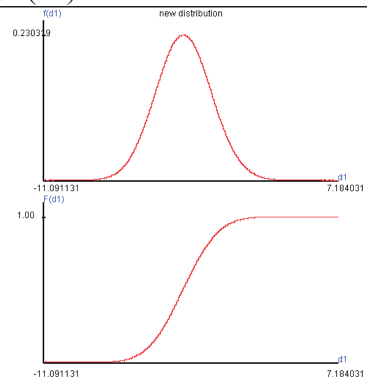


Coefficient

Mathematical Mean:		7.63116
Geometrical Mean :		7.54943
Harmonic Mean	:	7.46562
Variance	:	1.22393
S.D.	:	1.10632
Skewed Coef.	:	0.07987
Kurtosis Coef.	:	2.95576
MAD	:	0.88460
Range	:	10.86485
Mid_range	:	7.99507
Median	:	7.61610
Q1	:	6.87275
Q2	:	7.61610
Q3	:	8.37330
IQR	:	1.50055
C.V.	:	0.14497

d1=X1-X2,

f(d1), F(d1)

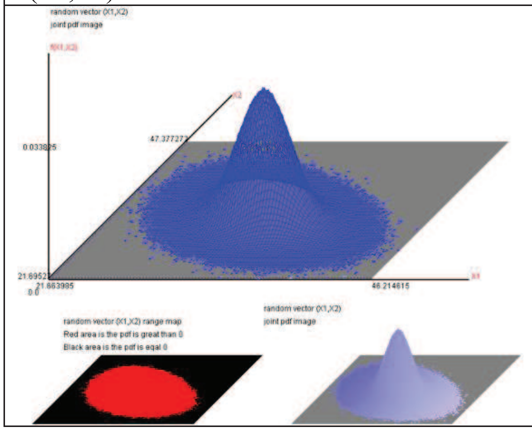
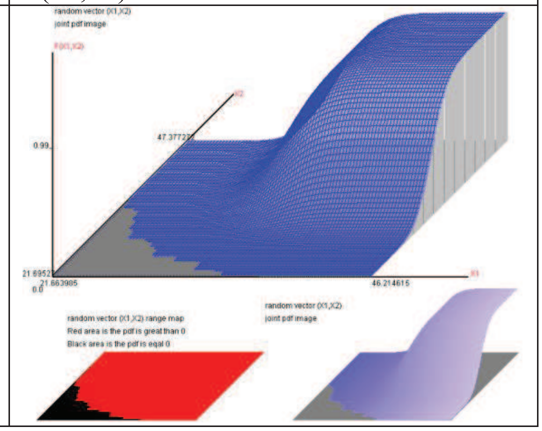


Coefficient

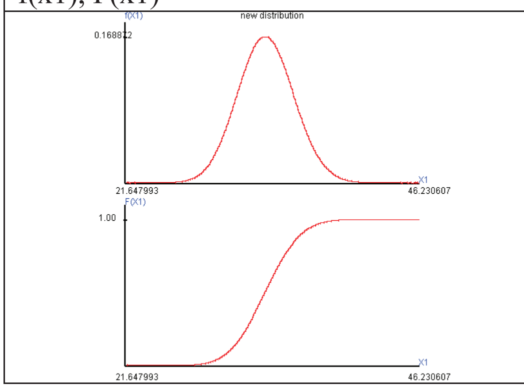
Mathematical Mean:		-2.45147
Geometrical Mean : none		
Harmonic Mean	: none	
Variance	:	2.99710
S.D.	:	1.73121
Skewed Coef.	:	0.03429
Kurtosis Coef.	:	2.97707
MAD	:	1.38267
Range	:	18.34310
Mid_range	:	-1.95355
Median	:	-2.46110
Q1	:	-3.62745
Q2	:	-2.46110
Q3	:	-1.28645
IQR	:	2.34100
C.V.	: none	

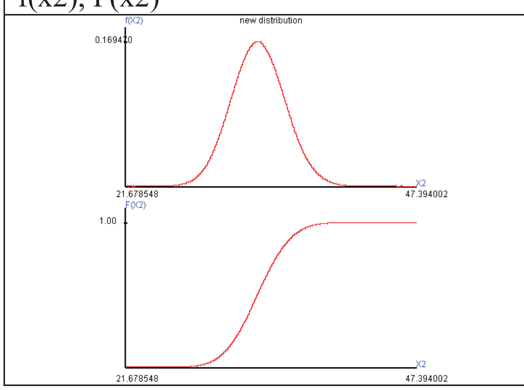
(5)The joint probability distribution of (x_1, x_2)',n=100,

(5-1) λ_1=0.3333, λ_2=0.3333,

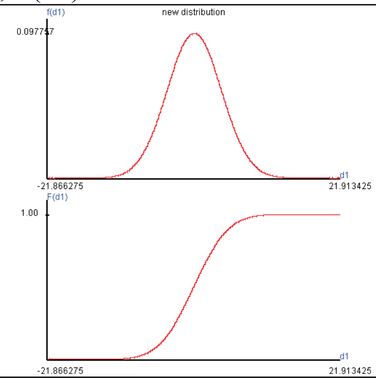
f(x1,x2)	F(x1,x2)
	

E(X1)= 33.3365, Var(X1)= 5.5602, E(X2)= 33.3325, Var(X2)= 5.5548,
Cov(X1,X2)= -2.7796, X1 and X2 correlation coefficient=-0.5002.

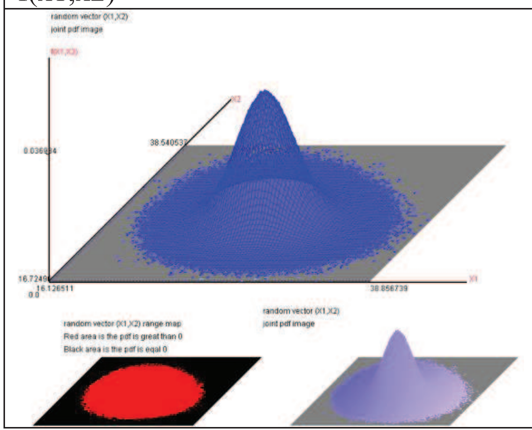
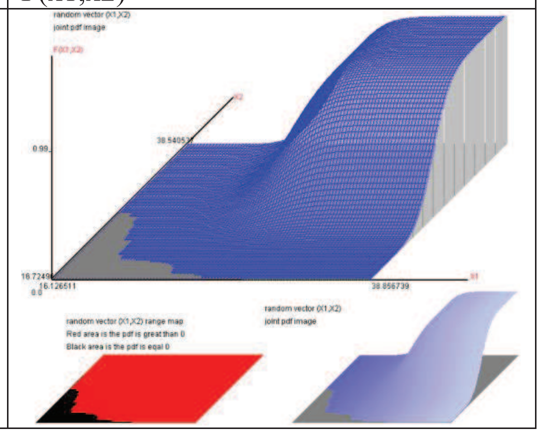
f(x1), F(x1)	Coefficient		
	Mathematical Mean:		33.33648
	Geometrical Mean	:	33.25278
	Harmonic Mean	:	33.16868
	Variance	:	5.56019
	S.D.	:	2.35800
	Skewed Coef.	:	0.05665
	Kurtosis Coef.	:	2.99376
	MAD	:	1.88222
	Range	:	24.67400
	Mid_range	:	33.93930
	Median	:	33.31460
	Q1	:	31.73190
	Q2	:	33.31460
	Q3	:	34.91630
	IQR	:	3.18440
	C.V.	:	0.07073

f(x2), F(x2)	Coefficient		
	Mathematical Mean:		33.33254
	Geometrical Mean	:	33.24892
	Harmonic Mean	:	33.16488
	Variance	:	5.55483
	S.D.	:	2.35687
	Skewed Coef.	:	0.05658
	Kurtosis Coef.	:	2.99170
	MAD	:	1.88133
	Range	:	25.81105
	Mid_range	:	34.53627
	Median	:	33.31020
	Q1	:	31.72855
	Q2	:	33.31020
	Q3	:	34.91185
	IQR	:	3.18330
	C.V.	:	0.07071

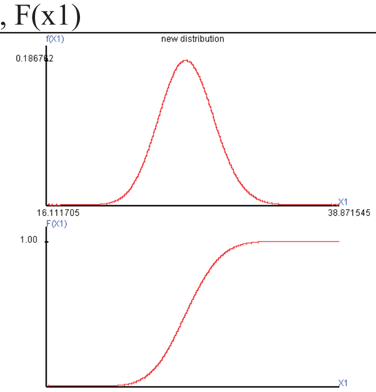
d1=X1-X2,

f(d1), F(d1)	Coefficient
	Mathematical Mean: 0.00394 Geometrical Mean : none Harmonic Mean : none Variance : 16.67426 S.D. : 4.08341 Skewed Coef. : 0.00032 Kurtosis Coef. : 2.99334 MAD : 3.25889 Range : 43.94245 Mid_range : 0.02357 Median : 0.00465 Q1 : -2.75305 Q2 : 0.00465 Q3 : 2.75965 IQR : 5.51270 C.V. : none

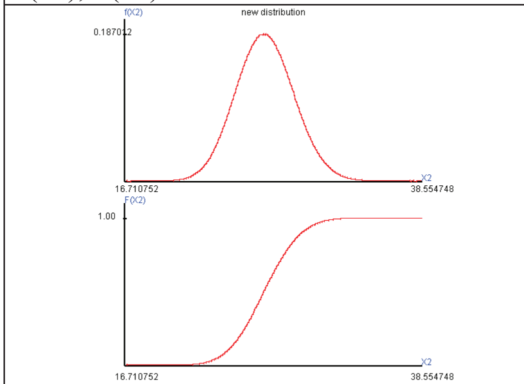
(5-2) λ_1=0.1, λ_2=0.1,

f(x1,x2)	F(x1,x2)
	

E(X1)= 26.9682, Var(X1)= 4.5775, E(X2)= 26.9591, Var(X2)= 4.5718,
Cov(X1,X2)= -1.3471, X1 and X2 correlation coefficient=-0.2945.

f(x1), F(x1)	Coefficient
	Mathematical Mean: 26.96821 Geometrical Mean : 26.88307 Harmonic Mean : 26.79751 Variance : 4.57745 S.D. : 2.13950 Skewed Coef. : 0.08754 Kurtosis Coef. : 2.99905 MAD : 1.70780 Range : 22.84445 Mid_range : 27.49162 Median : 26.93640 Q1 : 25.50680 Q2 : 26.93640 Q3 : 28.39580 IQR : 2.88900 C.V. : 0.07933

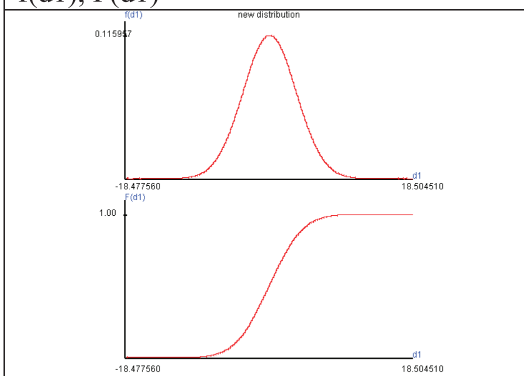
f(x2), F(x2)	Coefficient



Coefficient	
Mathematical Mean:	26.95906
Geometrical Mean :	26.87400
Harmonic Mean :	26.78852
Variance :	4.57180
S.D. :	2.13818
Skewed Coef. :	0.08787
Kurtosis Coef. :	2.99812
MAD :	1.70673
Range :	21.92520
Mid_range :	27.63275
Median :	26.92775
Q1 :	25.49760
Q2 :	26.92775
Q3 :	28.38540
IQR :	2.88780
C.V. :	0.07931

d1=X1-X2,

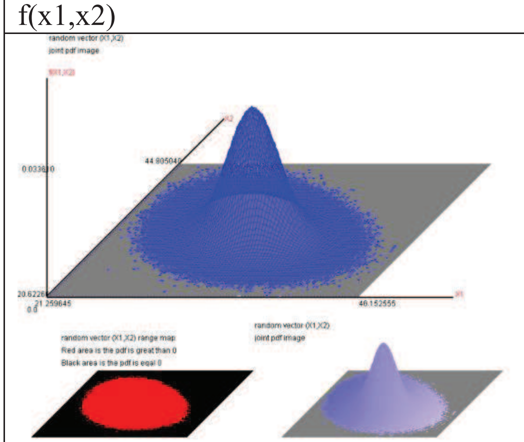
f(d1), F(d1)	Coefficient

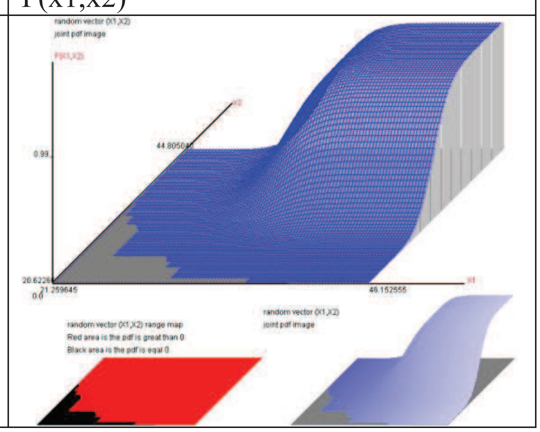


Coefficient	
Mathematical Mean:	0.00915
Geometrical Mean : none	
Harmonic Mean : none	
Variance :	11.84355
S.D. :	3.44145
Skewed Coef. :	-0.00023
Kurtosis Coef. :	2.99789
MAD :	2.74624
Range :	37.11955
Mid_range :	0.01348
Median :	0.00880
Q1 :	-2.31305
Q2 :	0.00880
Q3 :	2.33180
IQR :	4.64485
C.V. : none	

(5-3) λ_1=0.3, λ_2=0.3,

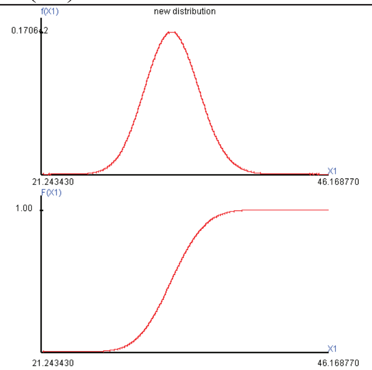
f(x1,x2)	F(x1,x2)





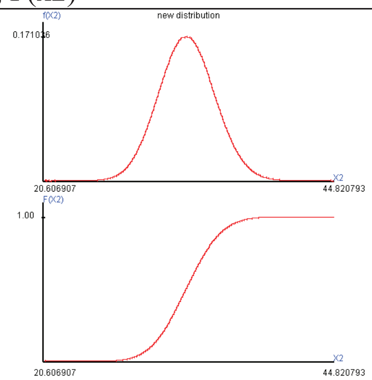
E(X1)= 32.5284, Var(X1)= 5.4489, E(X2)= 32.5079, Var(X2)= 5.4434,
Cov(X1,X2)= -2.5691, X1 and X2 correlation coefficient=-0.4717.

f(x1), F(x1)	Coefficient



Mathematical Mean:		32.52845
Geometrical Mean	:	32.44439
Harmonic Mean	:	32.35993
Variance	:	5.44889
S.D.	:	2.33429
Skewed Coef.	:	0.06084
Kurtosis Coef.	:	2.99397
MAD	:	1.86331
Range	:	25.01800
Mid_range	:	33.70610
Median	:	32.50480
Q1	:	30.93920
Q2	:	32.50480
Q3	:	34.09210
IQR	:	3.15290
C.V.	:	0.07176

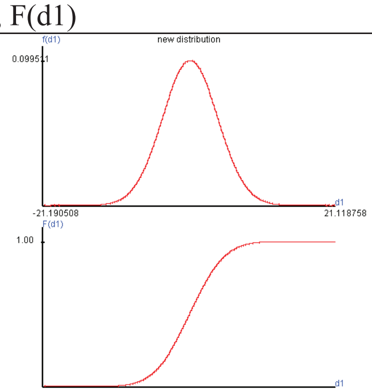
f(x2), F(x2)	Coefficient



Mathematical Mean:		32.50794
Geometrical Mean	:	32.42391
Harmonic Mean	:	32.33947
Variance	:	5.44344
S.D.	:	2.33312
Skewed Coef.	:	0.05999
Kurtosis Coef.	:	2.99436
MAD	:	1.86221
Range	:	24.30390
Mid_range	:	32.71385
Median	:	32.48495
Q1	:	30.91965
Q2	:	32.48495
Q3	:	34.07035
IQR	:	3.15070
C.V.	:	0.07177

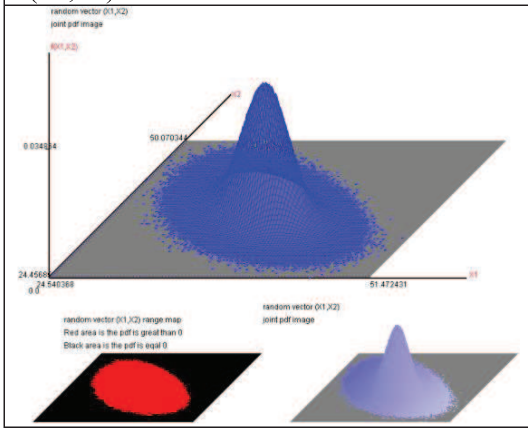
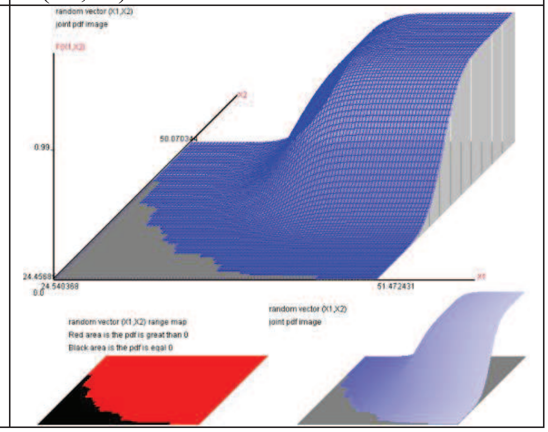
d1=X1-X2,

f(d1), F(d1)	Coefficient

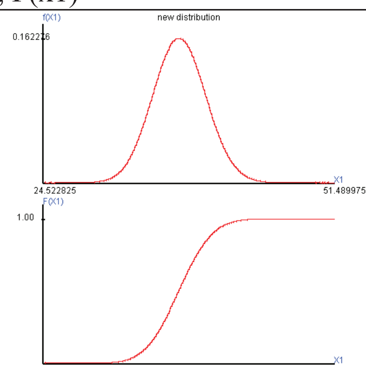


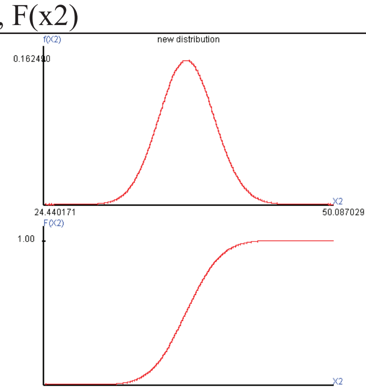
Mathematical Mean:		0.02051
Geometrical Mean	:	none
Harmonic Mean	:	none
Variance	:	16.03053
S.D.	:	4.00381
Skewed Coef.	:	0.00026
Kurtosis Coef.	:	2.99384
MAD	:	3.19558
Range	:	42.46655
Mid_range	:	-0.03588
Median	:	0.01965
Q1	:	-2.68185
Q2	:	0.01965
Q3	:	2.72280
IQR	:	5.40465
C.V.	:	195.24265

(5-4) λ_1=0.45, λ_2=0.45,

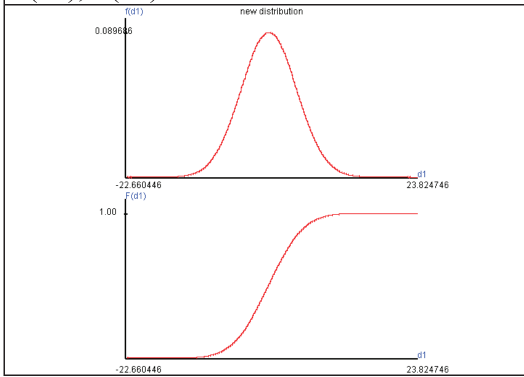
f(x1,x2)	F(x1,x2)
	

E(X1)= 37.0666, Var(X1)= 6.0207, E(X2)= 37.0442, Var(X2)= 6.0195,
Cov(X1,X2)= -3.8565, X1 and X2 correlation coefficient=-0.6406.

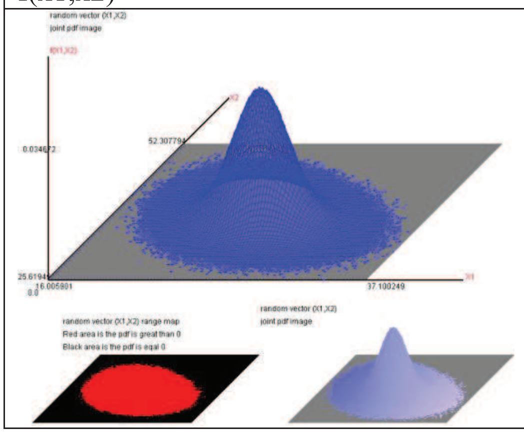
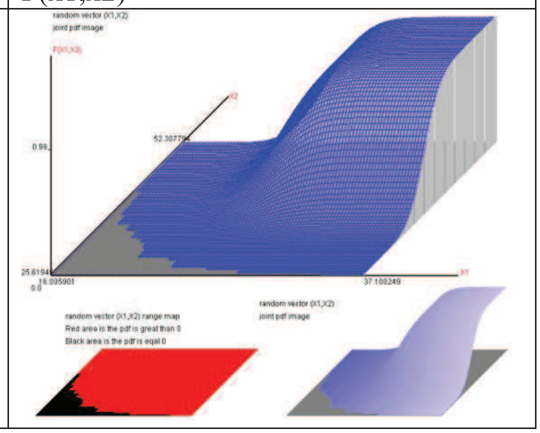
f(x1), F(x1)	Coefficient		
	Mathematical Mean:		37.06658
	Geometrical Mean	:	36.98506
	Harmonic Mean	:	36.90315
	Variance	:	6.02069
	S.D.	:	2.45371
	Skewed Coef.	:	0.03971
	Kurtosis Coef.	:	2.99081
	MAD	:	1.95864
	Range	:	27.06740
	Mid_range	:	38.00640
	Median	:	37.05070
	Q1	:	35.40085
	Q2	:	37.05070
	Q3	:	38.71435
	IQR	:	3.31350
	C.V.	:	0.06620

f(x2), F(x2)	Coefficient		
	Mathematical Mean:		37.04415
	Geometrical Mean	:	36.96260
	Harmonic Mean	:	36.88065
	Variance	:	6.01948
	S.D.	:	2.45346
	Skewed Coef.	:	0.03938
	Kurtosis Coef.	:	2.99251
	MAD	:	1.95841
	Range	:	25.74220
	Mid_range	:	37.26360
	Median	:	37.02860
	Q1	:	35.37860
	Q2	:	37.02860
	Q3	:	38.69225
	IQR	:	3.31365
	C.V.	:	0.06623

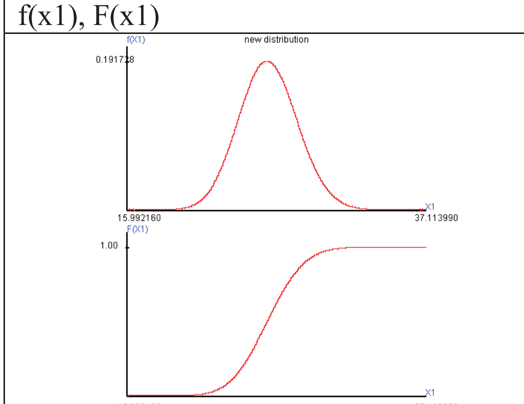
d1=X1-X2,

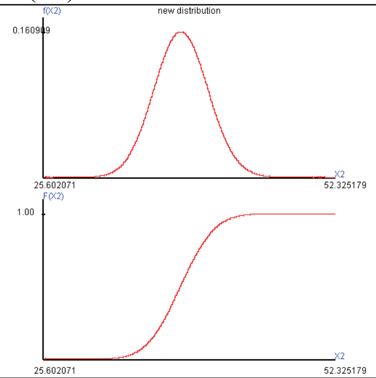
f(d1), F(d1)	Coefficient
	Mathematical Mean: 0.02243 Geometrical Mean : none Harmonic Mean : none Variance : 19.75308 S.D. : 4.44444 Skewed Coef. : -0.00006 Kurtosis Coef. : 2.99165 MAD : 3.54749 Range : 46.65800 Mid_range : 0.58215 Median : 0.02305 Q1 : -2.97850 Q2 : 0.02305 Q3 : 3.02250 IQR : 6.00100 C.V. : 198.15876

(5-5) λ_1=0.1, λ_2=0.5,

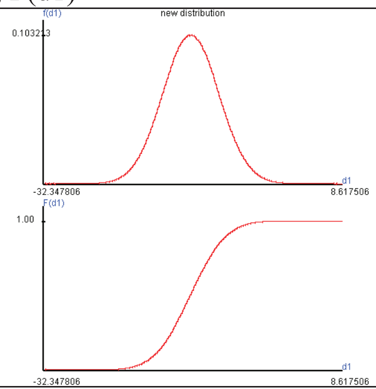
f(x1,x2)	F(x1,x2)
	

E(X1)= 25.9058, Var(X1)= 4.3309, E(X2)= 38.1510, Var(X2)= 6.1142, Cov(X1,X2)= -2.2661, X1 and X2 correlation coefficient=-0.4404.

f(x1), F(x1)	Coefficient
	Mathematical Mean: 25.90585 Geometrical Mean : 25.82200 Harmonic Mean : 25.73775 Variance : 4.33086 S.D. : 2.08107 Skewed Coef. : 0.09273 Kurtosis Coef. : 3.00341 MAD : 1.66080 Range : 21.20035 Mid_range : 26.55307 Median : 25.87360 Q1 : 24.48425 Q2 : 25.87360 Q3 : 27.29215 IQR : 2.80790 C.V. : 0.08033

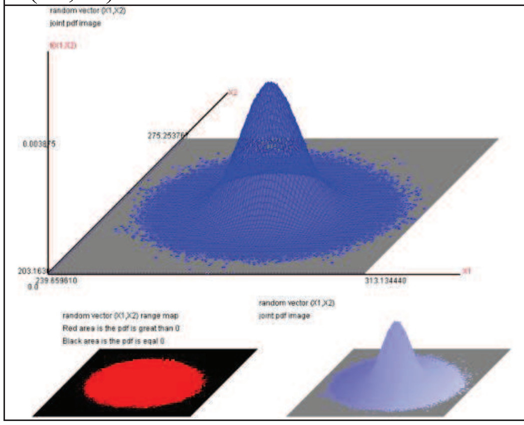
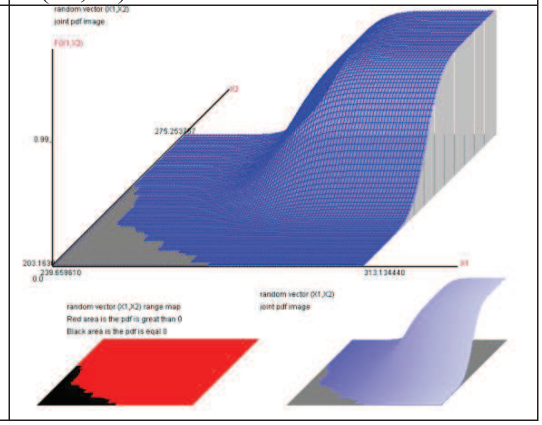
f(x2), F(x2)	Coefficient	
	Mathematical Mean:	38.15104
	Geometrical Mean :	38.07061
	Harmonic Mean :	37.98979
	Variance :	6.11417
	S.D. :	2.47268
	Skewed Coef. :	0.03589
	Kurtosis Coef. :	2.99116
	MAD :	1.97393
	Range :	26.82245
	Mid_range :	38.96362
	Median :	38.13615
	Q1 :	36.47300
	Q2 :	38.13615
	Q3 :	39.81295
	IQR :	3.33995
	C.V. :	0.06481

d1=X1-X2,

f(d1), F(d1)	Coefficient	
	Mathematical Mean:	-12.24519
	Geometrical Mean :	none
	Harmonic Mean :	none
	Variance :	14.97714
	S.D. :	3.87003
	Skewed Coef. :	0.01508
	Kurtosis Coef. :	2.99573
	MAD :	3.08860
	Range :	41.11760
	Mid_range :	-11.86515
	Median :	-12.25385
	Q1 :	-14.86285
	Q2 :	-12.25385
	Q3 :	-9.63860
	IQR :	5.22425
	C.V. :	none

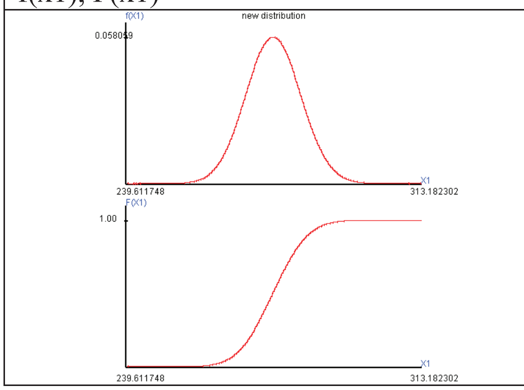
(6)The joint probability distribution of (x_1, x_2)',n=1,000,

(6-1) λ_1=0.1, λ_2=0.05,

f(x1,x2)	F(x1,x2)
	

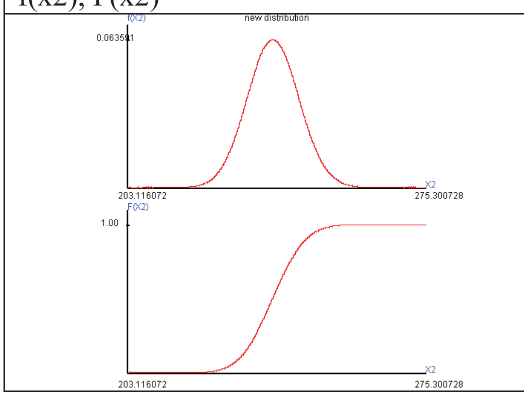
E(X1)= 276.1432, Var(X1)= 47.2775, E(X2)= 238.1263, Var(X2)= 39.5156,
Cov(X1,X2)= -11.2082, X1 and X2 correlation coefficient=-0.2593.

f(x1), F(x1)



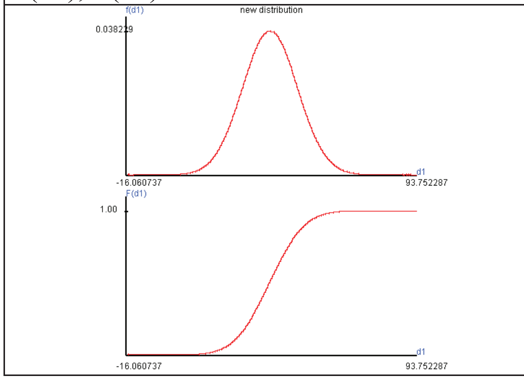
Coefficient		
Mathematical Mean:		276.14323
Geometrical Mean	:	276.05760
Harmonic Mean	:	275.97194
Variance	:	47.27754
S.D.	:	6.87587
Skewed Coef.	:	0.02999
Kurtosis Coef.	:	3.00141
MAD	:	5.48638
Range	:	73.84405
Mid_range	:	276.39702
Median	:	276.10955
Q1	:	271.48585
Q2	:	276.10955
Q3	:	280.76335
IQR	:	9.27750
C.V.	:	0.02490

f(x2), F(x2)



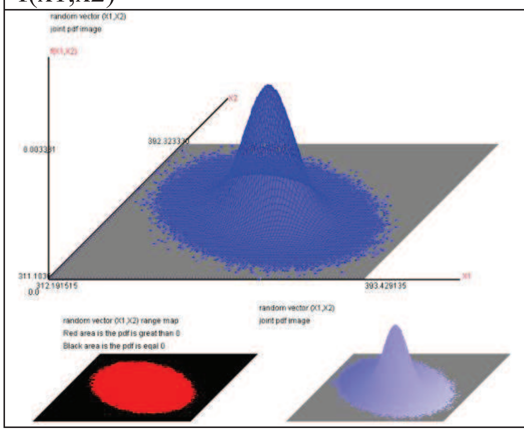
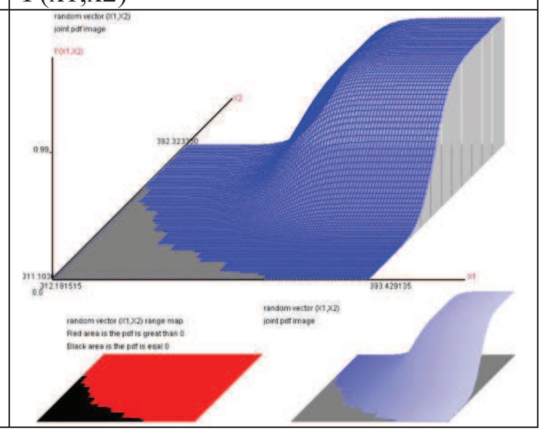
Coefficient		
Mathematical Mean:		238.12634
Geometrical Mean	:	238.04335
Harmonic Mean	:	237.96031
Variance	:	39.51559
S.D.	:	6.28614
Skewed Coef.	:	0.03196
Kurtosis Coef.	:	3.00072
MAD	:	5.01585
Range	:	72.45300
Mid_range	:	239.20840
Median	:	238.09265
Q1	:	233.86725
Q2	:	238.09265
Q3	:	242.34780
IQR	:	8.48055
C.V.	:	0.02640

d1=X1-X2,

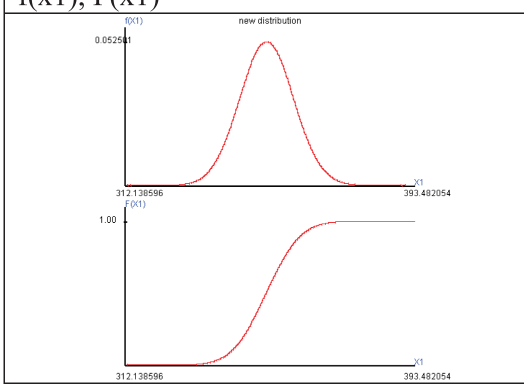
f(d1), F(d1)	Coefficient
	

Coefficient		
Mathematical Mean:		38.01689
Geometrical Mean : none		
Harmonic Mean	: none	
Variance	:	109.20962
S.D.	:	10.45034
Skewed Coef.	:	0.00316
Kurtosis Coef.	:	2.99959
MAD	:	8.33813
Range	:	110.22125
Mid_range	:	38.84577
Median	:	38.01360
Q1	:	30.96235
Q2	:	38.01360
Q3	:	45.06290
IQR	:	14.10055
C.V.	:	0.27489

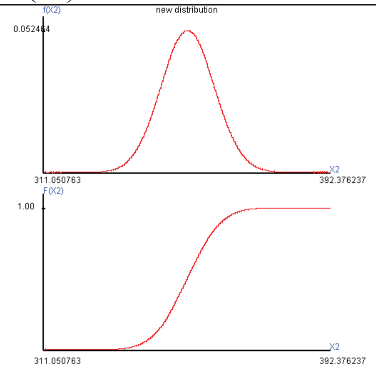
(6-2) λ_1=0.4, λ_2=0.4,

f(x1,x2)	F(x1,x2)
	

E(X1)= 351.6462, Var(X1)= 57.8745, E(X2)= 351.7337, Var(X2)= 57.9124, Cov(X1,X2)= -32.8920, X1 and X2 correlation coefficient=-0.5681.

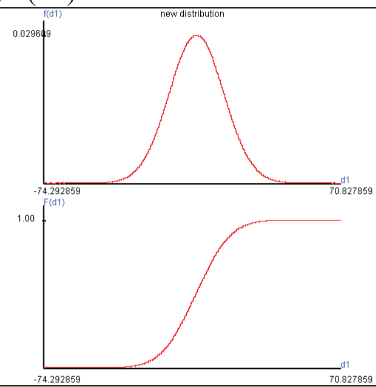
f(x1), F(x1)	Coefficient
	

Coefficient		
Mathematical Mean:		351.64623
Geometrical Mean :		351.56391
Harmonic Mean	:	351.48154
Variance	:	57.87447
S.D.	:	7.60753
Skewed Coef.	:	0.01491
Kurtosis Coef.	:	3.00100
MAD	:	6.06956
Range	:	81.64585
Mid_range	:	352.81032
Median	:	351.62825
Q1	:	346.50525
Q2	:	351.62825
Q3	:	356.76475
IQR	:	10.25950
C.V.	:	0.02163

f(x2), F(x2)	Coefficient
	

Coefficient		
Mathematical Mean	:	351.73371
Geometrical Mean	:	351.65135
Harmonic Mean	:	351.56896
Variance	:	57.91240
S.D.	:	7.61002
Skewed Coef.	:	0.01448
Kurtosis Coef.	:	2.99717
MAD	:	6.07242
Range	:	81.62780
Mid_range	:	351.71350
Median	:	351.71620
Q1	:	346.58965
Q2	:	351.71620
Q3	:	356.85650
IQR	:	10.26685
C.V.	:	0.02164

d1=X1-X2,

f(d1), F(d1)	Coefficient
	

Coefficient		
Mathematical Mean	:	-0.08748
Geometrical Mean	:	none
Harmonic Mean	:	none
Variance	:	181.57090
S.D.	:	13.47482
Skewed Coef.	:	-0.00091
Kurtosis Coef.	:	2.99825
MAD	:	10.75132
Range	:	145.66020
Mid_range	:	-1.73250
Median	:	-0.08725
Q1	:	-9.17590
Q2	:	-0.08725
Q3	:	8.99960
IQR	:	18.17550
C.V.	:	none