A 3SDP relaxation to solve vertex cover problem[1]

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Abstract

Vertex cover problem is a famous combinatorial problem, which its complexity has been heavily studied. It is known that it is hard to approximate to within any constant factor better than 2. In this paper, based on the addition of new constraints to the combination of 3 semidefinite programming (SDP) relaxations, we introduce a new stronger SDP relaxation for vertex cover problem which solve it exactly on general graphs. In this manner and by solving one of the NP-complete problems in polynomial time, we conclude that P=NP.

Keywords: Discrete Optimization, Vertex Cover Problem, Semi-definite Programming (SDP), Complexity Theory, NP-Complete Problems.

1. Introduction

In complexity theory, the abbreviation NP refers to "nondeterministic polynomial", where a problem is in NP if we can test whether a solution is correct in polynomial time. P and NP-complete problems are subsets of NP Problems. We can solve P problems in polynomial time while determining whether or not it is possible to solve NP-complete problems quickly (called the P vs. NP problem) is one of the principal unsolved problems in Mathematics and Computer science.

Interestingly, many results in complexity theory and computational optimization assume solidly based on the hypothesis P≠NP [4]. Moreover, due to this hypothesis and intractability of NP-complete problems, these problems were often addressed by using heuristic methods and approximation algorithms and as the field progressed, it became apparent that different NP-complete optimization problems have different approximation factors. But what if P=NP?

In this paper, we consider the vertex cover problem which is a famous NP-complete problem. It cannot be approximated within a factor of 1.36 [1], unless P=NP, while a 2–approximation for it can be trivially obtained by taking all the vertices of a maximal matching in the graph. However, improving this simple 2–approximation algorithm has been a quite hard task [3]. Here, we introduce a new semidefinite programming (SDP) relaxation for vertex cover problem which can solve it exactly on general graphs.

The rest of the paper is structured as follows. Section 2 is about the vertex cover problem and introduces the basic ingredients of a new SDP relaxation (we called it 3SDP) for vertex cover problem. In

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section 3, new theorems are introduced which conclude that the proposed 3SDP model produces the optimal solution of the vertex cover problem on general graphs. Finally, Section 4 concludes the paper.

2. A 3SDP model for vertex cover problem

In the mathematical discipline of graph theory, a vertex cover of a graph G=(V,E) is a set of vertices such that each edge of the graph is incident to at least one vertex of the set. The problem of finding a minimum vertex cover is a typical example of an NP-complete optimization problem. Aside from its theoretical interest, the vertex cover problem arises in many practical applications. This fact has encouraged considerable effort in finding good approximation solutions.

Despite many attempts to design approximation algorithms for vertex cover problem, the best-known approximation ratio is $2-o(1)$ and it is based on using SDP relaxation [2]. By assigning a unit vector $v_i \in \mathbb{R}^{n+1}$ to each vertex i∈V, a well known SDP formulation of the vertex cover problem is as follows [2]:

$$\min_{s.t.} \sum_{i\in V} \frac{1+v_o v_i}{2}$$

$$+v_o v_i + v_o v_j - v_i v_j = 1 \qquad ij \in E \qquad (1)$$

$$+v_i v_j + v_i v_k + v_j v_k \geq -1 \qquad i,j,k \in V \cup \{o\} \quad (2')$$

$$+v_i v_j - v_i v_k - v_j v_k \geq -1 \qquad i,j,k \in V \cup \{o\} \quad (2'')$$

$$-v_i v_j + v_i v_k - v_j v_k \geq -1 \qquad i,j,k \in V \cup \{o\} \quad (2''')$$

$$-v_i v_j - v_i v_k + v_j v_k \geq -1 \qquad i,j,k \in V \cup \{o\} \quad (2'''')$$

$$v_i v_i = 1 \qquad i \in V \cup \{o\} \quad (3)$$

$$v_i v_j \in \{-1,+1\} \qquad i,j \in V \cup \{o\} \quad (4)$$

By relaxing the constraints (4) as $-1 \leq v_i v_j \leq 1$ and using interior-point methods, we can solve the SDP relaxation in polynomial time. In an integral solution of this SDP relaxation, a vertex cover is composed of the vertices that their corresponding vectors are picked coincide with the vector v_o; i.e. $\{i\in V \mid v_o v_i = 1\}$. But in general though, an optimal solution of this SDP formulation will not be integral.

However, we can strengthen this SDP relaxation by considering 3 such SDP formulations, together. To do this, we assign 3 unit vectors $v_i^l \in \mathbb{R}^{3(n+1)}$ (l=1,2,3) to each vertex i∈V and in addition to the triangle inequalities (2), we add new constraints which we called them 120-degree and bisector equalities. The resulted SDP relaxation which we called it 3SDP, is as follows:

$$(3SDP) \qquad \min_{s.t.} \quad \frac{1}{3}\sum_{l=1}^{3}\sum_{i\in V} \frac{1+v_o^l v_i^l}{2}$$

$$+v_o^l v_i^l + v_o^l v_j^l - v_i^l v_j^l = 1 \qquad ij \in E\,, \ \ l = 1,2,3 \quad (5)$$

$$+v_i^l v_j^l + v_i^l v_k^l + v_j^l v_k^l \geq -1 \qquad i,j,k \in V \cup \{o\}\,,\ \ l = 1,2,3 \quad (6')$$

$$+v_i^l v_j^l - v_i^l v_k^l - v_j^l v_k^l \geq -1 \qquad i,j,k \in V \cup \{o\}\,,\ \ l = 1,2,3 \quad (6'')$$

$$-v_i^l v_j^l + v_i^l v_k^l - v_j^l v_k^l \geq -1 \qquad i,j,k \in V \cup \{o\}\,,\ \ l = 1,2,3 \quad (6''')$$

$$-v_i^l v_j^l - v_i^l v_k^l + v_j^l v_k^l \geq -1 \qquad i,j,k \in V \cup \{o\}\,,\ \ l = 1,2,3 \quad (6'''')$$

$$v_i^p v_i^q = \frac{-1}{2} \qquad i \in V \cup \{o\}\,,\ \ 1 \leq p \neq q \leq 3 \quad (7)$$

$$v_o^l v_i^p - v_o^l v_i^q = 0 \qquad i \in V\,,\ \ 1 \leq l \neq p \neq q \leq 3 \quad (8')$$

$$v_i^l v_o^p - v_i^l v_o^q = 0 \qquad i \in V\,,\ \ 1 \leq l \neq p \neq q \leq 3 \quad (8'')$$

$$v_i^l v_i^l = 1 \qquad i \in V \cup \{o\}\,,\ \ l = 1,2,3 \quad (9)$$

$$-1 \leq v_i^l v_j^l \leq +1 \qquad i,j \in V \cup \{o\}\,,\ \ l = 1,2,3 \quad (10)$$

Note that, in addition to the standard constraints of the last SDP formulation for each of the SDP combinations (SDP_l l=1,2,3), this 3SDP relaxation has: i) Constraints (7) as 120-degree equalities which cause that the angles $\widehat{v_o^p\ v_o^q}$ and $\widehat{v_k^p\ v_k^q}$ are 120° $k \in V, 1 \leq p \neq q \leq 3$, and ii) Constraints (8) as bisector equalities which cause that v_o^l to be on the bisector hyperplane of the vectors v_i^p and v_i^q, and likewise, v_i^l to be on the bisector hyperplane of the vectors v_o^p and v_o^q $\ 1 \leq l \neq p \neq q \leq 3$.

Here again, in an integral solution of the 3SDP relaxation, a vertex cover is composed of the vertices that their corresponding vectors are picked coincide with the vectors v_o^l ; i.e. {i | $v_o^l v_i^l = 1\ \ l = 1,2,3$}. But, it is interesting that an optimal solution of vertex cover problem can be produced by solving two such 3SDP models. To show this fact, it is sufficient to concentrate our attention on coplanarity and perpendicularity properties of the optimal vectors of the proposed 3SDP model, which we will prove them in the next section.

3. Coplanarity or perpendicularity property of optimal vectors of 3SDP model

Theorem 1. Suppose that there are 3 vectors V_1, V_2, $V_3 \in \mathbb{R}^n$ which satisfy 120-degree condition; i.e. $\widehat{V_1 V_2} = \widehat{V_2 V_3} = \widehat{V_3 V_1} = 120°$. Then, these vectors are coplanar.

Proof. We know that two arbitrary vectors are always coplanar. Then, we can assume that the vectors V_2 and V_3 in coordinates $V_2 = \begin{bmatrix} \frac{-\sqrt{3}}{2} & \frac{-1}{2} & 0 & \cdots & 0 \end{bmatrix}^t$ and $V_3 = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 & \cdots & 0 \end{bmatrix}^t$ have been fixed on the x_1x_2 plane, where $V_1 = [a_1 \ \ \cdots \ \ a_n]^t$. Then, it is sufficient to show that $a_2 > 0$, $a_1 = a_3 = \ \ \ldots \ \ = a_n = 0$.

We have $V_1V_2 = V_1V_3 = \frac{-1}{2}\|V_1\|$. Hence, $\frac{-\sqrt{3}}{2}a_1 - \frac{1}{2}a_2 = \frac{\sqrt{3}}{2}a_1 - \frac{1}{2}a_2$, and therefore $a_1 = 0$.

Moreover, based on the length of a vector and the law of cosine on triangles we have:

$$\left\|\overrightarrow{V_1V_3}\right\|^2 = \frac{3}{4} + (a_2 + \frac{1}{2})^2 + a_3^2 + \cdots + a_n^2$$

$$\left\|\overrightarrow{V_1V_3}\right\|^2 = \left(a_2^2 + \cdots + a_n^2\right) + (1) - 2\left(\sqrt{a_2^2 + \cdots + a_n^2}\right)(1)\cos\left(\widehat{V_1V_3}\right)$$

Therefore, $\cos\left(\widehat{V_1V_3}\right) = \frac{-a_2}{2\left(\sqrt{a_2^2+\cdots+a_n^2}\right)} =^? \frac{-1}{2}$ iff $a_2 > 0, a_1 = a_3 = \quad \ldots \quad = a_n = 0$ ■

Corollary 1. Let V_1, V_2, $V_3 \in \mathbb{R}^n$ satisfy a 120-degree condition, where $\|V_i\| = 1$ (i=1,2,3). Then we have $V_l = -\left(V_p + V_q\right) \quad 1 \le l \ne p \ne q \le 3$.

Theorem 2. Suppose that there are 6 vectors V_1, V_2, V_3, U_1, U_2, $U_3 \in \mathbb{R}^n$ which satisfy 120-degree and bisector conditions; i.e. $\widehat{V_1V_2} = \widehat{V_2V_3} = \widehat{V_3V_1} = \widehat{U_1U_2} = \widehat{U_2U_3} = \widehat{U_3U_1} = 120^o$, V_l is on the bisector hyperplane of U_p and U_q , and likewise U_l is on the bisector hyperplane of V_p and V_q, $1 \le l \ne p \ne q \le 3$. Then, these six vectors are coplanar or $U_iV_j=0$ $(1 \le i,j \le 3)$.

Proof. Based on Theorem 1, vectors V_1, V_2, V_3 are coplanar and vectors U_1, U_2, U_3 are coplanar, too. Then, we can assume that vectors V_1, V_2, V_3 have been fixed on the x_1x_2 plane and in coordinates $V_1 = [0 \quad 1 \quad 0 \quad \cdots \quad 0]^t$, $V_2 = \left[\frac{-\sqrt{3}}{2} \quad \frac{-1}{2} \quad 0 \quad \cdots \quad 0\right]^t$ and $V_3 = \left[\frac{\sqrt{3}}{2} \quad \frac{-1}{2} \quad 0 \quad \cdots \quad 0\right]^t$, and vectors U_1, U_2, U_3 are on another unknown plane and in coordinates $U_1 = [a_1 \quad \ldots \quad a_n]^t$, $U_2 = [b_1 \quad \ldots \quad b_n]^t$ and $U_3 = [c_1 \quad \ldots \quad c_n]^t$, where $t\|U_1\| = \|U_2\| = \|U_3\| = 1$, $t > 0$, $|a_2| \in \{0,1\}$. Then, it is sufficient to show that $U_i = \pm V_i$ (i=1,2,3) or $U_iV_j=0$ $(1 \le i,j \le 3)$.

We have $U_1V_2=U_1V_3$. Hence, $\frac{-\sqrt{3}}{2}a_1 - \frac{1}{2}a_2 = \frac{\sqrt{3}}{2}a_1 - \frac{1}{2}a_2$, and therefore $a_1 = 0$. Moreover, $U_2V_1=U_2V_3$ and $U_3V_1=U_3V_2$. Hence, $b_2 = \frac{\sqrt{3}}{2}b_1 - \frac{1}{2}b_2$ and $c_2 = \frac{-\sqrt{3}}{2}c_1 - \frac{1}{2}c_2$. Therefore, $\frac{b_2}{b_1} = \frac{\sqrt{3}}{3}$ and $\frac{c_2}{c_1} = \frac{-\sqrt{3}}{3}$. Then, we can assume that $b_1 = \frac{\sqrt{3}}{2}b$, $b_2 = \frac{1}{2}b$ and $c_1 = \frac{-\sqrt{3}}{2}c$, $c_2 = \frac{1}{2}c$.

We have $U_2V_1=U_3V_1$ and $U_2V_3= tU_1V_3$. Hence, b= c, and $\frac{3}{4}b - \frac{1}{4}b = -(\frac{1}{2})ta_2$. Therefore, $b = -ta_2$. Note that the vectors U_1, U_2, U_3 satisfy a 120-degree condition, where $t\|U_1\| = \|U_2\| = \|U_3\| = 1$, then we have $U_1= -t(U_2+U_3)$. Hence, $\begin{cases} a_2 = -t\left(\frac{1}{2}b + \frac{1}{2}c\right) = -tb = t^2a_2 \\ a_i = -t(b_i + c_i) \qquad i = 3, \ldots, n \end{cases}$. Therefore, $t = 1$, $b = c = -a_2$.

Now, if $a_2 = 0$ then we have $U_iV_j=0$ $(1 \le i,j \le 3)$. Otherwise, if $a_2 \in \{-1,+1\}$ then due to $\|U_1\| = 1$ we have $a_3 = \quad \ldots \quad = a_n = 0$. This concludes that, $b_i = -c_i$ i=3,…,n. Finally, b=±1, $U_2 = \left[\pm\frac{\sqrt{3}}{2} \quad \pm\frac{1}{2} \quad b_3 \quad \cdots \quad b_n\right]^t$ and $\|U_2\| = 1$. Hence, $b_3 = c_3 = \quad \ldots \quad = b_n = c_n = 0$. This concludes that $U_i = \pm V_i$ (i=1,2,3) ■

Corollary 2. Six optimal vectors v_o^1, v_o^2, v_o^3, v_i^1, v_i^2 and v_i^3 of the 3SDP model satisfy coplanarity or perpendicularity property. Hence, we have $v_o^l v_i^l \in \{0, \pm1\}$ (l=1,2,3).

Moreover and based on the optimal solution of the 3SDP model and Corollary 2, the vertex set V is decomposed to 3 sets as follows:

$V_1 = \{i | v_o^l v_i^l = 1 \ \ l = 1,2,3\}$, $V_{-1} = \{i | v_o^l v_i^l = -1 \ \ l = 1,2,3\}$, $V_o = \{i | v_o^l v_i^l = 0 \ \ l = 1,2,3\}$,

Corollary 3. By using constraints (6), called triangle inequalities, the subgraph G' on vertex set V_o has not any odd cycle and then solving the vertex cover problem on G' is not hard. In other words, by solving 3SDP on G' we have a decomposition $V_o = V_{o,1} \cup V_{o,-1}$, where $|V_{o,1}| \leq \frac{1}{2}|V_o|$ and $V_{o,o} = \emptyset$.

Theorem 3. By solving 3SDP relaxation on vertex set V and then on vertex set V_o, an optimal solution of vertex cover is obtained as $V_1 \cup V_{o,1}$.

Proof. Let F(S) be the optimal value of solving the 3SDP model on vertex set S. Then, $Z^*_{vertex\ cover} \geq F(V) = |V_1| + \frac{1}{2}|V_o| \geq |V_1| + |V_{o,1}|$. Therefore, $Z^*_{vertex\ cover} = |V_1| + |V_{o,1}|$ ■

Corollary 4. Due to the polynomial solvability of vertex cover problem on general graphs by the proposed 3SDP model, we have P = NP.

4. Conclusions

The P versus NP problem is one of the principal unsolved problems in complexity theory. In this paper, we present a new SDP relaxation for solving vertex cover problem, which indeed returns an optimal solution of the problem on general graphs in polynomial time. In this manner and by solving one of the NP-complete problems in polynomial time, we can conclude that P=NP.

However, many results in complexity theory and computational optimization assume solidly based on the hypotheses P≠NP. But, now that we know P=NP, we should make fundamental modifications in many of the results discussed in literatures.

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