A relationship between exponential growing and a basic asymptotic function

by

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1.- Abstract

In this paper we study a simple exponential growing problem which leads to a basic asymptotic function. It shows a hidden property of asymptotic functions.

2.- Introduction

Nowadays basic asymptotic functions are largely studied at high-school levels, in order to present the notion of limit of a function.

Some centuries ago, a relationship between exponential functions and a rectangular hyperbola, which has an asymptote, has been found indirectly through the inverse function of an exponential function. It lead to the definition of the natural logarithm function :

$$\ln(x)=\int_1^x \frac{1}{t}dt$$

This relationship is the result of researches that began during the XVIIth century thanks to the mathematician Grégoire de Saint-Vincent. He was working on the quadrature of the hyperbola $y=\frac{1}{x}$. He was applying Fermat's method and he noticed that « when the bases form a geometric progression, the rectangles have equal areas ; thus the area is proportional to the logarithm of the horizontal distance . », as E. Maor wrote in [1]. There is also interesting information in Hairer-Wanner [2].

In this article, we will neither work with areas nor with logarithms. Instead, we will find a direct relationship between exponential growing and a basic asymptotic function.

3.- Simple problems leading to exponential functions

In simple problems of that kind, there is growing at a certain rate per unit of time. For exemple, we begin with an amout A_0, and, say, this amount grows at the rate of 2 % per month. So the amount we get in time is :

$$A(t)=A_0\cdot\left(\frac{102}{100}\right)^t$$

where t is the number of months.

1 Independent researcher

In that kind of problem, after every fixed period of time, the amount grows at a fixed rate.

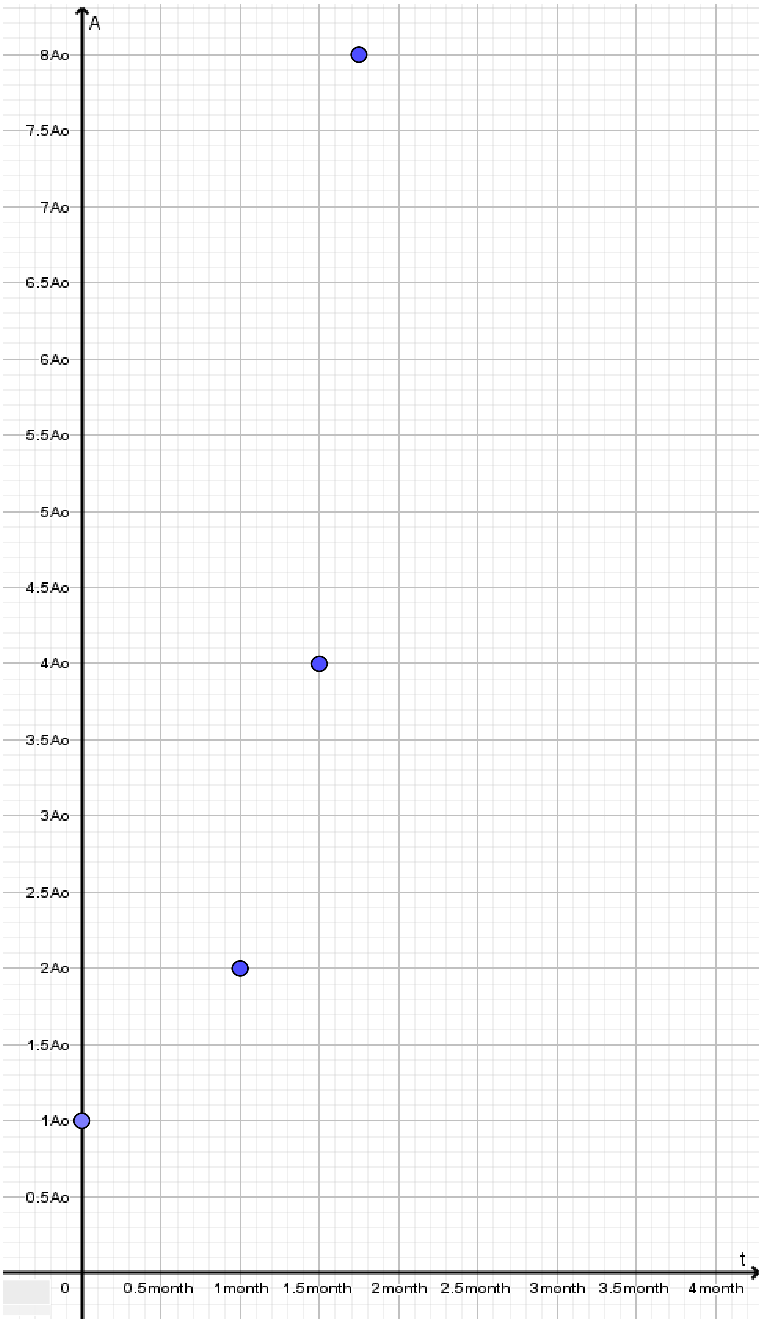
Let's call p that fixed period of time.

4.- What happens if the period of time p drecreases as the rate remains unchanged ?

One simple problem of this kind could be the following one :

We begin with an amount A_0 and, after a month, it doubles, and after 15 days it doubles again, and after 7,5 days it doubles again, and so on…

So we get the following graph :



5.- Finding a function which describes that kind of growing

In order to solve this problem, we will write a parametric equation, where n is the number of decreasing periods of time and t is the number of months.

For n=0 we have :

$$\begin{cases} t = 0 \\ A = A_0 \end{cases}$$

And $for\ n \geq 1, n \in \mathbb{N}$:

$$\begin{cases} t = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \ldots + \left(\frac{1}{2}\right)^{n-1} \\ A = A_0 \cdot 2^n \end{cases}$$

We can check that :

If n=0 we have t=0 and $A=A_0$

If n=1 we have t=1 and $A=A_0 \cdot 2$

If n=2 we have t=1,5 and $A=A_0 \cdot 2^2$

If n=3 we have t=1,75 and $A=A_0 \cdot 2^3$

And so on…

We can rewrite the parametric equation :

$For\ n \geq 1, n \in \mathbb{N}$:

$$\begin{cases} t = \left(\frac{1}{2}\right)^0 + \left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \ldots + \left(\frac{1}{2}\right)^{n-1} \\ A = A_0 \cdot 2^n \end{cases}$$

Or :

$$\begin{cases} t = \sum_{k=0}^{n-1} \left(\frac{1}{2}\right)^k \\ A = A_0 \cdot 2^n \end{cases}$$

And we remark that t is given by a geometric series whose common ratio is $\frac{1}{2}$. So we can use the formula for the sum of the first n terms as we find it in [3]:

$$\begin{cases} t=\dfrac{1-(\frac{1}{2})^n}{1-(\frac{1}{2})} \\ A=A_0\cdot 2^n \end{cases}$$

So we get :

$$\begin{cases} t=2-\dfrac{2}{2^n} \\ A=A_0\cdot 2^n \end{cases}$$

And now, if we look for a relation between the variables t and A, we must eliminate the variable n :

$$\begin{cases} 2-t=\dfrac{2}{2^n} \\ A=A_0\cdot 2^n \end{cases}$$

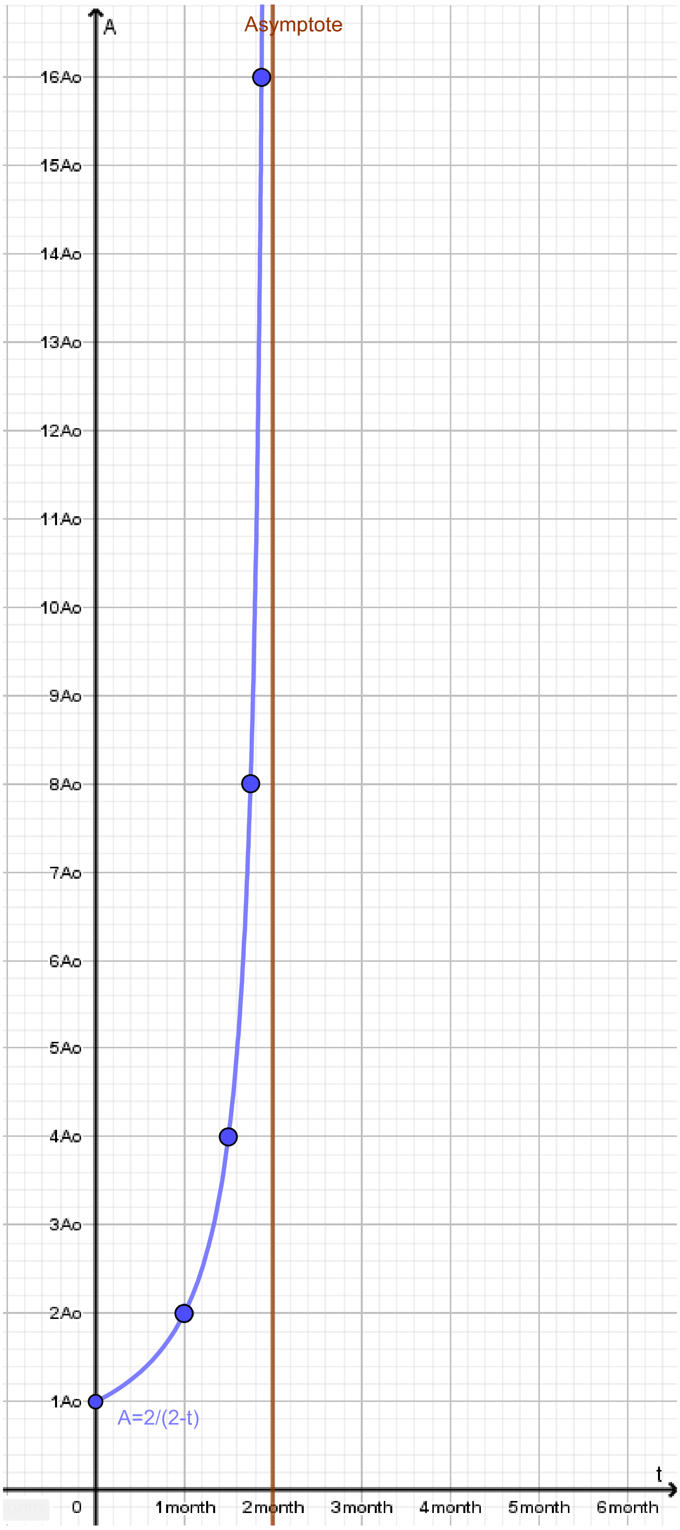
$$\begin{cases} \dfrac{2-t}{2}=\dfrac{1}{2^n} \\ A=A_0\cdot 2^n \end{cases}$$

$$\begin{cases} \dfrac{2}{2-t}=2^n \\ A=A_0\cdot 2^n \end{cases}$$

So finally we get :

$$A=\frac{2\cdot A_0}{2-t}$$

which means that A is given by a basic asymptotic function depending on the variable t…
Here is the graph of the function :


A
Asymptote
16Ao
15Ao
14Ao
13Ao
12Ao
11Ao
10Ao
9Ao
8Ao
7Ao
6Ao
5Ao
4Ao
3Ao
2Ao
1Ao
A=2/(2-t)
t
0
1month
2month
3month
4month
5month
6month

6.- Conclusions

So we got directly a relationship between exponential growing and an hyperbola. This fact will contribue to increase our knowledge about exponential functions and asymptotic functions.

References

[1] E. Maor, e THE HISTORY OF A NUMBER, Princeton University Press, 2015, p. 66

[2] G. Wanner and E. Hairer, L'analyse au fil de l'histoire, Springer, 2001

[3] https://en.wikipedia.org/wiki/Geometric_series