Proof of Riemann hypothesis

By Toshihiko Ishiwata

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Abstract. This paper is a trial to prove Riemann hypothesis according to the following process. 1. We create the infinite number of infinite series from one equation that gives $\zeta(s)$ analytic continuation to $Re(s) > 0$ and 2 formulas $(1/2 + a + bi,\ 1/2 - a - bi)$ which show zero point of $\zeta(s)$. 2. We find that a cannot have any value but zero from the above infinite number of infinite series. Therefore zero point of $\zeta(s)$ must be $1/2 \pm bi$.

1. Introduction

The following (1) gives Riemann zeta function $\zeta(s)$ analytic continuation to $Re(s) > 0$. "$+\cdots\cdots$" means infinite series in all equations in this paper.

$$1 - 2^{-s} + 3^{-s} - 4^{-s} + 5^{-s} - 6^{-s} + \cdots\cdots = (1 - 2^{1-s})\zeta(s) \tag{1}$$

The following (2) shows the zero point of the left side of (1) and also non-trivial zero point of $\zeta(s)$.

$$S_0 = 1/2 + a + bi \tag{2}$$

The range of a is $0 \le a < 1/2$ by the critical strip of $\zeta(s)$. The range of b is $b > 0$ due to the following reasons. And i is $\sqrt{-1}$.

1.1 There is no zero point on the real axis of the critical strip.

1.2 [Conjugate complex number of S_0] $= 1/2 + a - bi$ is also zero point of $\zeta(s)$. Therefore $b > 0$ is necessary and sufficient range for investigation.

The following (3) also shows zero point of $\zeta(s)$ by the functional equation of $\zeta(s)$.

$$S_1 = 1 - S_0 = 1/2 - a - bi \tag{3}$$

We have the following (4) and (5) by substituting S_0 for s in the left side of (1) and putting both the real part and the imaginary part of the left side of (1) at zero respectively.

$$1 = \frac{\cos(b\log 2)}{2^{1/2+a}} - \frac{\cos(b\log 3)}{3^{1/2+a}} + \frac{\cos(b\log 4)}{4^{1/2+a}} - \frac{\cos(b\log 5)}{5^{1/2+a}} + \cdots\cdots \tag{4}$$

$$0 = \frac{\sin(b\log 2)}{2^{1/2+a}} - \frac{\sin(b\log 3)}{3^{1/2+a}} + \frac{\sin(b\log 4)}{4^{1/2+a}} - \frac{\sin(b\log 5)}{5^{1/2+a}} + \cdots\cdots \tag{5}$$

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We also have the following (6) and (7) by substituting S_1 for s in the left side of (1) and putting both the real part and the imaginary part of the left side of (1) at zero respectively.

$$1 = \frac{\cos(b\log 2)}{2^{1/2-a}} - \frac{\cos(b\log 3)}{3^{1/2-a}} + \frac{\cos(b\log 4)}{4^{1/2-a}} - \frac{\cos(b\log 5)}{5^{1/2-a}} + \cdots\cdots \tag{6}$$

$$0 = \frac{\sin(b\log 2)}{2^{1/2-a}} - \frac{\sin(b\log 3)}{3^{1/2-a}} + \frac{\sin(b\log 4)}{4^{1/2-a}} - \frac{\sin(b\log 5)}{5^{1/2-a}} + \cdots\cdots \tag{7}$$

2. Infinite number of infinite series

We define $f(n)$ as follows.

$$f(n) = \frac{1}{n^{1/2-a}} - \frac{1}{n^{1/2+a}} \geq 0 \qquad (n = 2, 3, 4, 5, \cdots\cdots) \tag{8}$$

We have the following (9) from (4) and (6) with the method shown in item 1 of [Appendix 1: Equation construction].

$$0 = f(2)\cos(b\log 2) - f(3)\cos(b\log 3) + f(4)\cos(b\log 4) - f(5)\cos(b\log 5) + \cdots\cdots \tag{9}$$

We also have the following (10) from (5) and (7) with the method shown in item 2 of [Appendix 1: Equation construction].

$$0 = f(2)\sin(b\log 2) - f(3)\sin(b\log 3) + f(4)\sin(b\log 4) - f(5)\sin(b\log 5) + \cdots\cdots \tag{10}$$

We can have the following (11) (which is the function of real number x) from the above (9) and (10) with the method shown in item 3 of [Appendix 1: Equation construction]. And the value of (11) is always zero at any value of x.

$$\begin{aligned}
0 \equiv\, & \cos x\{\text{right side of } (9)\} + \sin x\{\text{right side of } (10)\} \\
=\, & \cos x\{f(2)\cos(b\log 2) - f(3)\cos(b\log 3) + f(4)\cos(b\log 4) - f(5)\cos(b\log 5) + \cdots\cdots\} \\
& + \sin x\{f(2)\sin(b\log 2) - f(3)\sin(b\log 3) + f(4)\sin(b\log 4) - f(5)\sin(b\log 5) + \cdots\cdots\} \\
=\, & f(2)\cos(b\log 2 - x) - f(3)\cos(b\log 3 - x) + f(4)\cos(b\log 4 - x) \\
& - f(5)\cos(b\log 5 - x) + f(6)\cos(b\log 6 - x) - \cdots\cdots
\end{aligned} \tag{11}$$

We have (12-1) by substituting $b\log 1$ for x in (11).

$$\begin{aligned}
0 =\, & f(2)\cos(b\log 2 - b\log 1) - f(3)\cos(b\log 3 - b\log 1) + f(4)\cos(b\log 4 - b\log 1) \\
& - f(5)\cos(b\log 5 - b\log 1) + f(6)\cos(b\log 6 - b\log 1) - \cdots\cdots
\end{aligned} \tag{12-1}$$

We have (12-2) by substituting $b\log 2$ for x in (11).

$$\begin{aligned}
0 =\, & f(2)\cos(b\log 2 - b\log 2) - f(3)\cos(b\log 3 - b\log 2) + f(4)\cos(b\log 4 - b\log 2) \\
& - f(5)\cos(b\log 5 - b\log 2) + f(6)\cos(b\log 6 - b\log 2) - \cdots\cdots
\end{aligned} \tag{12-2}$$

We have (12-3) by substituting $b\log 3$ for x in (11).

$$\begin{aligned}0 =& f(2)\cos(b\log 2 - b\log 3) - f(3)\cos(b\log 3 - b\log 3) + f(4)\cos(b\log 4 - b\log 3)\\ &- f(5)\cos(b\log 5 - b\log 3) + f(6)\cos(b\log 6 - b\log 3) - \cdots\cdots \end{aligned}\tag{12-3}$$

In the same way as above we can have (12-N) by substituting $b\log N$ for x in (11). $(N = 4, 5, 6, 7, 8, \cdots\cdots)$

$$\begin{aligned}0 =& f(2)\cos(b\log 2 - b\log N) - f(3)\cos(b\log 3 - b\log N) + f(4)\cos(b\log 4 - b\log N)\\ &- f(5)\cos(b\log 5 - b\log N) + f(6)\cos(b\log 6 - b\log N) - \cdots\cdots \end{aligned}\tag{12-N}$$

3. Proof

We define $g(k)$ as follows. $(k = 2, 3, 4, 5, \cdots\cdots)$

$$\begin{aligned}g(k) &= \cos(b\log k - b\log 1) + \cos(b\log k - b\log 2) + \cos(b\log k - b\log 3) + \cdots\cdots\\ &= \cos(b\log 1 - b\log k) + \cos(b\log 2 - b\log k) + \cos(b\log 3 - b\log k) + \cdots\cdots\\ &= \cos(b\log 1/k) + \cos(b\log 2/k) + \cos(b\log 3/k) + \cos(b\log 4/k) + \cdots\cdots \end{aligned}\tag{13}$$

In [Appendix 2: Investigation of $g(k)$] we have the following (26).

$$g(k) = \frac{\lim\limits_{N\to\infty} N\sin(b\log N/k + \tan^{-1} 1/b)}{\sqrt{1+b^2}}\tag{26}$$

We can have the following (14) from the above (26) and the infinite equations of (12-1), (12-2), (12-3), $\cdots\cdots$, (12-N), $\cdots\cdots$ with the method shown in item 4 of [Appendix 1: Equation construction].

$$\begin{aligned}0 =& f(2)\{\cos(b\log 2 - b\log 1) + \cos(b\log 2 - b\log 2) + \cos(b\log 2 - b\log 3) + \cdots\cdots\}\\ &- f(3)\{\cos(b\log 3 - b\log 1) + \cos(b\log 3 - b\log 2) + \cos(b\log 3 - b\log 3) + \cdots\cdots\}\\ &+ f(4)\{\cos(b\log 4 - b\log 1) + \cos(b\log 4 - b\log 2) + \cos(b\log 4 - b\log 3) + \cdots\cdots\}\\ &- f(5)\{\cos(b\log 5 - b\log 1) + \cos(b\log 5 - b\log 2) + \cos(b\log 5 - b\log 3) + \cdots\cdots\}\\ &+ f(6)\{\cos(b\log 6 - b\log 1) + \cos(b\log 6 - b\log 2) + \cos(b\log 6 - b\log 3) + \cdots\cdots\}\\ &- \cdots\cdots\\ =& f(2)g(2) - f(3)g(3) + f(4)g(4) - f(5)g(5) + f(6)g(6) - f(7)g(7) + \cdots\cdots\\ =& f(2)\frac{\lim\limits_{N\to\infty} N\sin(b\log N/2 + \tan^{-1} 1/b)}{\sqrt{1+b^2}} - f(3)\frac{\lim\limits_{N\to\infty} N\sin(b\log N/3 + \tan^{-1} 1/b)}{\sqrt{1+b^2}}\\ &+ f(4)\frac{\lim\limits_{N\to\infty} N\sin(b\log N/4 + \tan^{-1} 1/b)}{\sqrt{1+b^2}} - f(5)\frac{\lim\limits_{N\to\infty} N\sin(b\log N/5 + \tan^{-1} 1/b)}{\sqrt{1+b^2}}\\ &+ \cdots\cdots\\ =& (1/\sqrt{1+b^2})\lim_{N\to\infty} N\{f(2)\sin(b\log N/2 + \tan^{-1} 1/b) - f(3)\sin(b\log N/3 + \tan^{-1} 1/b)\\ &+ f(4)\sin(b\log N/4 + \tan^{-1} 1/b) - f(5)\sin(b\log N/5 + \tan^{-1} 1/b) + \cdots\cdots\} \end{aligned}\tag{14}$$

As shown in [Appendix 3: Sum of infinite series of sine waves] sum of infinite series of sine waves in the above (14) converges as follows.

$$\begin{aligned}&f(2)\sin(b\log N/2+\tan^{-1}1/b)-f(3)\sin(b\log N/3+\tan^{-1}1/b)\\&+f(4)\sin(b\log N/4+\tan^{-1}1/b)-f(5)\sin(b\log N/5+\tan^{-1}1/b)+\cdots\cdots\\&=f(2)\sin(b\log N-b\log 2+\tan^{-1}1/b)-f(3)\sin(b\log N-b\log 3+\tan^{-1}1/b)\\&\quad+f(4)\sin(b\log N-b\log 4+\tan^{-1}1/b)-f(5)\sin(b\log N-b\log 5+\tan^{-1}1/b)+\cdots\cdots\\&=A(a,b)\sin\{b\log N-B(a,b)+\tan^{-1}1/b\} \qquad (15)\end{aligned}$$

In the above (15) $A(a,b)$ and $B(a,b)$ are constants which depend on a and b. If $A(a,b)=0$ is true, $a=0$ holds true and if $A(a,b)\neq 0$ is true, $0<a<1/2$ holds true as shown in [Appendix 3: Sum of infinite series of sine waves]. From (14) and (15) we have the following (16).

$$0=\lim_{N\to\infty}N[A(a,b)\sin\{b\log N-B(a,b)+\tan^{-1}(1/b)\}] \qquad (16)$$

If $A(a,b)\neq 0$ is true, the right side of (16) diverges to $\pm\infty$. Therefore $A(a,b)=0$ must be true for (16) to hold true. a cannot have any value but zero due to $A(a,b)=0$. From (2) and (3) non-trivial zero point of Riemann zeta function $\zeta(s)$ must be $1/2\pm bi$ and other zero point does not exist. Riemann hypothesis which says "All non-trivial zero points of Riemann zeta function $\zeta(s)$ exist on the line of $Re(s)=1/2$." is true.

Appendix 1. Equation construction

We can construct (9), (10), (11) and (14) by applying the following Theorem 1[1].

Theorem 1

On condition that the following (Series 1) and (Series 2) converge, the following (Series 3) and (Series 4) are true.

$$\begin{aligned}&(\text{Series 1})=a_1+a_2+a_3+a_4+a_5+\cdots\cdots=A\\&(\text{Series 2})=b_1+b_2+b_3+b_4+b_5+\cdots\cdots=B\\&(\text{Series 3})=(a_1+b_1)+(a_2+b_2)+(a_3+b_3)+(a_4+b_4)+\cdots\cdots=A+B\\&(\text{Series 4})=(a_1-b_1)+(a_2-b_2)+(a_3-b_3)+(a_4-b_4)+\cdots\cdots=A-B\end{aligned}$$

1. Construction of (9)

We can have the following (9) as (Series 4) by regarding (6) and (4) as (Series 1) and (Series 2) respectively.

$$(\text{Series 1})=\frac{\cos(b\log 2)}{2^{1/2-a}}-\frac{\cos(b\log 3)}{3^{1/2-a}}+\frac{\cos(b\log 4)}{4^{1/2-a}}-\frac{\cos(b\log 5)}{5^{1/2-a}}+\cdots\cdots=1 \qquad (6)$$

$$(\text{Series 2})=\frac{\cos(b\log 2)}{2^{1/2+a}}-\frac{\cos(b\log 3)}{3^{1/2+a}}+\frac{\cos(b\log 4)}{4^{1/2+a}}-\frac{\cos(b\log 5)}{5^{1/2+a}}+\cdots\cdots=1 \qquad (4)$$

$$(\text{Series 4})=f(2)\cos(b\log 2)-f(3)\cos(b\log 3)+f(4)\cos(b\log 4)-f(5)\cos(b\log 5)$$

$$+\cdots\cdots = 1 - 1 = 0 \tag{9}$$

Here

$$f(n) = \frac{1}{n^{1/2-a}} - \frac{1}{n^{1/2+a}} \geq 0 \qquad (n = 2, 3, 4, 5, \cdots\cdots) \tag{8}$$

2. Construction of (10)

We can have the following (10) as (Series 4) by regarding (7) and (5) as (Series 1) and (Series 2) respectively.

$$\text{(Series 1)} = \frac{\sin(b\log 2)}{2^{1/2-a}} - \frac{\sin(b\log 3)}{3^{1/2-a}} + \frac{\sin(b\log 4)}{4^{1/2-a}} - \frac{\sin(b\log 5)}{5^{1/2-a}} + \cdots\cdots = 0 \tag{7}$$

$$\text{(Series 2)} = \frac{\sin(b\log 2)}{2^{1/2+a}} - \frac{\sin(b\log 3)}{3^{1/2+a}} + \frac{\sin(b\log 4)}{4^{1/2+a}} - \frac{\sin(b\log 5)}{5^{1/2+a}} + \cdots\cdots = 0 \tag{5}$$

$$\begin{aligned}\text{(Series 4)} =& f(2)\sin(b\log 2) - f(3)\sin(b\log 3) + f(4)\sin(b\log 4) - f(5)\sin(b\log 5)\\ &+\cdots\cdots = 0 - 0\end{aligned} \tag{10}$$

3. Construction of (11)

We can have the following (11) as (Series 3) by regarding the following equations as (Series 1) and (Series 2).

$$\begin{aligned}\text{(Series 1)} &= \cos x\{\text{right side of (9)}\}\\ &= \cos x\{f(2)\cos(b\log 2) - f(3)\cos(b\log 3) + f(4)\cos(b\log 4) - f(5)\cos(b\log 5)\\ &\quad +\cdots\cdots\} \equiv 0\\ \text{(Series 2)} &= \sin x\{\text{right side of (10)}\}\\ &= \sin x\{f(2)\sin(b\log 2) - f(3)\sin(b\log 3) + f(4)\sin(b\log 4) - f(5)\sin(b\log 5)\\ &\quad +\cdots\cdots\} \equiv 0\\ \text{(Series 3)} =& f(2)\cos(b\log 2 - x) - f(3)\cos(b\log 3 - x) + f(4)\cos(b\log 4 - x)\\ &- f(5)\cos(b\log 5 - x) + \cdots\cdots \equiv 0 + 0 \qquad (11)\end{aligned}$$

4. Construction of (14)

4.1 We can have the following (12-1*2) as (Series 3) by regarding (12-1) and (12-2) as (Series 1) and (Series 2) respectively.

$$\begin{aligned}\text{(Series 1)} =& f(2)\cos(b\log 2 - b\log 1) - f(3)\cos(b\log 3 - b\log 1)\\ &+ f(4)\cos(b\log 4 - b\log 1) - f(5)\cos(b\log 5 - b\log 1)\\ &+ f(6)\cos(b\log 6 - b\log 1) - \cdots\cdots = 0 \qquad (12\text{-}1)\\ \text{(Series 2)} =& f(2)\cos(b\log 2 - b\log 2) - f(3)\cos(b\log 3 - b\log 2)\\ &+ f(4)\cos(b\log 4 - b\log 2) - f(5)\cos(b\log 5 - b\log 2)\\ &+ f(6)\cos(b\log 6 - b\log 2) - \cdots\cdots = 0 \qquad (12\text{-}2)\\ \text{(Series 3)} =& f(2)\{\cos(b\log 2 - b\log 1) + \cos(b\log 2 - b\log 2)\}\\ &- f(3)\{\cos(b\log 3 - b\log 1) + \cos(b\log 3 - b\log 2)\}\\ &+ f(4)\{\cos(b\log 4 - b\log 1) + \cos(b\log 4 - b\log 2)\}\\ &- f(5)\{\cos(b\log 5 - b\log 1) + \cos(b\log 5 - b\log 2)\}\end{aligned}$$

$$+\cdots\cdots = 0 + 0 \tag{12-1*2}$$

4.2 We can have the following (12-1*3) as (Series 3) by regarding (12-1*2) and (12-3) as (Series 1) and (Series 2) respectively.

$$\begin{aligned}(\text{Series } 2) =& f(2)\cos(b\log 2 - b\log 3) - f(3)\cos(b\log 3 - b\log 3)\\ &+ f(4)\cos(b\log 4 - b\log 3) - f(5)\cos(b\log 5 - b\log 3)\\ &+ f(6)\cos(b\log 6 - b\log 3) - \cdots\cdots = 0 \qquad (12\text{-}3)\\ (\text{Series } 3) =& f(2)\{\cos(b\log 2 - b\log 1) + \cos(b\log 2 - b\log 2) + \cos(b\log 2 - b\log 3)\}\\ &- f(3)\{\cos(b\log 3 - b\log 1) + \cos(b\log 3 - b\log 2) + \cos(b\log 3 - b\log 3)\}\\ &+ f(4)\{\cos(b\log 4 - b\log 1) + \cos(b\log 4 - b\log 2) + \cos(b\log 4 - b\log 3)\}\\ &- f(5)\{\cos(b\log 5 - b\log 1) + \cos(b\log 5 - b\log 2) + \cos(b\log 5 - b\log 3)\}\\ &+\cdots\cdots = 0 + 0 \qquad (12\text{-}1^*3)\end{aligned}$$

4.3 We can have the following (12-1*4) as (Series 3) by regarding (12-1*3) and (12-4) as (Series 1) and (Series 2) respectively.

$$\begin{aligned}(\text{Series } 2) =& f(2)\cos(b\log 2 - b\log 4) - f(3)\cos(b\log 3 - b\log 4)\\ &+ f(4)\cos(b\log 4 - b\log 4) - f(5)\cos(b\log 5 - b\log 4)\\ &+ f(6)\cos(b\log 6 - b\log 4) - \cdots\cdots = 0 \qquad (12\text{-}4)\end{aligned}$$

$$\begin{aligned}&(\text{Series } 3)\\ &= f(2)\{\cos(b\log 2 - b\log 1) + \cos(b\log 2 - b\log 2) + \cdots + \cos(b\log 2 - b\log 4)\}\\ &\quad - f(3)\{\cos(b\log 3 - b\log 1) + \cos(b\log 3 - b\log 2) + \cdots + \cos(b\log 3 - b\log 4)\}\\ &\quad + f(4)\{\cos(b\log 4 - b\log 1) + \cos(b\log 4 - b\log 2) + \cdots + \cos(b\log 4 - b\log 4)\}\\ &\quad - f(5)\{\cos(b\log 5 - b\log 1) + \cos(b\log 5 - b\log 2) + \cdots + \cos(b\log 5 - b\log 4)\}\\ &\quad +\cdots\cdots = 0 + 0 \qquad (12\text{-}1^*4)\end{aligned}$$

4.4 In the same way as above we can have the following (12-1*N) as (Series 3) by regarding (12-1*N-1) and (12-N) as (Series 1) and (Series 2) respectively.

$$\begin{aligned}&f(2)\{\cos(b\log 2 - b\log 1) + \cos(b\log 2 - b\log 2) + \cdots + \cos(b\log 2 - b\log N)\}\\ &- f(3)\{\cos(b\log 3 - b\log 1) + \cos(b\log 3 - b\log 2) + \cdots + \cos(b\log 3 - b\log N)\}\\ &+ f(4)\{\cos(b\log 4 - b\log 1) + \cos(b\log 4 - b\log 2) + \cdots + \cos(b\log 4 - b\log N)\}\\ &- f(5)\{\cos(b\log 5 - b\log 1) + \cos(b\log 5 - b\log 2) + \cdots + \cos(b\log 5 - b\log N)\}\\ &+\cdots\cdots = 0 + 0 \qquad (12\text{-}1^*\text{N})\end{aligned}$$

4.5 We define $g(k, N)$ as follows.

$$\begin{aligned}g(k, N) =& \cos(b\log k - b\log 1) + \cos(b\log k - b\log 2) + \cdots + \cos(b\log k - b\log N)\\ &(k = 2, 3, 4, 5, \cdots\cdots)\end{aligned}$$

From (13) we can say $\lim_{N\to\infty} g(k, N) = g(k)$. Performing the operation in item 4.4

once increases all N in (12-1*N) by 1. If we repeat this operation infinitely i.e. we do $N \to \infty$, from (13) and (26) in [Appendix 2: Investigation of $g(k)$] we can have the following (14) .

$$
\begin{aligned}
0 &= \lim_{N\to\infty}\{\text{left side of (12-1*N)}\} \\
&= \lim_{N\to\infty}\{f(2)g(2,N) - f(3)g(3,N) + f(4)g(4,N) - f(5)g(5,N) + \cdots\cdots\} \quad \text{(21-1)} \\
&= f(2)g(2) - f(3)g(3) + f(4)g(4) - f(5)g(5) + f(6)g(6) - f(7)g(7) + \cdots\cdots \\
&= f(2)\frac{\lim_{N\to\infty} N\sin(b\log N/2 + \tan^{-1} 1/b)}{\sqrt{1+b^2}} - f(3)\frac{\lim_{N\to\infty} N\sin(b\log N/3 + \tan^{-1} 1/b)}{\sqrt{1+b^2}} \\
&\quad + f(4)\frac{\lim_{N\to\infty} N\sin(b\log N/4 + \tan^{-1} 1/b)}{\sqrt{1+b^2}} - f(5)\frac{\lim_{N\to\infty} N\sin(b\log N/5 + \tan^{-1} 1/b)}{\sqrt{1+b^2}} \\
&\quad + \cdots\cdots \quad \text{(21-2)} \\
&= \lim_{N\to\infty}\{f(2)\frac{N\sin(b\log N/2 + \tan^{-1} 1/b)}{\sqrt{1+b^2}} - f(3)\frac{N\sin(b\log N/3 + \tan^{-1} 1/b)}{\sqrt{1+b^2}} \\
&\quad + f(4)\frac{N\sin(b\log N/4 + \tan^{-1} 1/b)}{\sqrt{1+b^2}} - f(5)\frac{N\sin(b\log N/5 + \tan^{-1} 1/b)}{\sqrt{1+b^2}} \\
&\quad + \cdots\cdots\} \quad \text{(21-3)} \\
&= (1/\sqrt{1+b^2})\lim_{N\to\infty} N\{f(2)\sin(b\log N/2 + \tan^{-1} 1/b) - f(3)\sin(b\log N/3 + \tan^{-1} 1/b) \\
&\quad + f(4)\sin(b\log N/4 + \tan^{-1} 1/b) - f(5)\sin(b\log N/5 + \tan^{-1} 1/b) + \cdots\cdots\} \quad \text{(14)}
\end{aligned}
$$

In (21-1) all N become $N \to \infty$ simultaneously and synchronously each other because all N are controled by only one $\lim_{N\to\infty}$. In (21-2) all $\lim_{N\to\infty}$ must work simultaneously and synchronously each other because (21-2) is equal to (21-1) and all N in (21-2) also must become $N \to \infty$ simultaneously and synchronously each other. This situation in (21-2) is same as the situation where the whole (21-2) is conroled by only one $\lim_{N\to\infty}$ like (21-3). Therefore we can combine all $\lim_{N\to\infty}$ in (21-2) into one $\lim_{N\to\infty}$ and make (21-3).

Appendix 2. Investigation of $g(k)$

In item 4.5 of [Appendix 1: Equation construction] we defined $g(k, N)$ as the partial sum from the first term of $g(k)$ to the N-th term of $g(k)$. $(k = 2, 3, 4, 5, \cdots\cdots)$
From (13) $g(k, N)$ is as follows.

$$
\begin{aligned}
g(k,N) &= \cos(b\log 1/k) + \cos(b\log 2/k) + \cos(b\log 3/k) + \cdots + \cos(b\log N/k) \\
&= N\frac{1}{N}\{\cos(b\log\frac{1}{N}\frac{N}{k}) + \cos(b\log\frac{2}{N}\frac{N}{k}) + \cos(b\log\frac{3}{N}\frac{N}{k}) + \cdots + \cos(b\log\frac{N}{N}\frac{N}{k})\} \\
&= N\frac{1}{N}\{\cos(b\log\frac{1}{N} + b\log\frac{N}{k}) + \cos(b\log\frac{2}{N} + b\log\frac{N}{k}) + \cos(b\log\frac{3}{N} + b\log\frac{N}{k}) \\
&\quad + \cdots\cdots + \cos(b\log\frac{N}{N} + b\log\frac{N}{k})\}
\end{aligned}
$$

$$= N\frac{1}{N}\{\cos(b\log\frac{N}{k})\}\{\cos(b\log\frac{1}{N}) + \cos(b\log\frac{2}{N}) + \cdots + \cos(b\log\frac{N}{N})\}$$
$$- N\frac{1}{N}\{\sin(b\log\frac{N}{k})\}\{\sin(b\log\frac{1}{N}) + \sin(b\log\frac{2}{N}) + \cdots + \sin(b\log\frac{N}{N})\}$$

Here we do $N \to \infty$ as follows. $\lim_{N\to\infty} g(k, N)$ means $g(k)$.

$$\begin{aligned}
&\lim_{N\to\infty} g(k,N) = g(k) \\
&= \lim_{N\to\infty}\{\cos(b\log 1/k) + \cos(b\log 2/k) + \cos(b\log 3/k) + \cdots + \cos(b\log N/k)\} \\
&= \lim_{N\to\infty}[N\frac{1}{N}\{\cos(b\log\frac{N}{k})\}\{\cos(b\log\frac{1}{N}) + \cos(b\log\frac{2}{N}) + \cdots + \cos(b\log\frac{N}{N})\} \\
&\quad - N\frac{1}{N}\{\sin(b\log\frac{N}{k})\}\{\sin(b\log\frac{1}{N}) + \sin(b\log\frac{2}{N}) + \cdots + \sin(b\log\frac{N}{N})\}] \qquad (22\text{-}1) \\
&= \lim_{N\to\infty}\{N\cos(b\log\frac{N}{k})\}\lim_{N\to\infty}\frac{1}{N}\{\cos(b\log\frac{1}{N}) + \cos(b\log\frac{2}{N}) + \cdots + \cos(b\log\frac{N}{N})\} \\
&\quad - \lim_{N\to\infty}\{N\sin(b\log\frac{N}{k})\}\lim_{N\to\infty}\frac{1}{N}\{\sin(b\log\frac{1}{N}) + \sin(b\log\frac{2}{N}) + \cdots + \sin(b\log\frac{N}{N})\} \\
&\qquad (22\text{-}2)
\end{aligned}$$

In (22-1) all N become $N \to \infty$ simultaneously and synchronously each other because all N are controled by only one $\lim_{N\to\infty}$. If in (22-2) 4 $\lim_{N\to\infty}$ work simultaneously and synchronously each other, (22-2) becomes equal to (22-1) because all N in (22-2) also become $N \to \infty$ simultaneously and synchronously each other.
We define A and B as follows.

$$\begin{aligned}
A &= \lim_{N\to\infty}\frac{1}{N}\{\cos(b\log\frac{1}{N}) + \cos(b\log\frac{2}{N}) + \cos(b\log\frac{3}{N}) + \cdots + \cos(b\log\frac{N}{N})\} \\
&= \int_0^1 \cos(b\log x)dx \\
B &= \lim_{N\to\infty}\frac{1}{N}\{\sin(b\log\frac{1}{N}) + \sin(b\log\frac{2}{N}) + \sin(b\log\frac{3}{N}) + \cdots + \sin(b\log\frac{N}{N})\} \\
&= \int_0^1 \sin(b\log x)dx
\end{aligned}$$

We calculate A and B by Integration by parts.

$$A = [x\cos(b\log x)]_0^1 + bB = 1 + bB$$
$$B = [x\sin(b\log x)]_0^1 - bA = -bA$$

Then we can have the values of A and B from the above equations as follows.

$$A = \frac{1}{1+b^2} \qquad B = \frac{-b}{1+b^2} \tag{23}$$

From (23) we can have the following (22-3) and (22-4).

$$\begin{aligned}
g(k) &= (22\text{-}1) = (22\text{-}2) \\
&= \lim_{N\to\infty}\{N\cos(b\log\frac{N}{k})\}A - \lim_{N\to\infty}\{N\sin(b\log\frac{N}{k})B \qquad (22\text{-}3)
\end{aligned}$$

$$= \lim_{N\to\infty}\{N\cos(b\log\frac{N}{k})\}\frac{1}{1+b^2} + \lim_{N\to\infty}\{N\sin(b\log\frac{N}{k})\}\frac{b}{1+b^2} \tag{22-4}$$

We define $\lim_{N\to\infty} m(N)$ as follows.

$$\lim_{N\to\infty} m(N) = \lim_{N\to\infty}[\{N\cos(b\log\frac{N}{k})\}\frac{1}{1+b^2} + \{N\sin(b\log\frac{N}{k})\}\frac{b}{1+b^2}] \tag{24-1}$$

$$= \lim_{N\to\infty}\{N\cos(b\log\frac{N}{k})\}\frac{1}{1+b^2} + \lim_{N\to\infty}\{N\sin(b\log\frac{N}{k})\}\frac{b}{1+b^2} \tag{24-2}$$

In (24-1) all N become $N\to\infty$ simultaneously and synchronously each other because all N are controled by only one $\lim_{N\to\infty}$. If in (24-2) 2 $\lim_{N\to\infty}$ work simultaneously and synchronously each other, (24-2) becomes equal to (24-1) because all N in (24-2) also become $N\to\infty$ simultaneously and synchronously each other.
From (22-4) and (24-2) we have the following (25).

$$g(k) = \lim_{N\to\infty} m(N) \tag{25}$$

From (25) we have the following (26).

$$\begin{aligned} g(k) &= \lim_{N\to\infty} m(N) \\ &= \lim_{N\to\infty}[\{N\cos(b\log\frac{N}{k})\}\frac{1}{1+b^2} + \{N\sin(b\log\frac{N}{k})\}\frac{b}{1+b^2}] \\ &= \frac{\lim_{N\to\infty} N\{\cos(b\log\frac{N}{k}) + b\sin(b\log\frac{N}{k})\}}{1+b^2} = \frac{\lim_{N\to\infty} N\sin(b\log\frac{N}{k} + \tan^{-1}\frac{1}{b})}{\sqrt{1+b^2}} \end{aligned} \tag{26}$$

Appendix 3. Sum of infinite series of sine waves

Sum of infinite series of sine waves in the following (15) converges to one sine wave like the rightmost side of (15) due to the following reasons.

$$\begin{aligned} & f(2)\sin(b\log N/2 + \tan^{-1}1/b) - f(3)\sin(b\log N/3 + \tan^{-1}1/b) \\ & + f(4)\sin(b\log N/4 + \tan^{-1}1/b) - f(5)\sin b\log N/5 + \tan^{-1}1/b) + \cdots\cdots \\ & = f(2)\sin(b\log N - b\log 2 + \tan^{-1}1/b) - f(3)\sin(b\log N - b\log 3 + \tan^{-1}1/b) \\ & \quad + f(4)\sin(b\log N - b\log 4 + \tan^{-1}1/b) - f(5)\sin(b\log N - b\log 5 + \tan^{-1}1/b) + \cdots\cdots \\ & = A(a,b)\sin\{b\log N - B(a,b) + \tan^{-1}1/b\} \end{aligned} \tag{15}$$

1 +term and −term are defined as follows. $(b\log N - b\log n + \tan^{-1}1/b = \alpha\ )$

$$\begin{aligned} +\text{term} :\ & +f(n)\sin\alpha && \text{when the sign of } \sin\alpha \text{ is “}+\text{”}. \\ & -f(n)\sin\alpha && \text{when the sign of } \sin\alpha \text{ is “}-\text{”}. \\ -\text{term} :\ & +f(n)\sin\alpha && \text{when the sign of } \sin\alpha \text{ is “}-\text{”}. \\ & -f(n)\sin\alpha && \text{when the sign of } \sin\alpha \text{ is “}+\text{”}. \end{aligned}$$

The general term of the infinite series in (15) is
$(-1)^n f(n)\sin\{b\log N - b\log n + \tan^{-1}1/b\}$ $(n = 2, 3, 4, 5, \cdots\cdots)$.
If n is large natural number, the value of $b\log n$ increases very slowly with increase of

n and the sign of $\sin\{b\log N - b\log n + \tan^{-1} 1/b\}$ seldom change with increase of n. Therefore +term and −term appear alternately with increase of n and 2 +terms or 2 −terms appear in succession only when the sign of $\sin\{b\log N - b\log n + \tan^{-1} 1/b\}$ changes. If n_0 is large natural number and we regard the sum of these 2 +terms or 2 −terms that exist in succession as one +term or one −term respectively, we can consider the infinite series in $n_0 \le n$ as alternating series and this alternating series converges due to $\lim_{n\to\infty} f(n) = 0$.

$$f(n) = \frac{1}{n^{1/2-a}} - \frac{1}{n^{1/2+a}} \ge 0 \qquad (n = 2, 3, 4, 5, \cdots\cdots) \tag{8}$$

The partial sum of infinite series in (15) from $n = 2$ to $n = n_0 - 1$ is finite value. Therefore the infinite series in (15) converges.

2 $\beta = b\log N + \tan^{-1} 1/b$ does not depend on n in $f(n)\sin\{b\log N - b\log n + \tan^{-1} 1/b\}$. If we calculate sum or difference of 2 sine waves which have the same β, the result becomes another sine wave which has the same β as shown in the following (27).

$$\begin{aligned} &f(2)\sin\{b\log N - \theta(2) + \tan^{-1} 1/b\} \pm f(3)\sin\{b\log N - \theta(3) + \tan^{-1} 1/b\} \\ &= f(2\pm 3)\sin\{b\log N - \theta(2\pm 3) + \tan^{-1} 1/b\} \end{aligned} \tag{27}$$

The following (27-1) and (27-2) or (Figure 1) and (Figure 2) in the next page show $f(2\pm 3)$ and $\theta(2\pm 3)$ of another sine wave in the above (27). $f(2\pm 3)$ can be calculated by Cosine theorem. Sum or difference of 2 sine waves takes " + " or " − " sign of " ± " respectively in (27) and the following (27-1) and (27-2).

$$f(2\pm 3) = \sqrt{f(2)^2 + f(3)^2 \pm 2f(2)f(3)\cos\{\theta(2) - \theta(3)\}} \tag{27-1}$$

$$\theta(2\pm 3) = \tan^{-1} \frac{f(2)\sin\theta(2) \pm f(3)\sin\theta(3)}{f(2)\cos\theta(2) \pm f(3)\cos\theta(3)} \tag{27-2}$$

Therefore the partial sum of the infinite series in (15) becomes one sine wave which has the same β as each term of the partial sum like the following (28).

$$\begin{aligned} &f(2)\sin(b\log N - b\log 2 + \tan^{-1} 1/b) - f(3)\sin(b\log N - b\log 3 + \tan^{-1} 1/b) \\ &+ f(4)\sin(b\log N - b\log 4 + \tan^{-1} 1/b) \\ &- \cdots\cdots + (-1)^n f(n)\sin(b\log N - b\log n + \tan^{-1} 1/b) \\ &= f_n(a,b)\sin\{b\log N - \theta_n(a,b) + \tan^{-1} 1/b\} \end{aligned} \tag{28}$$

$f_n(a,b)$ and $\theta_n(a,b)$ converge to $A(a,b)$ and $B(a,b)$ with $n \to \infty$ respectively because as confirmed in item 1 the left side of (28) converges with $n \to \infty$.

Figure 1: Sum of 2 sine waves

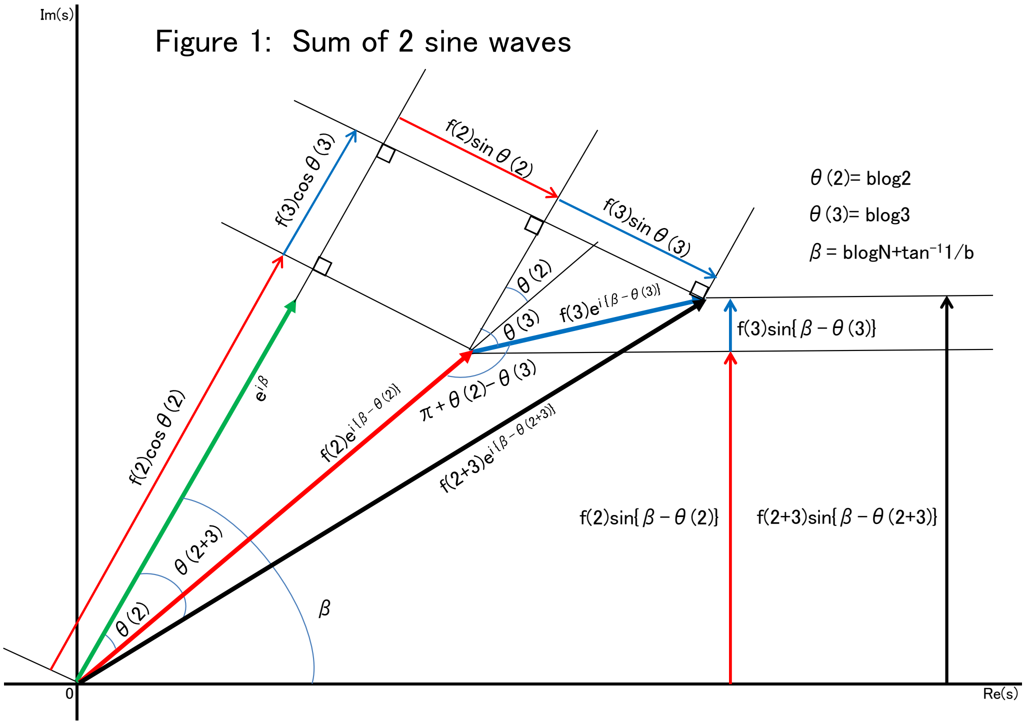
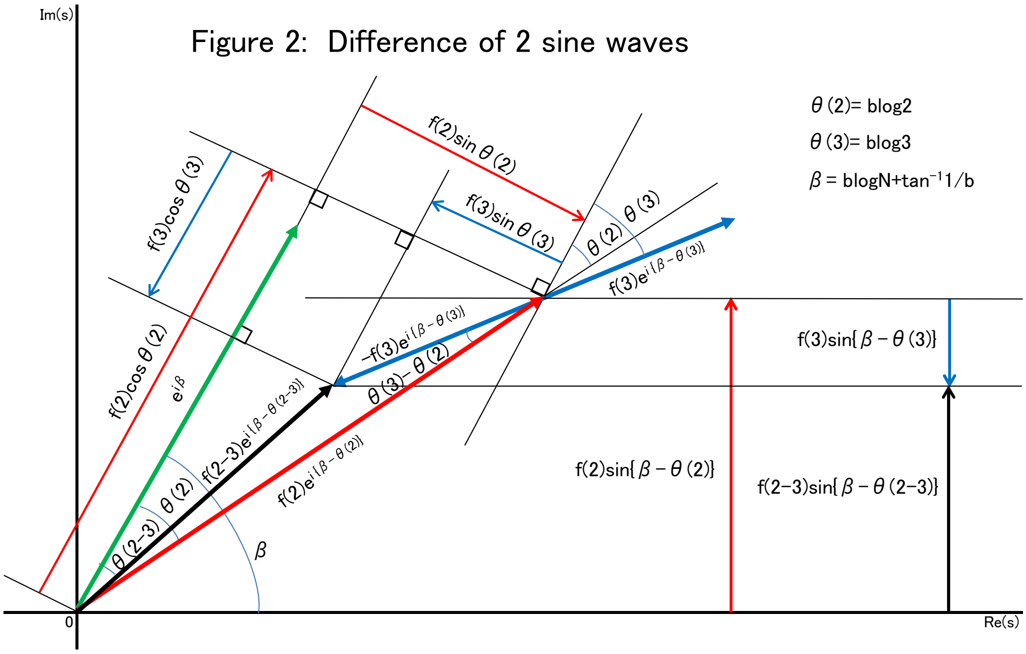


Figure 2: Difference of 2 sine waves



In (15) $A(a,b)$ and $B(a,b)$ are constants which depend on a and b. If $A(a,b) = 0$ is true, we have the following (15-1) which is identity regarding N.

$$f(2)\sin(b\log N - b\log 2 + \tan^{-1} 1/b) - f(3)\sin(b\log N - b\log 3 + \tan^{-1} 1/b)$$

$$+ f(4)\sin(b\log N - b\log 4 + \tan^{-1} 1/b) - f(5)\sin(b\log N - b\log 5 + \tan^{-1} 1/b) + \cdots\cdots$$
$$= A(a,b)\sin\{b\log N - B(a,b) + \tan^{-1} 1/b\} \equiv 0 \qquad (15\text{-}1)$$

For the value of the leftmost side of (15-1) to be zero at any value of N the value of $f(n)$ must be zero at any value of n like the following (8-1). From (8-1) $a = 0$ must hold true.

$$f(n) = \frac{1}{n^{1/2-a}} - \frac{1}{n^{1/2+a}} \equiv 0 \qquad (n = 2, 3, 4, 5, \cdots\cdots) \qquad (8\text{-}1)$$

Conversely if $a = 0$ is true, $A(a,b) = 0$ must be true from (8-1) and (15-1). Therefore if $A(a,b) \neq 0$ is true, $a \neq 0$ i.e. $0 < a < 1/2$ holds true by the contraposition.
Now we can say if $A(a,b) = 0$ is true, $a = 0$ holds true and if $A(a,b) \neq 0$ is true, $0 < a < 1/2$ holds true.

References

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Toshihiko ISHIWATA
E-mail: toshihiko.ishiwata@gmail.com