The Fundamental Physics of de Broglie's Waves, Part One

Nicolae Mazilu, PhD

Abstract. We uphold a general viewpoint that the ideas have their own 'objective' dynamics, that is a dynamics independent of personalities. Every now and then, the ideas are accessible to individuals. When it comes to social expression, though, it is only a genius, among individuals, who can reach such an accessible idea as ultimate truth, even to the point where he/she succeeds to casting it into proper words, available to society at large. Beyond those words no one can 'rationalize', not even the genius himself. We illustrate this observation with Erwin Schrödinger's interpretation of the wave function introduced by him to common knowledge, which has been cast into proper words by another genius, Richard Feynman, who, nevertheless, honestly acknowledged mistrusting that interpretation. In fact, Feynman always openly admitted, and effectively exercised the probabilistic interpretation of Max Born. To wit, it is guided by this kind of interpretation, that he has built the well-known version of wave mechanics, which is based entirely on the concept of probability. We argue that Feynman was actually right on both accounts – Schrödinger's and Born's interpretations – and show here why and how, thus revealing a few missing points in the contemporary natural philosophy. The whole argument, and the accompanying illustration, stands completely by *the words* of still another genius of modern physics: Louis de Broglie, whose physics is documented and developed in the present work. The construction of this physics gives us the opportunity of a historical review of significant moments of the physical knowledge.

Keywords: interpretation concept, Schrödinger's interpretation, Feynman's interpretation, de Broglie's light ray, de Broglie's application function, confinement, holography, scale transitions, Haas' quantization, Bohr's quantization, Thomson's natural philosophy

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Introduction

The fundamental physics intended in this work is entirely connected with the concept of interpretation, brought about by the the wave mechanics at the beginning of the previous century. The word 'interpretation' has in this instance an explicit meaning: it is not the usual 'clarification', 'elucidation' and such, of the common social language, but a well defined concept whose necessity became undeniably overbearing with the first signs of upcoming wave mechanics, in the last century. Let us explain the meaning, by getting into a little history concerning the wave mechanics, while following the stars of this important drama of human knowledge as they played their parts. The drama itself was centered, as it were, around the concept of wave function, which was introduced to common knowledge by the genius of Erwin Schrödinger.

The first in the order of things catalyzed by the concept of wave function of Schrödinger, was, of course, this very function. Schrödinger had, apparently, quite a down-to-earth reason for starting his wave-mechanical description of the fundamental physical structure of the universe around us: he believed that the integers involved in the Bohr-Sommerfeld quantization conditions should not be a primary concept, inasmuch as our mathematical knowledge allows the awareness that they can naturally originate from apparently deeper first principles. Quoting:

> In this paper I wish to consider, first, the simple case of the hydrogen atom (non-relativistic and unperturbed), and show that the customary quantum conditions can be replaced by another postulate, in which the notion of "whole numbers", merely as such, is not introduced. Rather when integralness does appear, it arises in the same natural way as it does in the case of the *node-numbers* of a vibrating string. The new conception is capable of generalisation, and strikes, I believe, very deeply at the true nature of the quantum rules. [(Schrödinger, 1928), p.1; *original emphasis*]

True to the resolution about the content of his work as declared in this excerpt, Schrödinger followed the very same technical trait along the first three of his communications on the wave mechanics, and only in concluding the fourth of them, he gave an explanation concerning the wave function he introduced. This explanation was actually extracted from the manner he was compelled to work with the function itself, in solving the various physical problems encountered as part and parcel of the eigenvalue viewpoint in the case of hydrogen atom model. Thus, the main general conclusion does not involve the wave function *per se*, but a quadratic expression constructed based on it:

> $\psi\psi^*$ is a kind of *weight-function* in the system's configuration space. The *wave-mechanical* configuration of the system is a *superposition* of many, strictly speaking of *all*, point-mechanical configurations kinematically possible. Thus, each point-mechanical configuration contributes to the true wave-mechanical configuration with a certain *weight*, which is given precisely by $\psi\psi^*$. If we like paradoxes, we may say that the system exists, as it were, simultaneously in all the positions kinematically imaginable, but not "equally strongly" in all. In macroscopic motions, the weight-function is practically concentrated in a small region of positions, which are practically indistinguishable. The centre of gravity of this region in configuration space travels over distances which are macroscopically perceptible. In problems of microscopic motions, we are in any case interested *also*, and in certain cases even *mainly*, in the varying *distribution* over the region. [(Schrödinger, 1928), p. 120; *original emphasis*]

We took the liberty of replacing here the original Schrödinger's *overbar* in the notation of the complex conjugate, by an upper star index, both from technical necessities related to our possibilities of producing this document, but mainly to be in line with the modern notation in physics. Now, for all intents and purposes, the concept of a *weight function* in describing the hydrogen atom, has brought the space extension of the matter to bear on physical components of the classical planetary model described as a Kepler problem: the nucleus of the hydrogen atom, as well as the electron revolving around it, are extended in space. However, in order to make sense, the weight function Schrödinger is talking about, must be referred to a space *ensemble of positions* – 'weight-function is concentrated in a region of positions' – that should be identified as one single position in the case of a macroscopic system in motion like the one described by the classical Kepler problem. This concept of ensemble of positions comes at odds with the fact that the weight function is defined in 'system's configuration space', which is actually an 'enclosed space'.

Indeed, soon after the publication of Schrödinger's ideas, and due to this very notion of weight function, it became quite obvious that we just have to take the idea of *interpretation* face-value, so to speak. In fact, we need such a concept, and this was indeed defined ever since, in the good tradition that started settling in physics with the philosophy of Copenhagen School, *i.e*. by the *experimental necessities*. Quoting:

> It is almost impossible to describe the result of any experiment *except in terms of particles* – a scintillation, a deposit on a plate, etc. – and this language is quite *incompatible with the language of waves*, which is used in the solution. A necessary part of the discussion of any problem is therefore *the translation of the formal mathematical solution, which is in wave form, into terms of particles*. We *shall call this process the interpretation*, and only use the word in this technical sense [(Darwin, 1927); *our Italics*]

In other words, with the upcoming wave mechanics the interpretation had to be considered a concept by itself, that should consist of 'rendering' the wave forms – some continua – in terms of particles. As Charles Galton Darwin presents it in the excerpt above, this process is mandatory only from an experimental point of view, specifically for the *recording purposes* that necessarily go along with an experiment, and is undoubtedly referring to the wave function itself – the wave form – not to its magnitude squared, or anything else. However, all along the Schrödinger's original elaboration, this process has been taken under a fundamental objective connotation, classical we should say, indicated by that "point-mechanical configuration" from the excerpt we presented before. In this take, Darwin's 'translation' is referring not to some 'recording', but to the very physical structure described by the wave function, which, while mathematically defined in the "system's configuration space", is nevertheless interpreted as an *ensemble of positions*, which cannot be constructed but in a reference frame, therefore in a 'ordinary space'. This is, again, quite a classical feature of a theory of continua for which we shall settle in the present work. It is, obviously, at odds with the orthodox interpretation of the wave function, which is referring, almost exclusively to a weight function *representing probabilities*, not the 'strength' of existing of a physical system in space, or anything else.

In thus accepting it – that is, by the idea of 'strength' – the concept of interpretation raises quite a significant concern, in hindsight even critical, which tends to show up especially if we forget about that objective connotation that Schrödinger assigned to his wave function, the one involving the notion of a 'charge cloud'. Incidentally, we have to recognize, though, that this interpretation is, in fact, the largest level of acceptance ever possible, inasmuch as it includes, as only a particular case, that experimental level invoked by Darwin in the excerpt above, a fact that shall be obvious as we go along with our work. The concern we are talking about is conspicuous, if we may say so, and comes with that notable dichotomy regarding the problem of space, which we just have mentioned above, but, nevertheless, needs to be clearly pinpointed as such. That dichotomy actually confounded the whole human knowledge of all times – no matter if natural philosophical, purely philosophical or simply technical in general – being, in fact, by and large unrecognized as such even today. It is the difference between what is *philosophically* accepted as 'the ordinary space', and what is *scientifically* accepted as 'the coordinate space'. Quoting Darwin again:

> In dealing with the interpretation we have touched on one of the great difficulties which have made it hard to gain physical insight into the wave theory. This is the fact that *the wave equation is not in ordinary space*, *but in a co-ordinate space*, and the question arises *how this co-ordinate space is to be transcribed into ordinary space*. It would appear that most of the difficulty has arisen from *an attempt to apply it illegitimately to enclosed systems*, *which are really outside the idea of space*. In most of the problems we shall discuss the question hardly arises, but where it does *the correct procedure is so obvious that there is no need to deal with it in advance*. It is tempting to believe *that this will be found to be always the case*. [(Darwin, 1927); *our emphasis*]

It is 'tempting', indeed, to assume that the problem will not pop up, but the evolution of physics proved that the case is quite contrary: Darwin was way too optimistic! We 'need to deal with this issue in advance', indeed: more precisely, even before we start anything physical, just because the very existence of wave mechanics and quantum mechanics is conditional on the measurement. As it turns out, the fact of the matter rests with the *identification* of the *length*, involving the "coordinate space" only, with the *distance*, which involves "the ordinary space" only. And this identification is just a special instance of a universal relationship, fundamental in the description of any universe according to actual human knowledge, which has to be recognized as such.

Indeed, it would appear in the above excerpts from Darwin's work, that the coordinate space is intimately connected with the idea of enclosed physical systems, which is 'outside', as it were, the space concept derived from our intuition. However, inasmuch as, physically speaking, we have always to deal only with 'enclosed systems', we need either to bring the 'ordinary space' under this concept of coordinate space, or to bring this last concept under that of ordinary space. This is to be done here, as it is done everywhere for that matter, with the aid of two instruments: a *clock*, to regularize our perception of *one* physical body, and a *coordinate system*, to regularize our perception of *many* physical bodies. These are the two essential features – or *differentiae*, using a philosophical label – of a general concept of *reference frame*. The whole physics is built around this concept, and what we have to say here makes no exception. Let us, therefore, commence our story.

Chapter 1 Turning Laxity into Preeminence

Schrödinger started his program with a logarithmic transformation of the classical variable of action, which helped convert the Hamilton-Jacobi equation for the planetary hydrogen atom into a second order partial differential equation [(Schrödinger, 1928), pp. 1–2]. Specifically, by a classical variational principle, applied to the energy thus calculated from the Hamilton-Jacobi equation, he was able to obtain a Helmholtz-type partial differential equation as an eigenvalue equation, thus providing, at least in principle, the quantization condition for the *planetary hydrogen atom*. Now, as a practical eigenvalue problem, the quantization thus performed can be carried out in any foreseeable – and, when it comes to prediction, even contrived – detail, by placing the physical system described as hydrogen atom in some specific environments, *via* particular boundary conditions. However, such a detail *does not* satisfy the classical demand for a physical system's description, which should be more of a… cosmological type, aiming mostly for the physically apparent construction of that very system, independently of any specific environment, universal, as it were. In the words of Darwin from the excerpt right above, the physical details would belong to the 'coordinate space', while the classical rigor always asks for the description of system 'in ordinary space'. In hindsight, one can say that this is actually a general aspect of our synthetic knowledge – innate to human nature, we should say – implanted in it by the necessity of finding some fundamental 'bricks' in the 'construction' of a universe. Such a 'universal' condition to satisfy the classical rigor was, apparently, quite handy for the method that Schrödinger devised, so he wasted no time in adopting, and effectively using it: *getting rid of the eigenvalues in the very problem of description of the hydrogen atom*.

1.1 A Foible of the Human Nature

The eigenvalues of a specific mathematical problem are, indeed, the ones reflecting, among others, the *boundary conditions* in which a system is performing in a specific problem, so that getting rid of them in the description of that physical system may be taken as having the connotation of a description which is independent of at least some environments, if not all of them. This last kind of description is, indeed, a 'cosmological' description of the system, satisfying the classical rigor, so we can very well surmise an equivalence of this rigor with the 'elimination of the eigenvalues'. Fact is that Schrödinger proceeded this way, by using a special periodical time dependence of the wave function, which he introduced *via* the initial logarithmic transformation. The mathematical problem comes down to eliminating the energy E – the eigenvalue in question – between the following two conditions:

$$\frac{\hbar^2}{2m}\Delta\psi_0(\boldsymbol{x},t)+[E-U(\boldsymbol{x})]\psi_0(\boldsymbol{x},t)=0; \quad \psi(\boldsymbol{x},t)=\psi_0(\boldsymbol{x})\cdot e^{iEt/\hbar} \tag{1.1.1}$$

where $U(\boldsymbol{x})$ is the potential describing the system as a classical dynamical one. The fact of the matter, as Darwin expresses it, is that "a stationary state is merely a solution of the wave equation that happens to be *harmonic in time*" [(Darwin, 1927), p. 260, *our emphasis here*], so that such a time dependence seems to be the hallmark of the eigenvalue problem altogether. The method of elimination is just as obvious as its well-known and epoch-making result: the partial differential equation under (1.1.1) above can be multiplied by the time exponential factor, resulting in an identical equation

$$\frac{\hbar^2}{2m}\Delta\psi(\boldsymbol{x},t)+[E-U(\boldsymbol{x})]\psi(\boldsymbol{x},t)=0 \tag{1.1.2}$$

whence, by an obvious replacement, which suggests itself mathematically, one gets the *nonstationary Schrödinger equation* as we know it today:

$$-i\hbar\frac{\partial\psi}{\partial t}(\boldsymbol{x},t)=-\frac{\hbar^2}{2m}\Delta\psi(\boldsymbol{x},t)+U(\boldsymbol{x})\psi(\boldsymbol{x},t) \tag{1.1.3}$$

However, in procceeding this way, Erwin Schrödinger started, apparently, being embarrassed by a mathematically obvious fact. Duly reflected in the work cited here is his insistence on the circumstance that, along with the consideration of the time exponential factor, comes a significant ambiguity: the eigenvalue can be eliminated by a time derivative of *any* order, not just the first one. And issuing from this ambiguity, there are at least a couple of sources of embarrassment to be considered here.

In the first place, this raises questions on the very concept of energy as formally represented by a Hamiltonian. So, in order to maintain the 'first derivative elimination', so to speak, one should ask for their *identity* as a separate hypothesis of wave mechanics, a condition quite obvious in the classical framework (Poincaré, 1897), but which came only later under a special scrutiny from the point of view of the wave mechanics (Kennedy & Kerner, 1965). One might suspect that the complex nature of the wave function is the actual problem: after all, the classical optics works very well in the real domain with trigonometric functions, and everything goes just fine. But then, in trying to trigonometrically eliminate the eigenvalue, one is forced to multiple time derivatives – specifically two of them for now – in view of the duality of the trigonometric functions satisfying the condition of periodicity in the real range. So, it seems that the complex wave function with the first time derivative still remains the best way to eliminate the eigenvalue, if it is to describe the system from a classical point of view.

Be it as it may, fact is that the complex character of the wave function became so significant to Schrödinger, that he took it as a persisting 'flaw' in the theory, which would irrevocably exclude any idea of 'complete' classical rigor, as long as that rigor was to be satisfied in real mathematical terms: while the stationary problem of energy eigenvalues could be so conducted as to be accomplished with a function ψ in real terms, when assuming that it has to satisfy the new 'universal' nonstationary – that is to say time-dependent – equation (1.1.3), *that function had to be essentially complex*. Today this fact would hardly be a problem – people are already used to work with many apparent formalities, especially with the mathematical ones (the prototype of them all!) simply by establishing rules of working – and one would virtually accept the condition with not so much concern about it. However, a century ago the things were entirely different when talking about the classical rigor. Thus, in asking why, and how could this happen, Schrödinger found no satisfactory answer in just the obvious mathematical procedure imposed by the particular choice of the time-dependence of the wave function: he needed a more profound reason, perhaps even a physical reason. Which he could not find at the time, so he served the eternity only with a footnote joke regarding the human condition in general. This joke carries the unmistaken signature of Wolfgang Pauli, his friend, notorious for many such incidents in the epoch we are talking about, incidents that even earned him the nickname *Mephisto* among the fellow physicists. Quoting, then:

> The words «essentially complex» try to cover here a great difficulty. In his desire to visualize, at any rate, the propagation of the waves ψ as a real phenomenon in the classical acceptance of this term, the autor has refused for himself to frankly recognize that any development of the theory accentuated more and more clearly the essentially complex nature of the wave function. Yet, this function is determined by an equation [*the nonstationary Schrödinger equation (1.1.3), n/a*] whose coefficients are essentially complex!
>
> But how could $\sqrt{(-1)}$ bring itself into this equation? One answer, which I don't dare indicate here but the general idea, was given to this question by a physicist who once left Austria, but who, despite the long years spent abroad, has not lost completely his biting Viennese humor, and who, besides, is known for his ability to find the right word, all the more fair as it is cruder. Here is that

> answer: «√(–1) slipped into that equation as something we let accidentally escape, providing us however with an invaluable relief, even though we produced it involuntarily.» [(Schrödinger, 1933), footnote on pp. 166–167; *our translation, n/a*]

Joke aside, the fact remained unsettled until recently: the complex nature of the wave function seemed just an accidental blunder of the common kind suggested in this quotation, that we have to accept inasmuch as it provides 'invaluable relief'. In this particular instance we have to properly use the existing mathematical rules of handling functions, of course. It took almost a century and a simple observation made quite recently (Schleich, Greenberger, Kobe & Scully, 2013), in order to realize that the wave function, such as Schrödinger wanted it, *must be complex at any rate*, for there is no way around this condition. Besides, the consequences of such a condition are staggering, first from historical point of view, but mostly from a gnoseological point of view. To mention just the most important of those consequences, *this condition would mean that a wave-mechanical universe is necessarily holographic*! Let us show a few details of this issue. In order to do that, we have to start with the mathematics of a complex signal, among which category we can certainly include the wave function, at least in a few essential instances.

1.2 What's Natural Has to be Valued: Requirements of Interpretation

We have to start, therefore, with an obvious methodological question: what would an *interpretable* complex wave function involve? This means that everything in elaborating an answer to this question, should revolve around the concept of interpretation, according to Darwin's definition excerpted above, but in a simpler, more general form: *there should be an ensemble of particles describing a wave* in general. At the time when Schrödinger published his ideas there appeared already two wave-mechanical *objective* interpretations of the wave function, on which, by and large, we have elaborated recently (Mazilu, Agop, & Mercheș, 2019). When we say 'objective' here, we understand interpretations in the sense of Schrödinger, more to the point, interpretations having little or nothing to do with the idea of 'recording', which, by the way, was mainly promoted by the quantum mechanics arising at that very time. The interpretations in question where those of Louis de Broglie and Erwin Madelung. They can be easily differentiated from one another if we simply follow their details mathematically. However there is one essential difference between them that needs to be considered as *a matter of principle*, *i.e.* for the benefit of human knowledge in general: the difference in the *phase of the wave function* involved in the interpretation.

The first one of the two interpretations (de Broglie, 1926) abides by the Schrödinger's *phase linearity with respect to time*, filling in, as it were, for some missing points brought about by the phenomenology of light in the classical theory of *light rays*. This interpretation, issuing from optics, is implicitly referring even to a 'recording' manner, as it were, inasmuch as it proves that, according to the theory of diffraction of light, the density of the interpretative ensemble of particles along a light ray should be given by *the square of the amplitude of the optical signal*. This should be an essential quantity according to Schrödinger original interpretation, as illustrated in the above excerpts. On the other hand, the second interpretation (Madelung, 1927) *gives up the linearity of phase with respect to time*, pushing through, so to speak, the Hamilton-Jacobi equation – considered, in fact, by Schrödinger himself as a starting point in his enterprise – but in its classical nonstationary form. This last interpretation has, apparently, nothing to do with the idea of recording. While the de Broglie's interpretation, which is not using the Schrödinger equation, makes explicitly use of the mathematical reality condition for the signal representing the light field in a light ray – much to Schrödinger's own satisfaction we should say! – the Madelung interpretation takes the complex wave function defined by the nonstationary Schrödinger equation as a starting point. Being kind of an exotic fact, this last interpretation attracted an overwhelming general attention of theorists, from both the fundamental points of view of the physics and knowledge at large, as well as from the practical point of view, related to the examples of the physics initiated by Schrödinger himself. The story is well-known, and can be entrusted to a classical literature by now, so that we shall not insist here on Madelung interpretation, but only to the extent that it represents *a matter of fundamental principle*. The reader can follow the story by consulting the literature indicated in our recent work (Mazilu, Agop, & Mercheș, 2019, 2020). Here we need only to point out that a general kind of Madelung interpretation is implicit in the general complex form of the wave function, and that Louis de Broglie showed that the light phenomenon can be *interpreted* along this idea. Details follow, casting a new meaning upon these old facts.

Start with the observation that an algebraically complex form of a physical signal, but 'in ordinary space', if it is to use the terminology of Darwin – among the kind of which the wave function is to be considered – involves exhibiting *an amplitude* and *a phase* as functions of a position. As any position in space, this one is assigned by some coordinates $\boldsymbol{x}$ with respect to a physical reference frame at a time moment t, chosen from a time sequence as given by a certain clock:

$$\psi(\boldsymbol{x},t) = A(\boldsymbol{x},t)\cdot e^{i\phi(\boldsymbol{x},t)} \tag{1.1.4}$$

As it happens, this function is by default, if we may say so, the solution of a differential equation that can be recognized as a *free particle Schrödinger equation*. This identification can be carried out, indeed, as it depends on specific conditions to be revealed mathematically (Schleich, Greenberger, Kobe & Scully, 2013), which can be physically 'interpreted' in the sense of Darwin, thus providing a logical reason for the very concept of interpretation. To wit, just routine calculations on the complex form (1.1.4) above, produce *the identity*

$$\frac{i\dfrac{\partial\psi(\boldsymbol{x},t)}{\partial t}+\beta\cdot\nabla^2\psi(\boldsymbol{x},t)}{\psi(\boldsymbol{x},t)} = -\left(\frac{\partial\phi}{\partial t}+\beta\cdot(\nabla\phi)^2-\beta\cdot\frac{\nabla^2 A}{A}\right)+\frac{i}{2A^2}\cdot\left(\frac{\partial A^2}{\partial t}+2\beta\nabla\cdot(A^2\nabla\phi)\right) \tag{1.1.5}$$

Here β is a conveniently introduced constant, having the physical dimensions of a *rate of area* (m^2/s), a fact that may prove significant by itself in later considerations. The 'specific conditions' necessary in order to make $\psi(\boldsymbol{x},t)$ a solution of the time dependent Schrödinger-type equation *for the free particle*, that is to say the equation:

$$i\frac{\partial\psi(\boldsymbol{x},t)}{\partial t}+\beta\cdot\nabla^2\psi(\boldsymbol{x},t)=0 \tag{1.1.6}$$

are, mathematically, the two equations that guarantee the vanishing of the right hand side of equation (1.1.5), as long as the amplitude $A(\boldsymbol{x},t)$ is not *zero* or *infinity*. One can say that, in such general cases, the wave function can be declared complex with *apodictical* certainty, just like any mathematical fact of our knowledge. Indeed, (1.1.6) can take place if, and only if we accept that the right hand side in equation (1.1.5) is a *null complex quantity*, which means:

$$\frac{\partial A^2(\boldsymbol{x},t)}{\partial t}+2\beta\cdot\left[\nabla\cdot\left(A^2\nabla\phi(\boldsymbol{x},t)\right)\right]=0,\qquad \frac{\partial\phi(\boldsymbol{x},t)}{\partial t}+\beta\cdot[\nabla\phi(\boldsymbol{x},t)]^2-\beta\cdot\frac{\nabla^2 A(\boldsymbol{x},t)}{A(\boldsymbol{x},t)}=0 \tag{1.1.7}$$

In so doing, we set a little twist on Schrödinger's initial story: it is *not* the wave function itself the one which must be real, but only some mathematical expressions involving its constitutive parts, the amplitude and phase. Claiming reality for the wave function is a much too restrictive condition. The fact of the matter is that the reality was always employed in physics not on symbols, *but on expressions involving these symbols*. This much we have learned from the physics of the last century, and this is the reason why, as we said before, we are today hardly surprised by the complex nature of the wave function. What, apparently, we have *not* learned, though, is to accept the consequences *such as they are*, and this fact became critical, even to the point that the paradoxes are nowadays accepted as a way of life and science!

Chapter 2 … sicut Natura nil facit per Saltum, ita nec Lex…

In order to illustrate these last conclusions, we shall offer now a practical example of them: the only theory that ever proposed a proof for some of the steps of a genuine interpretation. This is Louis de Broglie's theory, originally designed to establish that the physical optics does not contradict the wave-mechanical precepts, inasmuch as these precepts are coming out of necessities settled by the quantum mechanics (de Broglie, 1926). The main trait of this example is, at least in what concerns us, first, the historical continuity of knowledge, and secondly providing a striking case of 'taking the consequences such as they are' in the actual facts of modern day theoretical physics. However, there are a few fundamental theoretical reasons for which we are taking the example to its intimate details.

Let us notice, once again that, contrary to the observation of Darwin, the equation (1.1.6) here is an equation 'in the ordinary space', and that, disagreeing with the usual requirement, one needs actually 'to transcribe the ordinary space into coordinate space', as Darwin would say. In setting the problem this way, one cannot but notice that the wave mechanics reproduces in fact the general pattern of classical knowledge: it is actually starting from the ordinary space – the space of positions – that we have to construct a coordinate space – a space occupied by the matter accessible to experiment. This process demands, first and foremost, a reference frame to locate the events in space, secondly, some clocks serving to arrange the events in the order necessary to describe the motions, and finally some measurements in order to put together those events as… events. The de Broglie's interpretation, to be undertaken here, is referring to light, and has most of the elements of an interpretation concept, that we need to learn and reproduce in any other cases.

To start with, the first one of the equations (1.1.7) can be taken as a continuity equation provided two essential conditions are secured: 1) *the square of the amplitude of complex function* $\psi(\mathbf{x},t)$ *can be taken as a density* of the interpretative ensemble of free particles, and 2) *the velocity field* of the interpretative ensemble of free particles *is provided by the gradient of the phase* of wave function. It is quite instructive, indeed, for the knowledge at large, and especially for physics, to analyze how this historical interpretation dealt with such requirements. It should be noticed *again*, though, from this very beginning, that Louis de Broglie was, and still is, *the only one* in the history of physics who ever had undertaken the task of mathematically *proving* these two conditions, otherwise always considered as axioms. He started from optics, for there was no physical situation involving particles in the physical optics of the day, a field of knowledge which, belonging to the science of optics, inspired, as a matter of fact, the upcoming wave-mechanical ideas. De Broglie's work goes on, logically we should say, in 'filling' some missing points that mark discontinuities from the epoch's very natural philosophy of light. We find it opportune telling the story of this philosophy.

The specific problem at that time was, in de Broglie's acceptance at least, that of proving the old idea that the light can be seen as a flux of particles. However, inasmuch as such a proof has had eluded any human attempt before, he wanted just to add to our experience the one fact turned critical at the time, namely to show that the particulate image of the light contradicts neither the optical, nor the mechanical rules of natural philosophy, for the particular case of *the experiments of diffraction*. Long known for light, and routinely used, we might say, in optics even from the times of Augustin Fresnel, such experiments were revived for electrons at the epoch we are in now – and especially illustrated through the work of Davisson and Germer – by de Broglie's own idea of associating a wave with a particle [see (Davisson & Germer, 1927) for the history and literature on the problem]. In hindsight, this idea gives a positive turn to the old belief expressed in the title of this chapter, that the knowledge should be just as continuous as the nature itself: this seems to be the message of the de Broglie's very work on interpretation, at least as we take it. Indeed, the Fresnel theory of light, the one that instated the diffraction as a routine experiment in optics, harmoniously completes, in fact, the classical line of the natural philosophy referring to the light phenomenon at large. The same can be said of de Broglie's theory of light ray: it harmoniously fills in for missing points in the very 'completion' of Fresnel's theory of light, and even suggests a further 'completion' of the phenomenology of light, as the history shows us. Let us summarize the main historical steps of this side of natural philosophy, in order to better understand the position of Louis de Broglie in his enterprise that we have under scrutiny at this point.

The first physical image of light phenomenon was that of Thomas Hobbes, a global concept as it were, involving the idea of "orb" (Hobbes, 1644). It was probably religiously inspired – in fact, religion has always inspired any natural philosophical idea – by an analogy with the heart. The analogy did not work quite properly: everyone can perceive, or at least logically infer, that the light is expanding only, never contracting, like the heart does in fulfilling its job. However, the idea of materiality of light settled this issue in an unexpected way, insofar as, based on it, Robert Hooke placed the *periodic motion* where it should belong, just by logic: it takes place within the 'orb' – read wave surface of light – otherwise the light would be able destroy the transparent materials it penetrates, and such an event has never been observed. The periodic motion is a 'pulse', more precisely an "orbicular pulse", if it is to use Hooke's own words (Hooke, 1665). This last concept improves, kinematically, but mostly dynamically in hindsight, upon Hobbes' purely geometrical "line of light", naturally filling in for the fact of expansion, through the idea of propagation of light along a direction. This was just about the first case in the Newtonian epoch, whereby the human spirit was starting filling the empty, ideal world image provided by geometry, with properties of the world of human experience. The recognized case of this start of the classical natural philosophy in the epoch we are talking about is, of course, the triad of Kepler laws, facilitating the invention of forces by Isaac Newton.

The phenomenology of light was limited in that epoch to just two specific phenomena: *reflection* and *refraction* of light. Newton 'fortified', we might say, this phenomenology technologically (see his *Opticks*), in order to avoid the 'invention of hypotheses', by creating an experimental basis which, under different circumstances of course, works even today. However, based exclusively on that phenomenology, and therefore abundantly still 'inventing hypotheses', as it were, Hooke created a concept of light ray (*loc. cit. ante*, pp. 53 – 69), in which he incorporated what we think of as the first rational theory of colors. In this concept, the color is controlled by the angle between orbicular pulse and the mathematical rays delimiting a plane construction that can be rightfully called *physical ray*. It is on this last concept, introduced originally by Thomas Hobbes, that the experimental basis created by Newton helped improve, adding one of the most important differentiae to it; and on this differentia we have to abide for a while, insofar as it is of essence in what we have to say here. Thomas Hobbes had insisted before on the important fact that, as a concept, *the physical ray is not a plane figure* from geometrical point of view, but *a solid one*, a cone or a cylinder, or even a more general *tube*. Quoting:

> PROPOSITION IV
>
> *The ray is a solid space*
>
> Since a ray is the path through which a motion is projected from a luminous object and there can be no motion except of a body, it follows that a ray is the place of a body and therefore has three dimensions. Therefore, a ray is a solid space.
>
> *Definition of direct and refracted rays*
>
> A *direct* ray is the one whose section by a plane passing through its axis, is a parallelogram…
>
> A *refracted* ray is the one composed of two direct rays making an angle along an intermediate part…
>
> *Definition of the line of light:*
>
> The line where the sides of a ray begin… [(Hobbes, 1944), *our translation; see also* (Shapiro, 1973), *and the Portuguese translation of* Tractatus Opticus *in* Scientiae Studia, Sao Paulo, Volume **14**(2), pp. 483 – 526 (2016)]

The whole classical discussion before Newton is done on that 'section by the plane through axis', defining the ray to Hobbes, where the implicit assumption is that the section thus defined is 'generic': there is no distinguished axial section of the ray. Hooke's definition of the colors is all done using exclusively properties of these plane figures [(Hooke, 1665), pp. 53 – 69]. However, by adding this differentia – that is the color – to their concept, such sections can hardly continue to remain generic, and, therefore, do not satisfy the fundamental requirement of a standard. This was the conclusion of the Newton's celebrated, detailed and careful experiments with a prism, or a set of prisms, which proved that the color is a property of light, varying directionally, indeed, but as a feature of homogeneity of the ray [*see his* Opticks; *see also* (Shapiro, 1973)]: different *homogeneous rays* are distributed in a certain direction, not the color *per se* within a ray. The 'theory' created by Newton extends, in fact, to details, the experimental observation that different homogeneous rays have different refrangibilities, *corresponding to different colors*, so that the homogeneity needs to be further characterized by a color. In other words, from the point of view of the color, the ray geometry cannot be plane, as the Hooke's theory would imply. A homogeneous ray still satisfies the Hobbes' original definition, and has to be described by a solid geometry: it is a collection of homogeneous rays – the spectrum – that only incidentally, *i.e.* depending on the presence of the prisms, extends directionally in the cross-sectional plane.

It should be illuminating again, we believe, especially in understanding the concept of physical ray in general, to notice that, eventually, it was discovered that the color must be described by a gauge group acting in the cross-section of the ray (Resnikoff, 1974), reproducing a group action of SL(2,R) type, and that Erwin Schrödinger pioneered this very discovery (Schrödinger, 1920). One can rightfully say that, with the theory of colors he created in 1920, Schrödinger was in fact completing an 'apprenticeship' for the physics which he started building six years later in the form of a wave mechanics (Mazilu, Agop & Mercheș, 2019). Be it as it may, what we think is worth retaining from the brief history we just presented, in the spirit of today's physics, of course, is the fact that the epoch of reflection and refraction phenomenology produced the concept of a *light ray as a tube*, having the color as *a transversal property described by a gauge group*.

Along this historical path, Augustin Fresnel started a new epoch in the phenomenology, marked by the introduction of *diffraction* of light as a new phenomenon, in the description of which the periodic properties of light – only *assumed* by Robert Hooke – were the usual observables, and thereby the wave nature of light came closer to our understanding. The obvious spatial periodical pattern in the *recordings* of diffraction phenomena could thus be explained physically, as a mechanical interference phenomenon. In so doing, the optics made reference to the harmonic oscillator, in order to understand the energetics of light for instance, to say nothing of some other physically fundamental necessities, like the very definition of the *intensity of light*. However, this reference is, by stretching a little the meaning of words, 'illegal' to say the least, in the case of light, inasmuch as the light phenomenon seems a far cry from exhibiting the kind of inertial properties required by a proper dynamics of the harmonic oscillator. For, as a purely dynamical system, the harmonic oscillator is a system described by forces proportional to displacements (those type of elastic forces, used initially by Hooke to explain the behavior of light), and in the case of physical optics the second principle of dynamics is quite incidental, as it were. It was introduced only by a natural mathematical property of transcendence of the second order ordinary differential equation: it describes *any* type of periodic processes. And the fact is, that in the foundations of modern physical optics, the periodic processes of diffraction have more to do with the theory of statistics than with the dynamics (Fresnel, 1827). We shall come back to these issues here, for the specific case of electric charge.

This is, however, not to say that the harmonic oscillator is to be abandoned altogether, as a model, because this is not the case, either from experimental point of view, or even theoretically, as we shall show here. All we want to say is that we need to find its right place and form of expression in the theory, and this is indicated, again, through the order imposed by the measure of things, this time as their mass. Indeed, dynamically, the second order differential equation expressing the principle of inertia, involves *a finite mass*. On the other hand, for light the mass is virtually inexistent, and if the second order differential equation is imposed by adding the diffraction to the phenomenology of light, this means that it actually describes *a transcendence between finite and infrafinite scales of mass* [(Georgescu-Roegen, 1971); for a closer description of the concept, one can also consult (Mazilu, Agop & Mercheș, 2019)]. As we shall see later, the mathematics of scale transitions between finite and infrafinite – in this specific case, infinitesimal – gauges in a scale relativity, fully respects the rules related to the harmonic oscillator model. In fact, the whole wave mechanics, as a science, can be constructed based on such rules, which appear to be universal.

Now, along with the settling of Fresnel's theory in physical optics, a few changes in the natural philosophy have taken place. First in the order of things, was making the dynamics 'lawful', as it were, in the case of light. The first step was *to identify the phase*, mathematically involved as an independent variable in the trigonometric functions describing the diffraction in the light phenomenon*, with the time of an evolution*: mathematically speaking, the phase had to be linear in time. A condition which brought *the frequency* front and center, and with it *the concept of wavelength*, thus generating right away a whole new experimental technology of the Newtonian kind, but leaving nevertheless behind what the dynamical principles really needed for a sound physical theory: first, the *elastic properties* of the medium supporting the light and, secondly, the *interpretation of light* itself, which obviously required the old idea of particle, and therefore the inescapable inertial properties. The Fresnel's ellipsoid of elasticities pretty much fills in for the first aspect of this issue, while the second one was delayed, and left in suspension ever since, being occasionally replaced with *ad hoc* creations of mind, and so is it, actually, even today to a large extent. A proper dynamical use of the second principle of dynamics in the matters of light came in handy only later on, with the advent of the electromagnetic theory of light. This theory of light has in common with the old Fresnel optics the one equation that models the space and time periodical properties no matter of their physical approach, as long as these properties are described by a frequency: *the d'Alembert equation*. Which brings us back to the de Broglie's interpretation, we were about to elaborate on.

2.1 Louis de Broglie's Interpretation

This was, indeed, the starting point of Louis de Broglie in developing his idea of interpretation (de Broglie, 1926). To start with, the optical rules, at the epoch we are discussing now, were considered all concentrated, in a way or another, within the description of propagation of light as a material medium-supported phenomenon, described, for the case of vacuum, by the d'Alembert equation:

$$\Delta u(\boldsymbol{x},t)-\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}(\boldsymbol{x},t)=0 \tag{2.1.1}$$

In this context, de Broglie took notice of the fact that a *classical optical solution* of the equation should be written in the form (1.1.4), but with a special choice of the amplitude and phase:

$$u(\boldsymbol{x},t)=A(\boldsymbol{x})\cdot e^{i\omega[t-\phi(\boldsymbol{x})]} \tag{2.1.2}$$

This signal then must be submitted to some *static* space constraints, obviously mandatory in optics by the presence of screens, diopters, slits or any other kind obstacles that can be incidentally met by light in space. This is exactly the general philosophy outlined a little later by Charles Galton Darwin, only simply applied here in the case of optics. As if in agreement with that philosophy, de Broglie was forced to consider the light as a fluid of particles, for the incarnation of which the best candidate was the idea of photon, 'floating in the air' as it were, for at that time the photon was just about to be 'baptized' (Lewis, 1926). We preserve here the notations of de Broglie himself, by not exhibiting the wave function in the customary ψ notation. The main reason is that the signal (2.1.2) *cannot be* a solution of a nonstationary Schrödinger equation of the type (1.1.6). The same goes for the signal given in the equation (2.1.3) following immediately here, which should be the suitable analogous of the Schrödinger's wave function, and is the one taken by de Broglie as *properly describing a fluid of particles*. And this circumstance is certainly due to the algebraic form of the phase, which is quite particular in fact, and can be considered as a consequence of identifying the phase with the time of propagation, as we have noticed before. However, this form of the signal reproduces that important pattern of periodicity of the waves in physical optics, apparently the only one leading to the eigenvalue equation from which the Schrödinger's idea started unfolding (Darwin, 1927). As a matter of fact, this idea of linear dependence of the phase on time, has its origin in the physical optics, which had the concept of wave built in its own way, utterly neglecting the dynamics, and being, in its turn, utterly neglected by the two kinds of new mechanics arising at the time: wave mechanics and quantum mechanics. This is, in fact, the reason that Louis de Broglie undertook his task of interpretation, to start with: if in the case of light we have the propagation as analogous to a motion, in the case of material points we have nothing analogous to a frequency or a wavelength.

Now, by taking the light quanta as those 'material particles' able to explain, from a classical point of view, the particulate structure of light, de Broglie was also compelled into taking notice of the fact that one needs to assume a solution of equation (2.1.1) having, nonetheless, not only the phase, but also the amplitude time dependent, but of a general manner, not especially linearly, like the phase:

$$u(\boldsymbol{x},t) = f(\boldsymbol{x},t)\cdot e^{i\omega[t-\phi(\boldsymbol{x})]} \tag{2.1.3}$$

Here ϕ is the same function as that from (2.1.2), embodying, as we see it, the earlier fundamental idea of Louis de Broglie that the corpuscles and the representative waves should have *the same phase* (de Broglie, 1923). According to this idea, the presence of one and the same phase is understandable, indeed, as it was given in the two expressions of solutions of the d'Alembert equation, but the question remains as to why should the amplitude be also variable with time in this case.

Louis de Broglie gives an explanation in the English version of the 1926 work (de Broglie, 1926c), and this can be summarized as follows: in keeping with the special relativity, from which his ideas were allegedly springing, an elementary particle helping in the interpretation of light must be mathematically described by a field satisfying the *Klein-Gordon equation*. This equation is what defines, choosing the very words of de Broglie himself, «the wave phenomenon called 'material point'», and it is:

$$\Delta u(\boldsymbol{x},t) - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}(\boldsymbol{x},t) = \frac{\omega^2}{c^2}u(\boldsymbol{x},t); \quad \hbar\omega = m_0 c^2 \tag{2.1.4}$$

The last identity here is the genuine relation put forward in 1923, which, again, was *apparently* prompted by the relativistic mechanics, thus defining a frequency for the classical material point: the *de Broglie's frequency*. As it happens, a fundamental solution of this equation has the general form:

$$u(\boldsymbol{x},t) = \frac{a_0}{\sqrt{x^2+y^2+\gamma^2(z-vt)^2}}\sin\left(\omega[t-(\beta/c)z]\right) \tag{2.1.5}$$

where a_0 is a constant, $\beta \equiv V/c$, $\gamma^2(1-\beta^2) \equiv 1$, and the direction of motion is chosen as an axis of the reference frame, along which the coordinate z is measured. No doubt, the general solution of equation (2.1.4) can be taken as a linear combination of waves of the form (2.1.5), having different amplitudes, which may even involve different velocities with respect to one and the same reference frame. Also no doubt, a general solution of the Klein-Gordon equation may have some other convenient forms, but the one from equation (2.1.5) is the best suited in illustrating a few issues that turned out historically to be essential.

First of all, it shows that, disregarding the problem of phase, an issue may arise from the general point of view of the theory of complex signals. This theory produces, for instance, the equation (1.1.6), but only under condition that the amplitude of such a signal has two values it *must not take*: zero and infinity; two *exceptional values* as they say in mathematics. However, our acceptance for the notion of *exceptional value* here may be in contradiction with that from the current mathematics. This is why we owe a brief explanation: the notion is taken by us in the old connotation, promoted mainly by the works of Paul Montel, of a value *excepted* by the function. As the number of exceptional values always decides the analytic properties of the function, so the idea may be entertained that the wave function proper may have certain exceptional values in order to behave in a certain way. In fact, according to this mathematical theory, in the *ideal case* of light, and with the phase linear in time, the amplitude of the wave (2.1.5) cannot have *but only the two exceptional values*. First, at the event that locates the 'material point' in motion with respect to origin of space coordinates and time, an amplitude like (2.1.5) becomes infinite. Thus, when considering the classical material point a 'wave phenomenon', if this wave phenomenon is classically located as an event, *i.e. interpreted* as in the definition of Darwin, the representative wave has a specific singularity at its location: its amplitude becomes infinite. The phase can, therefore, be anything there, or nothing at all, for it does not even matter in the overall picture. Which is, in fact the way things went with the classical mechanics, to start with.

In other words, by interpretation, the very concept of wave acquires here a differentia, inasmuch as the concept of particle itself may acquire new properties above and beyond its usual classical depiction as a position endowed with mass, or charge, or both: it may be a sharp *singularity of the wave amplitude*, whereby this one becomes infinite. This is a fact that, apparently, embarrassed Louis de Broglie continuously, even to the point where, at a subsequent point of his work, he advanced the ideas of *guiding wave* and *double solution*. The real incentive of this idea can be unveiled by the content of a motivational work, which was published some ten years later than the works we have under scrutiny right now (de Broglie, 1935). We will come back to this issue which, again, proves to be fundamental, later, as we go along with the present work.

Coming back to the problem of exceptional values, there may be, on the other hand, a 'phenomenological' point of view, as we would like to call it, whereby the things regarding the amplitude are to be presented in an entirely different manner, independently of the space and time locations. The wave is here a *light wave*, and it should be the space locus, in a proper geometrical sense, of the events representing the «wave phenomena called 'material points'». The linearity of the Klein-Gordon equation allows a superposition of wave phenomena represented by (2.1.5) *with different velocities*, but there is a problem: as, classically speaking at least, all of the material points move with the speed of light, one must have $\beta \rightarrow 1$, and thus $\gamma \rightarrow \infty$, for all the waves of this type. So, if represented by a linear combination of such 'wave phenomena', the generic wave described by the solution of Klein-Gordon equation, must rather have a *vanishing amplitude*, no matter where the material points interpreting the light phenomenon are located in space and time. In this case too, the phase can be anything, or in fact nothing, for it does not even matter in the picture. However here, it turns out that it matters: this is what differentiates the propagation from a motion!

Therefore, in a word summarizing these observations, one can conclude, indeed, that the classical light would have to be represented, within a d'Alembert equation environment, only by waves having either an amplitude zero or infinity. Seems a desperate situation, but it has a logical explanation from the point of view of *scale transitions*. In hindsight, this explanation can even be taken as the ontological reason that it turned out positively on an account of human knowledge, in general. Namely, the optical recording in diffraction phenomena involves *space accidents* in the way of light, having a space extension closely connected with the wavelength of that light: without such a connection the diffraction phenomenon does not even make its appearance in the recorded light. Now, introduce the point of view of interpretation: the light cannot be interpreted but by ensembles of free particles – photons, luxons, or whatever these may be, nevertheless *free* – at least in the de Broglie's view, allegedly issuing from the special relativity. The word 'free' involves, in the case of a particle, no such space accidents along the moving paths of the interpretative particles. A space accident, or *obstacle*, is the one concept appearing as a reflection of our own experience: it is universal in our experience, *transcending* between the two essential phenomenologies, namely that of light and that of ponderous world. Only, in the experience from which it springs, the obstacle, as a concept, has two fundamental differentiae: either the *absence* of ponderous matter, or the *presence* of ponderous matter. Whence the two extreme possibilities of amplitude involved in the concept of interpretation: either infinite amplitude of the wave attached to an ensemble of interpretative particles, in the case of presence of ponderous matter, or zero amplitude in the case of its absence. Therefore a purely classical description within the framework of the d'Alembert equation, may be taken as having the meaning that a particle is actually a space accident. Such a conclusion may further be taken to indicate the insufficiency of the classical approach of the light description.

Indeed, along this line, we need to come back for the notion of exceptional value, as presented previously, in view of a speculation we find interesting to share with the scientific community. The two exceptional values, zero and infinity, are, according to the discussion right above, referring to what in physical optics comes with the notion of physical obstacles in experiments. This fact may also be taken as meaning that the amplitude of those de Broglie waves actually describing the *physical objects* of the experiments involving the obstacles, must not assume the two values zero and infinity, forasmuch as the physical objects of experimental study are, logically speaking, not the obstacles in experiments, these being part and parcel of the experimental arrangement. In this case, the amplitude functions describing such physical objects have, mathematically speaking, definite analytic properties (Montel, 1912, 1916), whereby one might be able to recognize, for instance, the necessity of a Regge poles theory (Regge, 1959), or even the necessity of the analytic properties involved in the S-matrix theory (Chew, 1968), with the subsequent concept of 'emergence of space order' (Chew & Stapp, 1988). Our own concept of analyticity in connection with the de Broglie's waves, will take an entirely different direction, suggested by the d'Alembert equation, based, however, on the very same line of reasoning in an argument whose bottom line is that the classical description of matter is insufficient.

Coming back to the main story here, this natural philosophical view seems to be endorsed by the historical order of things referring to light. In arguing it, is quite important to take notice that Louis de Broglie's point of view generated the discovery of a *fourth phenomenon* to be added to the phenomenology of light: the *holography*, whose breakthrough followed almost to detail the logic outlined above. Indeed, this is the spirit of the initial work of Dennis Gabor that started the idea (Gabor, 1948). First, this work was precisely directed to the insufficient description of the very phenomenon of difraction by Fresnel theory, more to the point, to its inappropriate connection with the other two phenomena relating to the general classical wave theory: reflection and refraction. As it turns out, the 'inappropriateness' is, in fact, associated with that insufficiency of the classical description that we signaled above, in connection with the 'exceptional values'.

To wit, the operation of electronic devices is obviously based on diffraction, which can be described by de Broglie's idea of correspondence between wave and particle, but with a little twist on proceedings: to consider the obstacles encountered by de Broglie's waves not void of ponderous matter, as in the classical case of the physical optics of light, but, on the contrary, made out of matter formations. These electronic devices admit, through modifications of their objectives, a certain amount of improvement for the image that goes through them. Optimizing the images, means here reducing in aberation and, as Gabor has noticed, in most of the cases this becomes an impossible task from technological point of view: there is an inherent limit of the technological possibilities connected with the case. Dennis Gabor noticed, nevertheless, that this technological approach is, in fact, unnecessary, for there is a principle, even a physical principle actually, involved here, that was not respected *ad litteram*. Quoting:

> The new microscopic principle described below offers a way around this difficulty, as it allows one to dispense altogether with electron objectives. Micrographs are obtained in a two-step process, by electronic analysis, followed by optical synthesis, as in Sir Lawrence Bragg's 'X-ray microscope'. But while the 'X-ray microscope' is applicable only in very special cases, where the phases are known beforehand, the new principle provides a complete record of amplitudes *and* phases in one diagram, and is applicable to a general class of objects. (Gabor, 1948)

A few more words are needed in order to properly understand this excerpt. Notice first that in the association wave-particle here, the electrons of the electronic microscope device are supposed to correspond to electromagnetic waves of *X*-ray type, for which the difraction pattern is connected with the *presence* of matter – atoms, specifically – in periodic crystals (Gabor, 1949). And *X*-rays are, by comparison with the usually perceived light, high frequency, small amplitude waves. Remember that in the case of classical light, such diffraction patterns are obtained when passing the light through pinholes – *i.e.* in the *absence* of matter – and the question arises: what is the connection between the two physical situations? For, in view of the wave-corpuscle duality, the two situations must be the same from a natural philosophical point of view.

Now, Gabor took notice of the fact that an *X*-ray diffraction pattern for crystals can be explained by *the change in phase* of the reflected radiation waves, due to the interaction with the electromagnetic structure of the crystal lattice atoms. Therefore, such a pattern can be explained by the presence of matter, which is certainly not the case for a pinhole. In the case of a pinhole, obviously, we have quite a contrary situation: the matter should count as absent. Naturally, if the de Broglie duality is universal, then there should be a differentia of the wave concept, therefore a universal feature, that reproduces the properties of matter even when this one is absent. Dennis Gabor assumed, and even proved experimentally, that the general case occurs, indeed, when we have a "complete record of amplitudes *and* phases in one diagram". This last part of the proviso is usually overlooked for the particular arrays of atom in a crystal. To wit, it was always considered that only the amplitude of the signal is the one that counts experimentally and, as Gabor himself observes, the emphasis is "somewhat unlucky". Quoting:

> It is customary to explain this by saying that the diffraction diagrams *contain information on the intensities only, but not on the phases*. The formulation is somewhat unlucky, as it suggests at once that *since the phases are unobservables*, this state of affairs *must* be accepted. In fact, not only that part of the phase which is unobservable drops out of *conventional diffraction patterns*, but also *the part which corresponds to geometrical and optical properties of the object*, and which in principle could be *determined by comparison with a standard reference wave*. It was this consideration which led me finally to the new method. [(Gabor, 1949), *our emphasis*]

The fact that phases contain no information on the issues related to the interpretation concept, was the hallmark of theoretical physics until the work of Yakir Aharonov and David Bohm, which stirred up the idea of connection between *potential* and *phase* in wave-mechanical problems (Aharonov & Bohm, 1959). In view of our presentation here, one should be entitled to say that the Gabor's principle is actually a proof, *avant la lettre* as it were, of the Aharonov-Bohm effect. As a matter of fact, a kind of general type of Aharonov-Bohm effect, according to which, given a right theoretical approach…

> One might… expect wave-optical phenomena to arise which are due to the *presence of a magnetic field* but not due to the *magnetic field itself*, i.e. which arise whilst the rays are in field-free regions only [(Ehrenberg & Siday, 1949); *our Italics*]

has been voiced, partly based on Gabor's own previous work, just about the time when he introduced the idea of holography. We shall pursue here that difference between the *presence* and the *action* of the field, which was the mark of electrodynamics from its very beginning. It is instrumental in understanding in what sense has the classical theory of matter to be completed to a general description of it.

For the rest, in order to draw the right lesson from the presence of fundamental phenomenon of holography, we need to notice that the Gabor's target is 'the conventional diffraction pattern', which is incomplete from the point of view of de Broglie's duality wave-corpuscle. The apparent proposal is that the phase should be observable in a hologram, which, therefore, should be just as natural in the phenomenology of light as the diffraction phenomenon itself. The degree of generality of the holographic principle has been noticed by its author from the very beginning:

> *The new principle can be applied in all cases where the coherent monochromatic radiation of sufficient intensity is available to produce a divergent diffraction pattern, with a relatively strong coherent bacground*. While the application to electron microscopy promises the direct resolution of structures which are outside the range of ordinary electron microscopes, probably the most interesting feature of the new method *for light-optical applications is the possibility of recording in one photograph the data of three-dimensional objects*. In the reconstruction, one plane after the other can be focused as if the object were in position, though the disturbing effect of the parts of the object outside the sharply focused plane is stronger in coherent light than in incoherent illumination. *But it is very likely that in light optics, where beam splitters are available, methods can be found for providing the coherent background which will allow better separation of object planes, and more effective elimination of the effects of the 'twin wave' than simple arrangements which have been investigated*. [(Gabor, 1949), *our emphasis*]

The last emphasized part here, was the main object of technological development in the last times. However, the universality of this principle begs the question: *what is the relevant property of waves that makes it work in the world at large*? The answer was provided a long time ago by Hugh Christopher Longuet-Higgins, within the Fresnel's theory of light, as a property of ensembles of damped harmonic oscillators, in a particular take (Longuet-Higgins, 1968). However, in order to properly understand such an answer we need an equally proper view on the idea of scale invariance in the universe, a subject on which we shall dwell later along this work.

Coming back to our line of presentation, the general classical view of interpretation is not spoiled, at least in one of the two cases of extreme amplitude: the physical object under experimental scrutiny can still be described by the two cases of exceptional values of amplitude, and this was in fact one of the paths taken by the wave mechanics. Indeed, classically the trajectory of a material point is, according to its definition adopted also by Darwin in delineating the concept of interpretation, *the locus of successive positions of a material point in motion*.

In a wave representation of 'the phenomenon called material point', the concept of trajectory is still a locus, only 'redefined', as it were, by a 'duality': the locus of those events where the *amplitude of the associated wave vanishes*, no matter of the time sequence of moments and space sequence of locations of these events, connecting the events of a locus where *the amplitude becomes infinite*, therefore the particle is positively present. Therefore, along the *space line* representing (continuously or not: this is the only departure from the genuine classical view!) a trajectory of the 'wave phenomenon representing a classical material point', *the phase should also be arbitrary* according to this optical representation. One positive consequence of this observation, is that it can be taken as a necessity of abandoning the algebraical form of the phase: we may be compelled to adopt a general phase form, not just linear in time, which is what the Madelung interpretation accomplishes, indeed. As we shall see later, the arbitrariness of phase needs to be properly framed within a SL(2,R) group action, in order to account for colors and, more importantly, for describing *the memory* of matter: *the material universe must be holographic*.

For the moment being, we follow Louis de Broglie, who had another observation, more optimistic at that time we should say, in keeping with the positive side of the issues. It is significant that he would follow an old natural philosophical path quite appropriate to the new steps of science at the time, that abides by the words of Arthur Eddington, on the occasion of proposal of Uhlenbeck and Goudsmit, made just about the same historical time, on the spin of the electron taken as part of the complex structures (Uhlenbeck & Goudsmit, 1926). One of the critiques raised against the idea in question was that, in keeping with the classical meaning of the word 'spinning', the electron must be a spatially extended particle that may involve rotations with superluminal speeds. Again, the complex factor $\sqrt{-1}$ came into argument, but this time as a prototype acceptance worth following mathematically. Quoting:

> The mathematical definition of velocity (dx/dt) contains no special reference to motion in a dynamical sense; x is merely the co-ordinate of a selected succession of world-points, and there is in the definition no guarantee that dx is traversed by anything *except the thought of the mathematician*. In describing the electron as spinning, what happens is that, faced with a hitherto unimagined structure, *we make our thought skip faster than light round its boundary*, and by so doing succeed in seeing a correlation with a more familiar structure, namely, that of *an electron at rest*. The *correlating velocity has no more physical existence* than has the factor $\sqrt{-1}$ used to correlate the structure of the four-dimensional world to the more familiar structure of a four-dimensional Euclidean space. In a deeper analysis *we should not speak of a moving charge-element but of a charge-and-current vector*, motion being attributable only to *boundaries or analogous features of charge distribution* – not to *charge* (*original emphasis, n/a*) but to *a charge* (*original emphasis, n/a*). When in the cruder description the charge moves faster than light, the charge-and-current vector J^{μ} becomes *space-like* (*original emphasis, n/a*). [(Eddington, 1926); *our Italics, except as indicated, n/a*]

This philosophy is certainly a sign that, in interpretation, the physics should follow closely the consequences of a theory that may not ask for reality of its concepts described by mathematical symbols. De Broglie used plainly the mathematics of relativity but, on the occasion of the work we are talking about here, his reasoning took another turn, even though along the same general philosophical line. Going a little ahead of us, we can say that this philosophical line has strong historical roots in physics, as we shall se later along our story.

Thus, Louis de Broglie took notice of the fact that if the amplitude function f is to have any *mobile* singularities, like in the case of a proper particle, these have to move *across a surface of constant phase*, particularly normally to such a surface. It is on this occasion that he noticed an opportunity of introducing that improvement on the classical approach of the description of matter, mathematically mandatory in order to get away with the exceptional values of the amplitude, inasmuch as these are, apparently, referring to the experimental arrangement, and do not properly touch the actual object of experimental study. And he took this opportunity, carrying it in a special way that has not only classically sound roots in physics, but opens a case for the knowledge at large. First of all, he took notice of the fact that the speed of a material point in position M at the time t is only 'second hand', so to speak, related to the speed of light entering the Poisson equation: it cannot be identified with the speed of light, as in the classical considerations leading to the case for exceptional values of the amplitude. For once, the fundamental solution of the Klein-Gordon equation may not be necessarily of the form (2.1.5); it can very well be a function of that solution, whose form can be so chosen as to avoid the exceptional values for the amplitude. On the other hand, even if not a function of the form (2.1.5), there are a bunch of other functional forms avoiding the exceptional values for the fundamental solutions of the d'Alembert equation. However, we should not pay too much attention to a case which, from the point of view of the concept of interpretation, may well prove to be simply nonexistent.

Thus, as de Broglie noticed, the speed of a particle located on an 'amplitude surface', as it were, has a mathematical expression depending on the form (2.1.3) of the signal representing the particles:

$$\frac{\partial f}{\partial t}(\boldsymbol{x},t)+v(\boldsymbol{x},t)\frac{\partial f}{\partial n}(\boldsymbol{x},t)=0 \quad \therefore \quad v(\boldsymbol{x},t)=-\left(\frac{\partial_t f}{\partial_n f}\right)_{(M,t)} \tag{2.1.6}$$

with obvious symbols for the partial derivatives in the second expression here. This formula is a simple consequence of the implicit function theorem, allowing us to introduce in the definition a 'guarantee that $d\boldsymbol{x}$ is traversed by a particle', as Eddington would say. The variable denoted here by n is one famous case of 'coordinate' for a one-dimensional 'coordinate space' representing the ray. It is taken *along the very trajectory of the material point* – the symbol n is here intended to suggest the idea of a 'normal' to the surface in question – while the partial derivatives on time, $(\partial_t f)$ and along the normal direction, $(\partial_n f)$, are taken in the position of M at the moment t. In a word, we have a 'coordinate space' here defined as the environment of a point located on a certain surface that cuts the particle trajectories across the ray, inside the tube representing this ray, according to the classical image of this concept.

Now, let us put this definition to work: substitute (2.1.2) and (2.1.3) in equation (2.1.1) and analyze the results. Why not use the Klein-Gordon equation (2.1.4)? The reason is quite simple: light quanta seem to have no mass, and thus the equation (2.1.4) should be avoided momentarily, except for the idea of singularity it suggests, as we discussed above. Now, the results of our substitution are obtained by making the imaginary parts of the algebraic expressions thus obtained vanish: de Broglie respected *explicitly* the classical rigor, which always asks for *real algebraical expressions* in presenting the equations that represent phenomena! Thus, one can find the following equations connecting the optical amplitude A to phase ϕ:

$$\frac{2}{A}\frac{dA}{dn}\equiv\frac{1}{A^2}\frac{d(A^2)}{dn}=-\frac{\Delta\phi}{\partial_n\phi} \tag{2.1.7}$$

and particle amplitude f to phase ϕ:

$$c^2[2(\partial_n\phi)(\partial_n f)+f\Delta\phi]=-2(\partial_t f) \tag{2.1.8}$$

Here the value of the 'normal' derivative is to be identified with the magnitude of the gradient of function along the light ray. Then we simply have, as de Broglie noticed, that the equation (2.1.7) will describe the diffraction phenomena *according to classical physical optics*, while the equation (2.1.8) will describe the diffraction phenomena by interpretation according to wave-mechanical theory, *i.e. by an ensemble of particles*. It should be indeed all about diffraction, forasmuch as we have to deal here with a space locus of events distributed in a space, and not along a classical trajectory *per se*. That space is a 'coordinate space' comparable with the one locally constructed in a reference frame based on Earth's crust surface. Only, the local portion of Earth's surface is replaced in the de Broglie's construction with the portion of the wave surface inside the tube representing the physical light ray. Therefore, this is, indeed, an *interpretation of the wave* in the acceptance of the very definition given by Charles Galton Darwin, but 'served' with all the required cutlery, as it were.

Now, in the French version of his work, de Broglie comes with a strange statement, aparently out of nowhere, and this is what we take as the real point of departure from the classical description of the matter. Namely, he assumes that if, as one *approaches at constant time* a light particle following its trajectory, the function f – the amplitude associated with the particles in interpretation – varies *as the reciprocal distance to that particle*, then in the position M of the particle the *ratio between f and $(\partial_n f)$ vanishes*. We find the oddity not in the fact that, if this approach is to be considered as 'physical', it should be done with a speed way greater than that of light: after all, such an approach may take place, according to Eddington's philosophy, only 'in de Broglie's mind', as it were. This may be the case indeed, for what strikes us is that *precise characterization* of the variation of the amplitude of the wave representing the particle: 'as the *reciprocal distance* to the particle', taking place *at constant time*, *i.e. instantaneously*. According to classical philosophy, if this approach is to be physical, it should be done with an *infinite velocity*. Besides, from the same point of view we would have a possibility to describe the 'presence of the particle in the wave', as it were, by a singularity with respect to the distance to the wave surface along the ray. Which, by the way, was an idea closely followed later by de Broglie himself.

Otherwise, however, the requirement appears as not strange at all, for it is just natural, mathematically speaking. Indeed, *the reason* for such a condition is quite obvious, inasmuch as it comes inherently with the equation (2.1.8), and becomes obvious if we write it on the form:

$$c^2(\partial_n\phi)+\frac{1}{2}\left(\frac{f}{\partial_n f}\right)\Delta\phi=-\frac{\partial_t f}{\partial_n f}$$

Using equation (2.1.6), and de Broglie's requirement in the form: $f/(\partial_n f)\to 0$, involving, as it is, no specific space behavior of f – that is, none other than the one of satisfying this condition – the formula reveals that *the phase should be a potential of velocities of light quanta*, *i.e.* it should have *the very classical role of the variable of action*. Thus, we can have a proof of the second condition to be secured for an interpretation of the complex signal in optics, which we listed in the beginning of the present section:

$$v(\boldsymbol{x},t)=c^2(\partial_n\phi)_{(M,t)} \tag{2.1.9}$$

We can take this formula as saying that the stream velocity of the light particles along a ray is the gradient of the phase of signal representing the light wave, *irrespective of the space behavior of that phase*, provided it is an optical phase – *viz.* it is linear in time, in order to satisfy the d'Alembert equation – and the amplitude of the wave behaves in the manner claimed in the de Broglie's request.

Thus, the case of light quanta proves that Schrödinger was right in choosing a linear phase for his wave function, and the only thing left unfinished for a proper interpretation of optics would be the overall construction of *a physical light ray*. As the concept of ray is classically understood, that is, as it was left by Hobbes, Hooke and Newton – to wit, as a thin pencil of trajectories of classical material points – this construction can be performed right away. Indeed, de Broglie came up with the idea that a *thin tube* confining an ensemble of trajectories of light particles would be able to do the job. Thus, the classical Newtonian image – or to be more precise, the Hooke-Newton image [see (Mazilu & Porumbreanu, 2018) for an extended account of historical order of development of this idea] – of the physical light ray takes, with de Broglie's completion, a geometrically more precise modern shape: *a generalized cylinder – more to the point, a canal surface – whose area of any transversal section is variable with the position along the ray*. For instance, the classical planetary atom with space extended physical components is composed of two such classical rays: a *spherical* one representing the central nucleus, and a *toroidal* one, representing the electron.

And so it comes that, in order to make it 'physical', de Broglie literally assumes a physical tube: he *assimilates a physical ray with a capillary tube of variable cross-section* σ, and thus he is bound to describe this tube by the known physical principles of the theory of capillarity. For once, assuming that the flux of light particles is conserved along the ray – an assumption that can, in general, be taken as the fundamental attribute of interpretation *via* the concept of ray within the theory of fluids – the equation representing this situation:

$$\rho v\sigma = const \tag{2.1.10}$$

should be satisfied, where ρ means the Newtonian volume density of the particles of light. Taking the logarithmic derivative in the direction of the ray, one can find

$$\frac{1}{\rho}\frac{\partial\rho}{\partial n}+\frac{1}{v}\frac{\partial v}{\partial n}+\frac{1}{\sigma}\frac{\partial\sigma}{\partial n}=0 \tag{2.1.11}$$

By a "known theorem of geometry", as de Broglie says, one can calculate the last term of this sum, which proves to be all he needs for getting an expression of the variation of density along the ray. Now, a problem pops up, compelling us to a choice: inasmuch as the classical physical ray is a space construction, it would be hard to decide the meaning of $\partial/\partial n$. Is it effectively variation *along the ray* itself, or *along the normal to the wave surface* taken as reference, as de Broglie assumes!? We take the word of de Broglie *ad litteram*, *i.e.* choose the meaning given by a variation along the normal, and with a good reason at that. It is, indeed, only in this case that we can take advantage of that 'known theorem' de Broglie alluded to, and according to which the last term in (2.1.11) is the double of the mean curvature of the surface $\phi = const$ in a given position (Poincaré, 1895). That quantity has as expression the sum of the principal curvatures of the surface, and as the mean curvature is the average of these curvatures, we have (Mazilu, Agop, & Mercheș, 2019):

$$\frac{1}{R_1}+\frac{1}{R_2}=\frac{\Delta\phi-\partial_n^2\phi}{\partial_n\phi} \tag{2.1.12}$$

Here $R_{1,2}$ are the radii of curvature of the principal sections of the surface. With (2.1.9) and (2.1.12), the equation (2.1.11) now takes the form

$$\frac{1}{\rho}\frac{\partial\rho}{\partial n}=-\frac{\Delta\phi}{\partial_n\phi} \tag{2.1.13}$$

and comparing this with (2.1.7), one finds right away:

$$\rho = const \cdot A^2 \tag{2.1.14}$$

Therefore, Louis de Broglie has also a proof of the first one from the two conditions to be secured in order to give the right interpretation of the wave function in the particular case of light: the Newtonian density of material points – the 'quants', in words of de Broglie – is given by *the squared amplitude of the wave function*, as classically revealed in diffraction phenomena, with no reference to the state of particles serving for interpretation: free, bound etc. A continuity equation can then be written with the field of velocities (2.1.9) and the density (2.1.14), as a general expression of conservation of *the number of quanta*, thus generalizing flux conservation (2.1.10). His conclusion, expressed in the French version of the work is referring strictly to the result just presented, without any comment on the wave amplitude to be considered:

> *The density of the light quanta is proportinal to the intensity of the classical theory*. The interference and diffraction phenomena are therefore accurately explainable by the corpuscular conception of light. [(de Broglie, 1926b); *our translation*, *original emphasis*]

The very same observation can be made about the conclusion of the English version of the work, where we can find the de Broglie's final conclusion in complete accordance with the Darwin definition of interpretation:

> *The density of light quants is to be taken proportional to the classical intensity.* In the dark fringes of the classical theory, the density of quants will be zero, but in a bright fringe a great number of quants will pass. Now, the motion being permanent, this explanation of the experimental facts will still be available if the light is very weak (Taylor's experiment); we have only to define the density of quants by a time average instead of a space average. [(de Broglie, 1926c); *original emphasis*]

No mention of the nonclassical amplitude f, 'forgotten', as it were, as soon as it provided the definition of the field of velocities. The theory has no clue about the direct connection between the two amplitudes, even though such a connection is utterly necessary. However, the conclusion to English version of the work excerpted right above, together with that condition of de Broglie that we deemed 'strange', seem to indicate such a connection, appropriate enough in order to satisfy the explanation that will follow here right away.

The condition in question can be written in a closed form, *i.e.* a form that involves an equality and therefore an equation can be defined, as follows:

$$f(\boldsymbol{x},t) : \partial_n f(\boldsymbol{x},t)\Big|_{t=const} \propto r \quad \therefore \quad r\frac{\partial f(r)}{\partial r} = \alpha f(r) \tag{2.1.15}$$

Here α is a real constant, and the convention is made that the variable r is that "distance to the particle" along the ray, as required by de Broglie, which is to be calculated somehow in the reference frame centered on 'the particle', from the position $\boldsymbol{x}$ and the moment of time t when this particle is to be considered. What exactly has to be thus located, *remains unspecified* – is it a particle, is it another kind of event, or what?! – and, of course, in dire need to be properly specified later. Getting a little ahead of us here, we need to mention though that this is the place where we can see the necessity of that condition of holography that got through later in the form given to it by Dennis Gabor, whose approach to the theory seems to be the right one, once it was so brilliantly confirmed by the technology. Anyway, the equation (2.1.15) tells us that the amplitude $f(\boldsymbol{x},t)$ should be homogeneous of degree α in the distance r. The case first chosen by de Broglie in the French version of his work is $\alpha = -1$, but as the theory is well served with *any* real value of that constant, not to mention a host of other functional forms for *f*, we found this particular choice… 'strange'. It is the explanation of de Broglie that induced us in so characterizing the choice, in the first place:

> *By reasons that I cannot show here*, (*our emphasis, n/a*) it is likely that, if one approaches *at constant time* the particle of light following its trajectory, the function *f* goes as the inverse of the distance to the particle…

In the English version, those "reasons" are delegated to the "*analogy* with the solution for the free spherical material point". Certainly the procedure needs an explanation, which, we think, may go, up to a point, along the following lines, in need of further elaboration, to be done later on.

After 'making the imaginary parts vanish', a procedure which leads to the equations (2.1.7) and (2.1.8), all that is left from the equation (2.1.1) is the equation

$$\nabla^2 A(\boldsymbol{x}) = 0 \tag{2.1.16}$$

for $A(\boldsymbol{x})$, and the equation

$$\nabla^2 f(\boldsymbol{x},t) = \frac{1}{c^2}\frac{\partial^2 f(\boldsymbol{x},t)}{\partial t^2} \tag{2.1.17}$$

for $f(\boldsymbol{x},t)$. In other words, *A* must be a solution of the Laplace equation, while *f* must be a solution of the d'Alembert equation. $A(\boldsymbol{x})$ has to be identified with a solution representing the result of recording in a problem of diffraction, no doubt about that. Also no doubt, $f(\boldsymbol{x},t)$ is to be taken at a given moment of time *t*, which is to be kept constant. Certainly such a solution of d'Alembert equation can be identified with a solution of the Laplace equation, however, not in the place located by the $\boldsymbol{x}$ coordinates, *but in a place correlated with it through an* '*affinity*', as it were, represented by a Lorentz transformation. This suggests that the function $A(\boldsymbol{x})$ has to be connected with $f(\boldsymbol{x},t)$ by a sort of 'application'. In this respect, it is quite important to take notice that the idea of 'application' followed de Broglie all along his life and work, even to the point where he introduced the concept of *application function*, that we have to discuss here in due time (de Broglie, 1935). For the rest, de Broglie tells us that this kind of interpretation of the light continuum takes place in the case of dim light, with reference to a 'Taylor's experiment'. Again, no doubt, this reference sends us to the only work in epoch that replicates, in a usual environment, the conditions in which Albert Einstein says that 'radiation of low density behaves as though it consisted of a number of independent energy quanta' (Einstein, 1905b). The 'Taylor experiment' is the first and only work on physics of the celebrated fluid mechanics theorist Geoffrey Ingram Taylor, a pupil of J. J. Thomson at Cambridge, probably performed at the recommendation of his master (Taylor, 1909).

Now, any solution of equation (2.1.16), described in spherical polar coordinates, plainly satisfies the condition requested by de Broglie, even though maybe not exactly in the form given in equation (2.1.15), but through a more complicated 'application function'. A solution of equation (2.1.17) at constant time also does the job, indeed, however not directly but, as we have just noticed, only after a proper affinity represented by a Lorentz transformation is provided. The problem arises explicitly, about the connection between the two solutions *in the general case*. This is a big problem, indeed, judging by what we have to solve here: we are not only hindered by the necessity of finding a proper 'realization' of the Lorentz transformation that would provide *A* given *f*, or *vice versa*, but also by the fact that such a realization must *transcend the inertia*, in order to allow us using the equation (2.1.17) for what it is really intended. Let us see how this can happen.

2.2 A Fluid Description of de Broglie's Interpretation

It is time now to produce a first fact from the physics involved in the de Broglie's interpretation of waves by means of the concept of ray, as presented up to this point. If it is that the theory of fluids should be put to work in order to secure a foundation for physics, as some contemporary theories rightly emphasize (Jackiw, Nair, Pi & Polychronakos, 2004), then it certainly has to cover issues of classical extraction, among which, first and foremost comes the inertia. Now, the question is: where is the inertia entering the stage in the de Broglie's act of interpretation, and what it accounts for? According to classical natural philosophy, a particle serves for indicating a space location, and this is one of the essential points of the interpretation concept as defined by Darwin. From this point of view, the inertia may well be taken as that "innate force, whatever it is", to use one of Newton's expression, that makes the particle stick with the location it defines in a certain space. This may not be an exhaustive definition of a particle, as we shall show later in this work, but it is certainly rational. Therefore, the question that pops up right away should be the following: *is the theory of fluids able to replace the classical idea of inertia*, which appears as the only property that differentiates between physical structures from the classical point of view? As we have seen, in the definition of a general concept of obstacle, the presence or absence of matter is instrumental, and the phenomenology of light, as completed according to the de Broglie's line of ideas, proves it plainly. If this turns out to be the case, then there should be a general property of matter that reduces to inertia in particular cases. The de Broglie's interpretation above seems to indicate a proper approach to solving this problem.

Start with that condition of Louis de Broglie, that we labeled as 'strange' before, and consider it from the point of view of the concept of interpretation: the description of a 'particle' should serve, before anything else, interpretation purposes. Thus, as *one – whatever this 'one' is – approaches* the particle serving for interpretation, *at constant time*, the ratio between amplitude f and its derivative along the ray goes to zero. Now, if a particle goes with a wave along the ray, in the de Broglie's manner described above, we shall have to consider a coordinate space described as the portion of the wave inside the tube containing the ray. The geometry of this space can be described in what seems to be a natural system of coordinates: the two coordinates on the de Broglie surface, and one coordinate along the ray. It is in this coordinate system that we have to apply the considerations that follow.

From the point of view of the theory of fluids, the description of a certain ray goes as follows [(Pierce, 2007); see especially Section 3.16 on the *Ray Acoustics*]: a ray velocity is defined by the formula

$$\boldsymbol{v}_{ray} \equiv \frac{d\boldsymbol{r}}{dt} = \boldsymbol{V} + c\hat{\boldsymbol{u}} \qquad (2.2.1)$$

where $\boldsymbol{V}$ is the velocity of the matter as a whole, c is the speed of the perturbation described based on particles – to wit, the sound in the specific case described by Allan Pierce – and $\hat{\boldsymbol{u}}$ is the *unit* vector of the direction of propagation of that perturbation of fluid. If the wave front of perturbation is described by an equation of the form $t = \tau(\boldsymbol{r})$, then we can write

$$\boldsymbol{r}(t+dt) \simeq \boldsymbol{r}(t) + \boldsymbol{v}_{ray}(t)dt \qquad (2.2.2)$$

so that the equation of the wave front becomes

$$t + dt = \tau[\boldsymbol{r}(t) + \boldsymbol{v}_{ray}(t)dt] \simeq \tau(\boldsymbol{r}) + [\boldsymbol{v}_{ray}(\boldsymbol{r}) \cdot \nabla\tau(\boldsymbol{r})]dt + \ldots \qquad (2.2.3)$$

This shows that the gradient of the wave front is a vector that in the previous approximations should be the inverse to the ray velocity with respect to the unit sphere:

$$\boldsymbol{V}_{ray}(\boldsymbol{r})\cdot\boldsymbol{s}(\boldsymbol{r})=1;\quad \boldsymbol{s}(\boldsymbol{r})\equiv\nabla\tau(\boldsymbol{r}) \tag{2.2.4}$$

This vector is usually called *slowness* of the perturbation in the fluid, and its definition entitles us to consider it as a local *normal vector* to the wave front described – more to the point, *interpreted* – as a set of contemporary positions of the particles of fluid, by an equation of the form $\tau(\boldsymbol{r})$ = constant. In view of equation (2.2.1) the slowness vector can be written as

$$\boldsymbol{s}(\boldsymbol{r})=\frac{\hat{\boldsymbol{u}}}{c+\boldsymbol{V}\cdot\hat{\boldsymbol{u}}}\quad\therefore\quad s(\boldsymbol{r})=\frac{1}{c+\boldsymbol{V}\cdot\hat{\boldsymbol{u}}}=\frac{1-\boldsymbol{V}\cdot\boldsymbol{s}}{c} \tag{2.2.5}$$

where in the last equality we have used the definition (2.2.4) of the slowness field. In terms of this field, the equation (2.2.1) for the definition of the ray velocity becomes

$$\frac{d\boldsymbol{r}}{dt}=\frac{c^2}{\Gamma}\boldsymbol{s}+\boldsymbol{V};\quad \Gamma\equiv 1-\boldsymbol{V}\cdot\boldsymbol{s} \tag{2.2.6}$$

With this, we have, for each particle of the fluid, an equation of motion for the position $\boldsymbol{r}$ in terms of slowness. Now, if we succeed in writing another equation of motion for the time derivative of slowness in terms of position, then we can have a phase space for the fluid, and one can say that this phase space is enacted by the ray structure of the fluid. In order to do that, start with the observation that the general equation for slowness is

$$\frac{d\boldsymbol{s}}{dt}=(\boldsymbol{V}_{ray}\cdot\nabla)\boldsymbol{s}\quad\therefore\quad\frac{d\boldsymbol{s}}{dt}=c(\hat{\boldsymbol{u}}\cdot\nabla)\boldsymbol{s}+(\boldsymbol{V}\cdot\nabla)\boldsymbol{s} \tag{2.2.7}$$

In view of the fact that, by its definition $\boldsymbol{s}$ is a gradient, its curl should be zero, which means the equation

$$\frac{1}{2}\nabla(\boldsymbol{s}^2)=(\boldsymbol{s}\cdot\nabla)\boldsymbol{s} \tag{2.2.8}$$

a relation that can be obtained as a benefit of the general vector analysis identity: $\nabla(s^2)\equiv 2\,[\boldsymbol{s}\times(\nabla\times\boldsymbol{s})+(\boldsymbol{s}\cdot\nabla)\boldsymbol{s}]$. Using this relation in (2.2.7), we end up with the following equation for $\boldsymbol{s}$:

$$\frac{d\boldsymbol{s}}{dt}=-\Gamma\frac{\nabla c}{c}-\nabla(\boldsymbol{V}\cdot\boldsymbol{s})+(\boldsymbol{V}\cdot\nabla)\boldsymbol{s} \tag{2.2.9}$$

with Γ defined as in second of the equations (2.2.6). Now, the first equation (2.2.6) together with (2.2.9) describe the space filled with fluid. In case the fluid is *stationary* ($\boldsymbol{V}=\boldsymbol{0}$), this pair of equations comes down to

$$\frac{d\boldsymbol{r}}{dt}=c^2\boldsymbol{s};\quad\frac{d\boldsymbol{s}}{dt}=-\frac{\nabla c}{c} \tag{2.2.10}$$

giving us the possibility of a 'mind describing' – to use an expression close to the observation of Arthur Eddington, excerpted above – of the condition of de Broglie, namely of 'approaching the wave phenomenon called material point at constant time'. We can get a description in a *space of positions* by eliminating the slowness vector between these two equations. Assuming, as apparently Louis de Broglie did with his hypothesis of 'approaching at constant time', that the velocity c does not depend on time, but is only a function of position, we have the equation

$$\frac{d^2\boldsymbol{r}}{dt^2}=-\frac{1}{2}\nabla(c^2) \tag{2.2.11}$$

In a theory of interpretation, as this concept is defined by Darwin, this is an *acceleration merely due to the perturbation, viz. to a wave*: a kind of acceleration necessary in describing, for instance, the propagation of gravitation that Poincaré once wanted to formalize, pursuing the example of electromagnetic waves [(Poincaré, 1906), §9].

Therefore, between bodies stationary with respect to one another in a regular space – like in the case of star system of Newton, or of the molecules of a stationary ideal gas – *the acceleration is not decided exclusively by inertial properties*: there should be some more fundamental – if we may be allowed the license of such an expression – *decision principle* than the Mach's principle. This conclusion is, in fact, in line with the phenomenology of classical natural philosophy: mechanically, the acceleration is decided by collision forces, acting incidentally. This may even be taken as an incentive for a revision of both theory of gravitation and the classical mechanics concurrently (Bekenstein & Milgrom, 1984). It is thus quite fair to assume that the principle of inertia is nothing but only a particular case of a *universal natural-philosophical principle* allowing us to introduce the forces that Newton once invented, along with a universal kinematical condition. Such a universal natural-philosophical principle should be equally referring to the waves, as shown above, not only to particles.

Indeed, *de Broglie's concept of interpretation* – let us call it this name, inasmuch as it is certainly an interpretation according to the definition of Darwin, but specifically done with the aid of the ray concept – allows us to say that, proceeding in the exact manner Newton did, the gradient in the right hand side of (2.2.11), can be taken as a measure of the curvature of a certain portion of wave surface in the de Broglie capillary tube. Therefore, at least in some restricting hypotheses, if not anyway, we should be entitled, for instance, to assume that c^2 is analogous to a potential:

$$c^2 \equiv \frac{2K}{r} \tag{2.2.12}$$

where K is a physical constant, and r is the magnitude of the position vector $\boldsymbol{r}$. We thus get from (2.2.11) the precise equations of motion of the Newtonian dynamics, *with no mention of any inertia*. Before settling for a *modus operandi* of such a theory, let us put forward some geometrical speculations that may be in position to illuminate us on the nature of speed we denoted by c above. To be more precise, by comparison with the historical facts, such speculations can be taken at least as heuristical and, as we see it, this is actually their grand merit.

First of all, using the ansatz (2.2.12) in equation (2.2.11) that actually suggests it, we have the well-known equation resembling to the dynamical equation serving to model the classical Kepler motion:

$$\frac{d^2\boldsymbol{r}}{dt^2} = \frac{K}{r^3}\boldsymbol{r} \tag{2.2.13}$$

If we refer this acceleration to a force, then the force that would be able to generate it according to the second principle of dynamics must be a *force of repulsion*. The trajectory which results from the integration of equation (2.2.13) is a plane conic, indeed, just like in the case of the classical Kepler motion, however not an ellipse, but a branch of hyperbola for instance, as in the case of the alpha particles that once led to the idea of planetary atom (Rutherford, 1911). For once, this fact speaks plainly for the case of a purely Newtonian explanation of the classical concept of physical ray of light in terms of a 'center of force' located in the source of light (Mazilu & Porumbreanu, 2018). It is worth insisting on this important issue with an explanation, even at this point, in view of its importance of principle. The essential mathematics and the connected physical philosophy will be presented later on, in a deeper detail.

Fact is that the Newtonian forces defined by the Kepler motion as observed in the sky, can have *two* possible algebraical equations that define their magnitude [(Mazilu, Agop, & Mercheș, 2020), Chapter 5]:

$$F_1(\boldsymbol{r}) = \frac{\mu r}{(\alpha x^2 + 2\beta xy + \gamma y^2)^{3/2}} \quad \textit{and} \quad F_2(\boldsymbol{r}) = \frac{\mu r}{(\boldsymbol{a}\cdot\boldsymbol{r} + b)^3} \tag{2.2.14}$$

depending on how the equation of the orbit of that motion *is established*. Here x and y are the coordinates of the moving point with respect to the *center of force*; they are considered as components of the vector $\boldsymbol{r}$. Let us explain in a little more detail what we mean here by 'establishing' an equation for the orbit of motion. The trajectories corresponding to the central forces above, written in their implicit algebraical forms are, respectively:

$$\boldsymbol{\alpha}x^2 + 2\boldsymbol{\beta}xy + \boldsymbol{\gamma}y^2 = (ax+by+c)^2 \quad and \quad \boldsymbol{u}^2x^2 + 2(\boldsymbol{u}\cdot\boldsymbol{v})xy + \boldsymbol{v}^2y^2 = (\boldsymbol{a}\cdot\boldsymbol{r}+b)^2 \tag{2.2.15}$$

Here the parameters a, b, c, as well as the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, are determined by the *initial conditions* of the motion, specifically by the components of the initial velocity, while α, β, γ and the vector $\boldsymbol{a}$ are *innate characteristics of the field of force*. The vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ satisfy to a geometrical condition showing that their vector product is fixed. The *center of force* is the point of intersection of the tangent lines to the orbit, which are determined by the quadratic forms from the left hand side of these equations, according to equations

$$\boldsymbol{\alpha}x^2 + 2\boldsymbol{\beta}xy + \boldsymbol{\gamma}y^2 = 0 \quad and \quad \boldsymbol{u}^2x^2 + 2(\boldsymbol{u}\cdot\boldsymbol{v})xy + \boldsymbol{v}^2y^2 = 0 \tag{2.2.16}$$

respectively: the two tangent lines are given by the two linear forms, factors of these quadratic forms. Their point of intersection, marked in the plane of motion as the center of force, is *the pole* with respect to orbits, of the straight lines given by the right hand sides of the equations (2.2.15), and determined according to equations:

$$ax+by+c=0 \quad and \quad \boldsymbol{a}\cdot\boldsymbol{r}+b=0 \tag{2.2.17}$$

Now, the general geometrical theory of a conic, shows that its algebraic equation ca be established by a triangle: it is tangent to two of the sides of this triangle, in the point of their intersection with the third side. The above short review of the connection between the equations of Newtonian forces and those of the orbits allegedly determined by these forces – by the way, the determination we are talking about is done according to the laws of classical dynamics – shows that the equation of the magnitude of force can play different roles in establishing the equation of the orbit: it can either determine the two sides of the characteristic triangle (the *conjugate triangle*, as they usually say in geometry) tangents to orbit, or it can determine the straight line joining the two points intersection of the orbit with the tangents from the center of force (the *directrix*, as they call it in geometry). The rest of the necessary conditions for determining the algebraic equation of the orbit, is delegated to the initial conditions of the dynamical problem: in the first case these decide the directrix of the orbit, while in the second case they decide the tangents to the orbit, and therefore the position of the center of force. Therefore, according to classical dynamics, the Newtonian forces having the magnitude inversely proportional with the square of distance, are obtained only in the first case of (2.2.14) for $\alpha = \gamma$ and $\beta = 0$, in whereby the conical character of the orbit is exclusively decided by the initial conditions of the corresponding dynamical problem. On the other hand, the isotropic elastic forces come with the second case (2.2.14) for the vector $\boldsymbol{a}$ null. But there is a catch!

Indeed, in this last case we are under the dominance of the *a priori* condition:

$$(\boldsymbol{u}\cdot\boldsymbol{v})^2 - \boldsymbol{u}^2\cdot\boldsymbol{v}^2 \le 0 \tag{2.2.18}$$

This should be satisfied for any pair of real vectors whatsoever. Therefore the two tangents to orbit – established, in the corresponding dynamical problem, by the *initial conditions* of the motion – are complex straight lines: their coefficients are "essentially complex", as Schrödinger would say. The only thing real in the given case is the intersection of these two lines: two complex lines with complex conjugated coefficients always cut each other in a real point, as a simple calculation can assure us. Therefore, only *the center of force is real*, and is inside the region from the plane of motion delimited by the orbit. The directrix of the orbit is external to it, the tangency points are complex, more precisely complex conjugated to each other. This is plainly the case of Newton's starting point in the mathematics of natural philosophy. Mention should be made, however, that this is also the case of Fresnel's light, but with a *special gauging*, introduced to scientific comprehension a long tome ago, by James Mac Cullagh, considered by some as the forefather of Maxwellian electrodynamics, and for the right reasons at that [(Mac Cullagh, 1831); see also (Darrigol, 2002, 2010) and (Mazilu & Agop, 2012)].

Speaking of James Mac Cullagh and his gauging procedure, the the main point of this digression is the first case from equation (2.2.15), whereby the tangents can be real: this is the fashion Mac Cullagh himself imagined the action of matter on the light, and it completely satisfies the Newtonian geometrical philosophy of forces as sketched above. In this case, though, the center of force is outside the orbit, and one can imagine the two tangent lines intersecting each other in a point that is the source of light. Then the lines themselves are two mathematical rays of a conical surface delimiting a physical ray constructed in the manner of Thomas Hobbes. The plane figure involving the conic itself can be, indeed, obtained by the intersection of the conical congruence of lines 'with a plane through its axis'. This construction, made plainly in the Newtonian geometrical spirit, is not at all dynamical though, but can be described purely kinematically, and can be based on the equation (2.2.13), just as its classical dynamical counterpart. Thereby that equation needs to be framed into another way of thinking: *the quantization*.

First, considering a little geometry, the time surface $\tau(\boldsymbol{r})$ can be taken as an affine surface: the vector $\boldsymbol{s}$ defined in equation (2.2.4) is, as we said, simply the normal to this surface. Then the dot product $(\boldsymbol{s}\cdot d^2\boldsymbol{r})$ is just as simply the second fundamental form of this surface, up to a factor. This is a quadratic form in the local coordinates on time surface, representing the 'height' of that surface with respect to a local tangent plane. Consequently it may be considered a *third coordinate* of a system of coordinates to be used in the description of the space from the vicinity of a point of the time surface. This is why we deem as appropriate considering it a little closer here. We can calculate it from equation (2.2.13), by dot multiplying with $\boldsymbol{s}$:

$$\boldsymbol{s}\cdot d^2\boldsymbol{r} = \frac{K}{r^3}\boldsymbol{s}\cdot\boldsymbol{r}\cdot(dt)^2 \tag{2.2.19}$$

Then using again the first equation from (2.2.10) and the equation (2.2.12), we get a 'constant of motion' in the form:

$$\left(\frac{d\boldsymbol{r}}{dt}\right)^2 + c^2 = const \tag{2.2.20}$$

In order to discern an interpretation for this constant, we need to detail a three-dimensional calculus, mostly in order to reveal a few differentiae regarding the concepts of *reference frames* and *coordinate systems* in general.

2.3 An Appropriate Differential Geometry of Ordinary Space

The geometry of a three-dimensional space can be written in the language of position vectors. The set of all position vectors, with their origin in the same position in space, may be represented in the form

$$\boldsymbol{r} = r\cdot\hat{\boldsymbol{e}}_r \tag{2.3.1}$$

Here r is a length assigned to the position vector in any Cartesian reference frame having its origin in the given position, while $\hat{\boldsymbol{e}}_r$ is the unit vector orienting this length, in order to make a vector out of it. Now, in equation (2.3.1) we can, in fact, assume a spherical polar coordinate system, adapted to any one of the possible Cartesian reference frames in which this very equation is written. For, there are, indeed, a double infinity of such reference frames, differing from one another by arbitrary rotations in space. Therefore, in a matrix form, using implicitly one such generic Cartesian reference frame, the equation (2.3.1) should be written in the form

$$\boldsymbol{r} \equiv |x\rangle = \begin{pmatrix} x \\ y \\ z \end{pmatrix}; \quad \hat{\boldsymbol{e}}_r \equiv \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix} \tag{2.3.2}$$

In most of the applications connected to the physics of matter in space, like, for instance, in the kinematics generated by the equation (2.2.13) above, only the first- and second-order *symmetric* differentials are of importance. We use the term 'symmetric differential' here, only as in classical dynamics, *i.e.* as opposed to *exterior differential*; the meaning will become clearer as we go along with our argument. The first order differential of the position vector, which is always 'symmetric' by its very nature, can be calculated directly from (2.3.1), using the usual differentiation rules:

$$d\boldsymbol{r} = (dr)\hat{\boldsymbol{e}}_r + r(d\hat{\boldsymbol{e}}_r) \tag{2.3.3}$$

In order to calculate the differential of the unit vector of orienting the position vector, we use the second of the definitions in (2.3.2), with the result

$$d\hat{\boldsymbol{e}}_r = (d\theta)\hat{\boldsymbol{e}}_\theta + (\sin\theta d\varphi)\hat{\boldsymbol{e}}_\varphi \tag{2.3.4}$$

where the two new unit vectors from the right hand side here are two orthogonal vectors given by the matrices

$$\hat{\boldsymbol{e}}_\theta \equiv \begin{pmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta \end{pmatrix}; \quad \hat{\boldsymbol{e}}_\varphi \equiv \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix} \tag{2.3.5}$$

These two unit vectors, together with the one defined in (2.3.2) form a generic Euclidean reference frame, referred to spherical polar coordinates, the one in which the vector $\boldsymbol{r}$ is defined as the matrix $|x\rangle$ from equation (2.3.2). Using (2.3.4) in (2.3.3), the differential of the position vector becomes:

$$d\boldsymbol{r} = (dr)\hat{\boldsymbol{e}}_r + (rd\theta)\hat{\boldsymbol{e}}_\theta + (r\sin\theta d\varphi)\hat{\boldsymbol{e}}_\varphi \tag{2.3.6}$$

The square of this differential of the position vector is given by the inner product:

$$d\boldsymbol{r}\cdot d\boldsymbol{r} = (dr)^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \equiv (dr)^2 + r^2(d\Omega)^2 \tag{2.3.7}$$

with an obvious definition for $(d\Omega)^2$, as the metric of unit sphere. We recognize here the regular Euclidean metric, written in polar spherical coordinates. One can verify the Frenet-Serret equations, describing the variation of Euclidean frame made of the above unit vectors associated with the spherical coordinate system:

$$\begin{pmatrix} d\hat{\boldsymbol{e}}_r \\ d\hat{\boldsymbol{e}}_\theta \\ d\hat{\boldsymbol{e}}_\varphi \end{pmatrix} = \begin{pmatrix} 0 & d\theta & \sin\theta d\varphi \\ -d\theta & 0 & \cos\theta d\varphi \\ -\sin\theta d\varphi & -\cos\theta d\varphi & 0 \end{pmatrix} \cdot \begin{pmatrix} \hat{\boldsymbol{e}}_r \\ \hat{\boldsymbol{e}}_\theta \\ \hat{\boldsymbol{e}}_\varphi \end{pmatrix} \tag{2.3.8}$$

This equation helps in establishing the *second symmetric differential* of the position vector used in calculating the acceleration. Indeed, from equations (2.3.6) and (2.3.8) we have

$$\begin{aligned} d^2\boldsymbol{r} = {} & (d^2r - rd\Omega^2)\cdot\hat{\boldsymbol{e}}_r \\ & +(rd^2\theta + 2drd\theta - r\sin\theta\cos\theta d\varphi^2)\cdot\hat{\boldsymbol{e}}_\theta \\ & +(r\sin\theta\, d^2\varphi + 2\sin\theta dr d\varphi + 2r\cos\theta d\theta d\varphi)\cdot\hat{\boldsymbol{e}}_\varphi \end{aligned} \tag{2.3.9}$$

In the system of Newtonian dynamics, this vector represents the acceleration, of course when adequately related, as in equation (2.2.13), to a continuity parameter playing the part of the time of problem. Momentarily, we do not proceed like that, but will stay within the words of Eddington, keeping the time only *in our mind*, and discuss the general case of the differentials. However, by abusing a little of the advantage of classical terminology, the components of the vector (2.3.9) will still be designated as ‘accelerations’, just as the components of vector (2.3.6) will be called ‘velocities’.

In the case of classical free particle, the components of acceleration vanish, a condition that, considered as universal, *i.e. independent of any time parameter*, comes down to a system of three differential equations:

$$\begin{gathered} d^2r - rd\Omega^2 = 0 \\ rd^2\theta + 2drd\theta - r\sin\theta\cos\theta d\varphi^2 = 0 \\ r\sin\theta\, d^2\varphi + 2\sin\theta drd\varphi + 2r\cos\theta d\theta d\varphi = 0 \end{gathered} \tag{2.3.10}$$

This system can be solved by starting with the quadratic form $d\Omega^2$, which depends only on angles, and is needed in the first of the equations for the description of the second differential of the radial coordinate. It satisfies a simple differential equation; first, we have by direct symmetric-differentiation:

$$d(d\Omega^2) = 2(d\theta d^2\theta + \sin\theta\cos\theta d\theta d\varphi^2 + \sin^2\theta\, d\varphi d^2\varphi) \tag{2.3.11}$$

Now, using here the last two equations from (2.3.10) we can produce the result

$$rd(d\Omega^2) + 4dr(d\Omega^2) = 0 \tag{2.3.12}$$

and thus we can get a general aspect of the Kepler’s *area law*, but in space:

$$d\Omega^2 = \frac{R^4}{r^4}dt^2 \tag{2.3.13}$$

Here R^2 is a constant having the dimensions of a rate of area, in case the continuity parameter t is taken as time. Therefore, we just have introduced the ‘time parameter’ t, to be measured by the metric of the unit sphere. Digressing a little: this equation defines the spherical angle Ω itself – the continuity parameter on the unit sphere – *as a time parameter*. This is a fact known for ages in human history: the social time is usually measured by the positions of stars and the significant ‘orbs’ on the celestial canopy. Now, inserting the result (2.3.13) into the first equation (2.3.10) we get

$$d^2r = \frac{R^4}{r^3}dt^2 \tag{2.3.14}$$

The equation (2.3.14) can be solved to give the known solution:

$$r^2 = At^2 + 2Bt + C \tag{2.3.15}$$

with A, B and C constants satisfying to the constraint:

$$R^4 \equiv AC - B^2 \tag{2.3.16}$$

In words: the radial coordinate in a given position in space, is the coordinate of a classical free particle, provided the Kepler’s area law is verified.

Now, again, a little digression on the previous results seems necessary, in order to justify some incidental further proceedings, as far as they may involve stochastic or fractal applications. Notice that the equation (2.3.13) is a direct consequence of the last two equations (2.3.10). These last ones can be *formally integrated* in conjunction, as follows: first we have them properly arranged in the form of a homogeneous differential system having a skew-symmetric matrix:

$$d\begin{pmatrix} r^2 d\theta \\ r^2\sin\theta\, d\varphi \end{pmatrix} = \begin{pmatrix} 0 & \cos\theta\, d\varphi \\ -\cos\theta\, d\varphi & 0 \end{pmatrix}\begin{pmatrix} r^2 d\theta \\ r^2\sin\theta\, d\varphi \end{pmatrix} \tag{2.3.17}$$

which is, of course, easier to solve. Indeed, (2.3.17) can be solved by matrix exponentiation, with the result

$$\begin{pmatrix} r^2 d\theta \\ r^2 \sin\theta\, d\varphi \end{pmatrix} = \begin{pmatrix} \cos(\cos\theta\, d\varphi) & -\sin(\cos\theta\, d\varphi) \\ \sin(\cos\theta\, d\varphi) & \cos(\cos\theta\, d\varphi) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \tag{2.3.18}$$

where the *constant differentials a* and *b* – that can aptly be called *fractals* – are constrained by the condition:

$$R^4 \equiv a^2 + b^2 \tag{2.3.19}$$

The two differentials in the left hand side of equation (2.3.18) have also the meaning of *elementary area components* on the surface of sphere. Indeed, from equations (2.3.1) and (2.3.6) we get:

$$\boldsymbol{r} \times d\boldsymbol{r} = (r^2 d\theta)\cdot(\hat{\boldsymbol{e}}_r \times \hat{\boldsymbol{e}}_\theta) + (r^2 \sin\theta\, d\varphi)\cdot(\hat{\boldsymbol{e}}_r \times \hat{\boldsymbol{e}}_\varphi)$$

Therefore, in view of the properties of the generic Cartesian reference frame, the elementary area of the unit sphere is actually a vector in the plane (θ, ϕ), having as components the differentials from equation (2.3.18):

$$\boldsymbol{r} \times d\boldsymbol{r} = -(r^2 \sin\theta\, d\varphi)\cdot \hat{\boldsymbol{e}}_\theta + (r^2 d\theta)\cdot \hat{\boldsymbol{e}}_\varphi \tag{2.3.20}$$

Now it is time for introducing the dynamics, and this will be done here in the classical way.

Notice that, as long as we have to do with central accelerations, the last two of the equations (2.3.10) should not affected affected by anything: it is only the first equation that acquires, for instance, a term in the right hand side. Consequently, the integral (2.3.13) persists even in this case, but the equation (2.3.14) gets, in its right hand side, an expression depending on the magnitude of acceleration. So that instead of (2.3.14) we shall have:

$$d^2 r - r(d\Omega^2) = f(\boldsymbol{r})dt^2 \tag{2.3.21}$$

Here t is the time of the problem, as defined before, and $f(\boldsymbol{r})$ is the magnitude of impressed acceleration, up to a sign. This is a second order purely differential equation for the magnitude of the position vector, to be solved once we know the magnitude of the acceleration. In order to go over to the new time here, we usually take notice that the equation (2.3.20), offer the 'area constant' of the motion by relation

$$r^2 \dot{\Omega} = \dot{a} \equiv R^2 \tag{2.3.22}$$

with an obvious notation for the area constant, and R^2 given by (2.3.18). Using this, (2.3.21) becomes the classical *Binet's equation*:

$$u^2 \left(\frac{d^2 u}{d\Omega^2} + u \right) = \frac{1}{\dot{a}^2} f(\boldsymbol{r}); \quad u \cdot r = 1 \tag{2.3.23}$$

Now, that we have established the general mathematical frame of our discussion, it is time to appropriate it for the important case of equation (2.2.13), which is, of course, formally identical with the classical Kepler motion.

That case was historically meant to describe the physics of the planetary system, whereby the force generating the describing dynamics of the system is the Newtonian force: a central force, having the magnitude $f(\boldsymbol{r})$ inversely proportional with the square of distance r. For the present kinematical case that function represents just the acceleration, and its magnitude is $f(\boldsymbol{r}) = K^2 \cdot u^2$, where K is a constant; the equation (2.3.23) can be written as a second order differential equation in the 'celestial angle' parameter, as it were, *i.e.* the 'proper motion' as they call it in astronomy:

$$\frac{d^2 u}{d\Omega^2} + u = \kappa^2; \quad \kappa \equiv \frac{K}{\dot{a}}$$

This equation can be solved right away. It is known that the general solution of this equation has the form

$$u(\Omega) = \kappa^2 + u_0 \cos(\Omega + \Omega_0) \tag{2.3.24}$$

with u_0 and Ω_0 integration constants, to be assigned according to some initial conditions. In order to grasp the meaning of these initial conditions, from both mathematical and physical points of view, we need to write the solution (2.3.24) in velocites. Indeed, in view of the definition of variable u given in equation (2.3.23), the variable $(\dot{a}\cdot u)$ is a velocity, whose expression can be found from equation (2.3.24), transcribed as

$$\frac{1}{r}=\kappa^2+u_0\cos(\Omega+\Omega_0) \tag{2.3.25}$$

Therefore $(\kappa^2\dot{a})$ is a velocity, constant for the corresponding motion, as well as $(\dot{a}\cdot u_0)$ which varies with the initial conditions of the motion. Choosing a convenient trigonometric function for the equation of motion, and assuming that the initial velocity is taken at the direction Ω_0 from space, while the components of initial velocity in the plane of motion are (V_1,V_2), we then have the location of initial position in that plane, determined by the parameters:

$$\begin{matrix} V_1=\dot{a}u_0\cos(\Omega_0) \\ V_2=\dot{a}u_0\sin(\Omega_0) \end{matrix} \quad \therefore \quad \begin{matrix} \Omega_0=\tan^{-1}(V_2/V_1) \\ r_0=\dot{a}\big/\sqrt{V_1^2+V_2^2} \end{matrix} \tag{2.3.26}$$

i.e. the initial polar coordinates of the motion with respect to the center of force. The 'convenient choice' of the trigonometric function in equation of motion, is the one which makes the Kepler orbit in the Cartesian coordinates (ξ,η) of the plane of motion, referred to the center of force, appearing as the quadratic form:

$$\left(\frac{\kappa^2}{\dot{a}^2}-V_1^2\right)\xi^2-2V_1V_2\xi\eta+\left(\frac{\kappa^2}{\dot{a}^2}-V_2^2\right)\eta^2+2\dot{a}(V_1\xi+V_2\eta)=\dot{a}^2 \tag{2.3.27}$$

This is a conic in general, having its center at the location

$$\xi_0=-\frac{\dot{a}V_1}{\Delta},\quad \eta_0=-\frac{\dot{a}V_2}{\Delta};\quad \Delta\equiv\frac{\kappa^2}{\dot{a}^2}-V_1^2-V_2^2 \tag{2.3.28}$$

with respect to the center of force, in the plane of motion. When referring the orbit to its own center, the equation (2.3.27) becomes

$$\left(\frac{\kappa^2}{\dot{a}^2}-V_1^2\right)x^2-2V_1V_2xy+\left(\frac{\kappa^2}{\dot{a}^2}-V_2^2\right)y^2=\frac{\kappa^2}{\Delta} \tag{2.3.29}$$

where $x=\xi-\xi_0$ and $y=\eta-\eta_0$. One can see that this orbit is completely described by the matrix

$$\boldsymbol{a}\equiv\frac{\Delta}{\kappa^2}\begin{pmatrix} \kappa^2/\dot{a}^2-V_1^2 & -V_1V_2 \\ -V_1V_2 & \kappa^2/\dot{a}^2-V_2^2 \end{pmatrix} \tag{2.3.30}$$

as follows: the semi-axes are the eigenvalues a and b of the inverse of $\boldsymbol{a}$, *i.e.* the matrix

$$\boldsymbol{a}^{-1}\equiv\frac{\dot{a}^4}{\kappa^2}\frac{1}{(1-e_1^2-e_2^2)}\begin{pmatrix} 1-e_2^2 & e_1e_2 \\ e_1e_2 & 1-e_1^2 \end{pmatrix}\equiv\frac{\dot{a}^4}{\kappa^2\lambda^2}(\lambda^2\boldsymbol{1}+|e\rangle\langle e|) \tag{2.3.31}$$

and are given by

$$a^2=\frac{\dot{a}^4}{\kappa^2\lambda^2},\quad b^2=\frac{\dot{a}^4}{\kappa^2} \tag{2.3.32}$$

where $\boldsymbol{1}$ is the corresponding identity matrix, and we used the following notations:

$$|e\rangle\equiv\begin{pmatrix} e_1 \\ e_2 \end{pmatrix}=\frac{\dot{a}}{\kappa}\begin{pmatrix} V_1 \\ V_2 \end{pmatrix};\quad \lambda^2\equiv1-e_1^2-e_2^2 \tag{2.3.33}$$

The letter 'e' is intended to suggest an important fact here. Namely, if we calculate the eccentricity of orbit:

$$e^2\equiv\frac{a^2-b^2}{a^2}\quad\therefore\quad e^2=1-\lambda^2\equiv\langle e|e\rangle \tag{2.3.34}$$

We have thus an outstanding result, showing that the vector $|e\rangle$ completely characterizes the shape and orientation of the orbit in its plane. As its length is the eccentricity of the orbit, one can see at once that, firstly, not all the orbits are of the same kind, *i.e.* only ellipses, or only hyperbolas, or only parabolas. The outstanding property here is that the shape of orbit *depends on the initial conditions of the motion thus described*. It is like the fate of a certain particle in motion in a Newtonian force field is decided by its present. Secondly, another outstanding fact is that, even in the same class of orbits – ellipses, hyperbolas or parabolas – we can have different orbits describing the same motion, depending also on the initial conditions. Such is the case of Saturn rings, for instance, or that of a cloud of particles orbiting around the same force center (Larmor, 1900). The case is essential when it comes to the interpretation of the matter from an extended body orbiting around a center of force: the nature of cohesion forces between the particles of the interpretative ensemble of matter, is then at least partially dictated by the center of force.

However, the main point we want to make here is about the eccentricity vector $\boldsymbol{e}$: it roughly describes the position of the center of orbit with respect to the center of force, or vice versa, if one prefers. Let us assume the first case, in order to bring in an important argument related to confinement. The orbits described by the quadratic equations (2.3.27) or (2.3.29) are ellipses only if $0 < e < 1$, which means that the eccentricity vector should be confined to the interior of the unit disk, while the vector (ξ_0, η_0) must be confined to the region close to the center of force. In the case of planetary model of the hydrogen atom, this region is 'covered' by the positive charge of the model, so a problem arises to which Louis de Broglie offered a classical solution (de Broglie, 1935): *how is the charge spread on a given coordinate space*? This is, indeed, not an ordinary space, but a coordinate space in view of the constraint that the Kepler trajectory is an ellipse, which limits the eccentricity vector to such a space. It brings in, for a closer scrutiny, one of the most important ideas of Louis de Broglie, namely the idea of an *application function*, we have mentioned a few times before.

2.4 Louis de Broglie's Application Function

At the time when de Broglie opened the idea of application function (de Broglie, 1935), the probabilistic interpretation of the wave function was in full swing, steering, as it were, our spirit in an apparently different direction from the classical line. No wonder then, de Broglie was alone, again, on his path, and the idea remained burried in the literature, from its very birth. His reasoning started from the assumption that the fundamental interactions in the world of material particles involve just electromagnetic fields, but he went a little further, methodologically speaking. Namely he moved on to suggest a way of describing how *a field is applied on a physical quantity* that can be characterized by a volume density, and that way appears to us as having a universal connotation, insufficiently used by physics today, if occasionally used at all. To wit, it should be true for any kind of field, *in fact it should be the very definition of a field*. De Broglie used Poisson's equation in order to substantiate the idea that the field and the matter are generally defined in different positions in a space region. One could say that he developed such an idea, having in the background the necessity of identifying the particle amplitude with the optical amplitude, as we have noticed before in connection with his interpretation: there should be an application between the space range of the solutions of equation (2.1.16) and that of the solutions of equation (2.1.17). The line of thinking goes as follows.

If V is a physical magnitude describing the field, and this physical magnitude can be taken as a potential defined in a position, $\boldsymbol{x}$ say, from a space covered by an electron, with ρ the density of electricity of that electron, defined as a function of position $\boldsymbol{X}$ from that space, then, according to classical precepts, the interaction can be expressed by monomials having the following algebraic structure:

$$\rho(\boldsymbol{X})V(\boldsymbol{x})\delta(\boldsymbol{X}-\boldsymbol{x}) \tag{2.4.1}$$

where δ is the Dirac symbol. In view of the space extension of the electron, the interaction *per se* is thus expressed by a double space integral over the two space positions $\boldsymbol{x}$ and $\boldsymbol{X}$, whereby one of them can be dispensed with, due to the symmetry of the Dirac's symbol which allows a two-way result:

$$\begin{aligned}\oiiint d^3\boldsymbol{x}\oiiint d^3\boldsymbol{X}\cdot\rho(\boldsymbol{X})V(\boldsymbol{x})\delta(\boldsymbol{X}-\boldsymbol{x}) &= \oiiint d^3\boldsymbol{x}\cdot\rho(\boldsymbol{x})V(\boldsymbol{x}) \\ &= \oiiint d^3\boldsymbol{X}\cdot\rho(\boldsymbol{X})V(\boldsymbol{X})\end{aligned} \tag{2.4.2}$$

Here the special symbol of space integration is used to suggest the whole space, in order to be able to exploit the properties of the Dirac functional. We need to take proper notice of this mathematical observation, as it means physically *no space scale transition*: the electron belongs to the same world as the field. Thus, de Broglie takes note of the fact that in equations (2.4.1) and (2.4.2)…

> ... the factor δ is a ‹‹application function›› whose role is that of expressing the fact that in each point an *electromagnetic field is applied to the electricity from that point* [(de Broglie, 1935); *our translation and Italics*]

It is not hard then to take further notice, that starting from the mentioned property of symmetry of the Dirac's symbol, this ‹‹application function›› plays also a reciprocal role, so to speak, insofar as it also indicates the way in which, maintaining de Broglie's phrasing, the "electricity is applied to field". Thus *the field equation* – here the classical Poisson equation – for the field magnitude characterized by the function V, can be written as

$$\nabla^2 V(\boldsymbol{x}) = \iiint d^3\boldsymbol{X}\cdot\rho(\boldsymbol{X})\cdot\delta(\boldsymbol{X}-\boldsymbol{x}) \tag{2.4.3}$$

using the properties of the Dirac symbol. Since in the classical case this leads to the Coulombian potential which is singular at the position of charge, the classical physics itself was urged into assuming here a space extension of the material point, which in turn calls for *a physical structure for the electron*. However, at this point de Broglie takes notice of the fact that…

> … Unfortunately it *does not seem possible at all*, within present-day ideas, *to assign a structure to the electron*: the quantum theories of interaction between electromagnetic field and matter also find an infinite energy for the electron (except, however, the recent very interesting theory of Mr. Born). [(de Broglie, 1935); *our translation and Italics*]

The work of 'Mr. Born' referred to by de Broglie in this excerpt, is the one from 1934, setting the ground for what we know today as the Born-Infeld nonlinear electrodynamics [(Born, 1934); (Born & Infeld, 1934)], to which we shall refer later in this work, as being related to the issue at hand: the 'application' of the field on charge and the 'reciprocal application' of the charge on field.

This observation of impossibility, made by the great physicist, as, in fact, the whole spirit of physics to this very date, warrants our conclusion that, when talking of a structure, Louis de Broglie would have in mind a *physical structure* for the electron. More to the point, this meant two important things followed closely by theoretical physics thus far: first, the description of the electron had to be done *without any transition of space scale* and, secondly, as a consequence of this intransitivity of space scale, the internal space of an electron had to be described *by the motion of some constitutive elements*, like, for instance, the classical material points. In other words *the time scale remains the same* either, in any description of an elementary particle, be it electron or any other particle. For once, this was the reason that the planetary model was not properly applied as a model to different space and time scales.

Accordingly, de Broglie suggests a clever way out of impasse, based on statistics, specifically by replacing the symbol δ of Dirac with an isotropic Gaussian:

$$\exp\{-(\boldsymbol{X}-\boldsymbol{x})^2/\sigma^2\}$$

The point is that mathematically, the Gaussian reduces to δ in the classical limit of exactitude, when the standard deviation of the data is very small, which ideally means a vanishing standard deviation: $\sigma \to 0$. When implementing this replacement, the equation (2.4.3) takes the form:

$$\nabla^2 V(\boldsymbol{x}) = \iiint d^3\boldsymbol{X}\cdot\rho(\boldsymbol{X})\cdot\exp\{-(\boldsymbol{X}-\boldsymbol{x})^2/\sigma^2\} \tag{2.4.4}$$

Thus everything happens as if *the charge is normally distributed around the point* $\boldsymbol{x}$, or as if *the field is normally distributed around the point* $\boldsymbol{X}$, and we have, even from the classical point of view, a way of introducing the probabilities, without renouncing the classical ideas. In spite of the fact that it has the dimensions of a length, and even plays the classical part of the radius of a spatially finite electron, the quantity σ ...

> ... is nevertheless *not a structure parameter*; rather, it is a *parameter of uncertainty* of the position of application of the field upon charge (*or vice versa*). This seems to be in better agreement with the quantum ideas than the *introduction of a genuine radius* [(De Broglie, 1935), *our translation and Italics*]

Again, when de Broglie talks of ‘a structure’ here, we think we are legitimate in understanding ‘a physical structure’. That ‘vice versa’, however, which we specifically emphasized in the excerpt above, belongs to de Broglie himself, and expresses an essential point, when combined with his own observation that either the quantum mechanics or the wave mechanics do not support the idea of a “genuine radius” for the electron. It shows that the Gaussian thus introduced is not simply an entirely subjective element: both the application of the field upon charge but, more importantly, *the application of the charge upon field* need to be further documented and physically assessed, on an equal footing. Generally, we can replace here the word ‘charge’ with ‘matter’, and also extending along appropriately the concept of field – to a Yang-Mills field, for instance – we can make a physical law out of this observation of Louis de Broglie.

2.5 Application Function as a Matter of Principle

The main problem revealed by the de Broglie’s idea of application is to decide, first, *the position* of that application, and then *the region* of application. More to the point, if we have, say a position of field application to charge, how is the field distributed around that position; or if we need to apply the charge upon field to a certain position, how is the charge distributed around that position?! In physical terms this decision can be properly done only by a statistics, as de Broglie himself has taken notice, and, what is more important, by *a prior statistics* at that: if we can decide that an ensemble of physical points is congregated within a space region, then we know approximately the extension of the space region containing these points and where, in an ordinary space, are they accumulated. The problem of prior statistics can be thus set, in many of the physical problems, in the form of the *maximum information entropy principle*: find the function $\rho(\boldsymbol{x})$, representing the density of such a pile of points, that realizes the extreme – usually the maximum – of integral

$$I[\rho] = \iiint_{Space} \rho(\boldsymbol{x}) \cdot \ln \rho(\boldsymbol{x}) \cdot d^3\boldsymbol{x} \tag{2.5.1}$$

submitted to the two kinds of constraints, regarding the location and extension of the distribution of positions:

$$m^k = \iiint_{Space} x^k \cdot \rho(\boldsymbol{x}) \cdot d^3\boldsymbol{x}; \quad \Delta^{ij} = \iiint_{Space} (x^i - m^i) \cdot (x^j - m^j) \cdot \rho(\boldsymbol{x}) \cdot d^3\boldsymbol{x} \tag{2.5.2}$$

where $\boldsymbol{m}$ is a specification of mean position, and $\boldsymbol{\Delta}$ is a specification of the variance matrix. Here $\rho(\boldsymbol{x})$ has to be twice differentiable, and to satisfy the limit and normalization conditions of a proper density function:

$$\iiint_{Space} \rho(\boldsymbol{x}) \cdot d^3\boldsymbol{x} = 1; \quad \lim_{x \to \infty} \rho(\boldsymbol{x}) \sim 0 \tag{2.5.3}$$

The functional (2.5.1) is the well-known classical entropy of the distribution described by the density $\rho(\boldsymbol{x})$, and maximizing it should give 'the best non-commital' density describing the distribution (Jaynes, 1968, 1978, 1982). By Lagrange multipliers, ρ must realize the extremum of the functional (we owe the details of the following calculations connected to the Gaussian maximum entropy principle to an old friend, *Dr. Marian Soare*):

$$J[\rho] = \iiint_{Space} H\left(\boldsymbol{x}, \rho(\boldsymbol{x})\right) \cdot d^3\boldsymbol{x} \tag{2.5.4}$$

where the function H represents the update of the integrand of (2.5.1) with the constraints from equations (2.5.2) and (2.5.3), included via Lagrange multipliers (λ):

$$H\left(\boldsymbol{x}, \rho(\boldsymbol{x})\right) = \rho(\boldsymbol{x}) \ln \rho(\boldsymbol{x}) + \lambda_0 \rho(\boldsymbol{x}) + \sum_k \lambda_k x^k \rho(\boldsymbol{x}) + \sum_{ij} \lambda_{ij} (x^i - m^i)(x^j - m^j) \rho(\boldsymbol{x}) \tag{2.5.5}$$

The Euler-Lagrange equation is based on derivatives:

$$\frac{\partial H}{\partial \rho(\boldsymbol{x})} \equiv \ln \rho(\boldsymbol{x}) + 1 + \lambda_0 + \sum_k \lambda_k x^k + \sum_{ij} \lambda_{ij} (x^i - m^i)(x^j - m^j); \quad \frac{\partial H}{\partial \nabla_{\boldsymbol{x}} \rho(\boldsymbol{x})} \equiv \boldsymbol{0}$$

so that an extremum is realized for the following functional form of the density:

$$\rho(\boldsymbol{x}) = \exp\left\{-\left[1 + \lambda_0 + \sum_k \lambda_k x^k + \sum_{ij} \lambda_{ij} (x^i - m^i)(x^j - m^j)\right]\right\} \tag{2.5.6}$$

We need to evaluate the Lagrange multipliers, and this can be done by inserting the information we have, from the measurement data for instance. In order to do this, we need, in fact, to work a little bit on the quadratic form, reducing it to a sum of squares, and thus to evaluate the factors of $\rho(\boldsymbol{x})$ corresponding to each term of the two sums in the exponent. First, notice that, up to sign, this one can be written in the form:

$$\rho(\boldsymbol{x}) \equiv \exp\{-F(\boldsymbol{x})\} \quad \therefore \quad F(\boldsymbol{x}) \equiv \left\langle x - m \middle| \boldsymbol{\Lambda} \middle| x - m \right\rangle + \left\langle \lambda \middle| x \right\rangle + \lambda_0 + 1 \tag{2.5.7}$$

with obvious identifications for the 'bras' and 'kets'; this form is quite convenient in calculations. Performing a linear non-homogeneous, but proper transformation:

$$\left|x\right\rangle = \boldsymbol{\eta} \cdot \left|y\right\rangle + \left|x_0\right\rangle; \quad \det(\boldsymbol{\eta}) \neq 0 \tag{2.5.8}$$

we then have, first of all

$$\langle x-m|\boldsymbol{\Lambda}|x-m\rangle = \langle y|\boldsymbol{\eta}^T\cdot\boldsymbol{\Lambda}\cdot\boldsymbol{\eta}|y\rangle + 2\langle x_0-m|\boldsymbol{\Lambda}\cdot\boldsymbol{\eta}|y\rangle + \langle x_0-m|\boldsymbol{\Lambda}|x_0-m\rangle$$

and secondly

$$\langle\boldsymbol{\lambda}|x\rangle = \langle\boldsymbol{\lambda}|\boldsymbol{\eta}|y\rangle + \langle\boldsymbol{\lambda}|x_0\rangle$$

so that the quadratic form from equation (2.5.7) becomes

$$F(\boldsymbol{y}) = \langle y|\boldsymbol{\eta}^T\cdot\boldsymbol{\Lambda}\cdot\boldsymbol{\eta}|y\rangle + \left(2\langle x_0-m|\boldsymbol{\Lambda}+\langle\boldsymbol{\lambda}|\right)\cdot\boldsymbol{\eta}|y\rangle + \langle x_0-m|\boldsymbol{\Lambda}|x_0-m\rangle + \langle\boldsymbol{\lambda}|x_0\rangle + 1 + \boldsymbol{\lambda}_0 \tag{2.5.9}$$

Now, we conveniently restrict the Lagrange multipliers, by assuming that the following two conditions are to be satisfied

$$\boldsymbol{\eta}^T\cdot\boldsymbol{\Lambda}\cdot\boldsymbol{\eta} = |\boldsymbol{\Lambda}\rangle^k\langle e_k|;\quad \left(2\langle x_0-m|\boldsymbol{\Lambda}+\langle\boldsymbol{\lambda}|\right)\cdot\boldsymbol{\eta} = \langle 0| \tag{2.5.10}$$

with summation over repeated index k in the first equality. Let us detail the notations here, in view of the fact that they may come in handy later on along this work: $|\boldsymbol{\Lambda}\rangle^k$ is the column matrix having as entries the eigenvalues of the 3×3 matrix $\boldsymbol{\Lambda}$ on the corresponding place indicated by their index, with zeros on all the other places, while $|e_k\rangle$ are the unit vectors of an absolute Euclidean frame. More precisely, we have:

$$\begin{aligned}|\boldsymbol{\Lambda}\rangle^k\langle e_k| &\equiv \begin{pmatrix}\boldsymbol{\Lambda}_1\\0\\0\end{pmatrix}\begin{pmatrix}1&0&0\end{pmatrix} + \begin{pmatrix}0\\\boldsymbol{\Lambda}_2\\0\end{pmatrix}\begin{pmatrix}0&1&0\end{pmatrix} + \begin{pmatrix}0\\0\\\boldsymbol{\Lambda}_3\end{pmatrix}\begin{pmatrix}0&0&1\end{pmatrix}\\ &= \begin{pmatrix}\boldsymbol{\Lambda}_1&0&0\\0&0&0\\0&0&0\end{pmatrix} + \begin{pmatrix}0&0&0\\0&\boldsymbol{\Lambda}_2&0\\0&0&0\end{pmatrix} + \begin{pmatrix}0&0&0\\0&0&0\\0&0&\boldsymbol{\Lambda}_3\end{pmatrix}\\ &= \begin{pmatrix}\boldsymbol{\Lambda}_1&0&0\\0&\boldsymbol{\Lambda}_2&0\\0&0&\boldsymbol{\Lambda}_3\end{pmatrix}\end{aligned} \tag{2.5.11}$$

where Λ_k are the eigenvalues of the matrix $\boldsymbol{\Lambda}$. The second one of the conditions (2.5.10) gives directly

$$|\lambda\rangle = 2\boldsymbol{\Lambda}\cdot|m-x_0\rangle \tag{2.5.12}$$

while the first one of them shows that the matrix $\boldsymbol{\eta}$ must diagonalize $\boldsymbol{\Lambda}$. Then we have

$$F(\boldsymbol{y}) = \langle y|\boldsymbol{\Lambda}\rangle^k\cdot\langle e_k|y\rangle - \langle x_0-m|\boldsymbol{\Lambda}|x_0+m\rangle + 1 + \lambda_0 \tag{2.5.13}$$

Now, it is quite clear that in order to be able to speak of probabilities the exponential $exp\{-1-\lambda_0+\langle x_0-m|\boldsymbol{\Lambda}|x_0+m\rangle\}$ must be taken as a normalization factor. The factor cam be evaluated in terms of the eigenvalues of the matrix $\boldsymbol{\Lambda}$, and the final result is

$$\rho(\boldsymbol{y}) = \frac{\sqrt{\boldsymbol{\Lambda}_1\boldsymbol{\Lambda}_2\boldsymbol{\Lambda}_3}}{(2\pi)^{3/2}}e^{-\frac{1}{2}\sum\Lambda_k(y^k)^2} \tag{2.5.14}$$

This is now a Gaussian of *zero mean* and, consequently, by equation (2.5.8) we must have $|x_0\rangle = |m\rangle$, because the averaging operation is linear. So the normalization factor is actually only $exp\{-1-\lambda_0\}$, independently of the specifications for means and variances. The matrix $\boldsymbol{\Lambda}$ is the inverse of the matrix $\boldsymbol{\Delta}$ from equation (2.5.2), so that the previous result can be written exclusively in terms of initial specifications, and its general form is

$$\rho(\boldsymbol{x}\,|\,\boldsymbol{m},\boldsymbol{\Delta}) = \frac{1}{\sqrt{(2\pi)^3\det(\boldsymbol{\Delta})}}\exp\left(-\frac{1}{2}\langle x-m|\boldsymbol{\Delta}^{-1}|x-m\rangle\right) \tag{2.5.15}$$

which is a well-known and widely used formula.

Going back, for a while, to the de Broglie's construction of application function, an initial specification for the matrix of variances can be given, for instance, by the classical formula of spherical volumes, showing that the application of the field upon the charge of the electron – or of the charge of the electron on the central field of forces in the planetary model, for that matter – cannot be made outside of the volume of the electron. If we approximate the space region occupied by the electron with an ellipsoid, say, then we can have a specification of the matrix of variances in the form of an integral:

$$\Delta^{ik} \equiv \iiint_{\langle y|\boldsymbol{a}|y\rangle < c^2} y^i y^k d^3\boldsymbol{y} \quad \therefore \quad \Delta \equiv \iiint_{\langle y|\boldsymbol{a}|y\rangle < c^2} |y\rangle\langle y| d^3\boldsymbol{y} \tag{2.5.16}$$

This specification may also prove quite beneficent in the case of light, if we keep in mind, for instance, the Fresnel theory based on the elipsoid of elasticities: the space extension there, is the *a priori* extension necessary for the local space interpretation of the light. Incidentally, such a specification gives [(Cramer, 1962), §11.12]

$$\Delta = \frac{\pi^{3/2}}{5\cdot\boldsymbol{\Gamma}(5/2)}\cdot\frac{c^5}{\sqrt{\det\boldsymbol{a}}}\boldsymbol{a}^{-1} \quad \therefore \quad \Delta = \frac{4\pi}{15}\cdot\frac{c^5}{\sqrt{\det\boldsymbol{a}}}\boldsymbol{a}^{-1} \tag{2.5.17}$$

Thus, the maximum information entropy principle provides here the density from equation (2.5.15), which should be, in any possible mean position $|m\rangle$, a Gaussian density of the form:

$$\rho(\boldsymbol{x}) = \frac{15}{4\pi c^5}\left(\frac{\det\boldsymbol{a}}{2\pi}\right)^{3/2}\exp\left(-\frac{15}{8\pi}\frac{\sqrt{\det\boldsymbol{a}}}{c^5}\langle x-m|\boldsymbol{a}|x-m\rangle\right) \tag{2.5.18}$$

This historical example, as we see it, is most enlightening in revealing an issue which, in its essentials, is connected to that important problem of 'transcription' of the 'coordinate space' into 'ordinary space', if it is to use the terminology of Charles Galton Darwin. One can even say that Fresnel's ellipsoid of elasticities was the first sign of the pressing necessity of considering a coordinate space, in the particular form of an 'enclosed system' as it were, in order to endow the ordinary space with the physical properties of light. One can also say the de Broglie's application idea is, indeed, a natural extension of the classical theory of light, as we have already discussed before, but taken to details of interpretation. While we reserve a later development of the concept of application as we go along with the present work, the observation just made, raises a problem of principle that needs to be settled right away.

Indeed, the Gaussian produced by the maximum entropy principle, is an example of *mixed* statistical inference, involving, as we have seen, in its specifications from equation (2.5.2) both constraints related to ordinary space – for the mean of the distribution – and constraints related to coordinate space – for the matrix of variances. The algebraic possibility of reduction of the Gaussian to its center without affecting the variances is a reflection of the real fact that the principle is valid *regardless of the mean position of distribution*. In view of the examples above, of Fresnel's theory of light, and that of de Broglie's application function, one can say that physics was made possible, at least to a certain extent, by an invariance of such a distribution with respect to the mean position. To wit, the light can be measured by diffraction independently of the location in space, the electron can be in motion, and located anywhere, etc. However, one can also get the feeling that such a kind of invariance is limited, and this limitation is due to a possibility of scale transition.

A practical example is persistently recurring in our mind and, as a matter of fact, may prove itself quite illuminating for everyone as it was for the present author. Think of the light coming from a distant star: according to the present view in astronomy, this light is a phenomenon concerning the *transfinite* scale of our universe. However, as the development of physics proves it, such a phenomenon can only be quantified at the *infrafinite* – specifically, microscopic – scale of that universe: this is how the modern physics was built, as a matter of fact. In other words, the phenomenon of light can be taken as the one *real phenomenon undeniably transcending the scales*, and in fact this is how the modern physics was possible. Not the same thing can be said of a flux of electrons though: *the inertia adds limitations to the scale transition*. According to these observations, one needs a special delineation of the *a priori* possibility of description of a finite scale, accomplished, as above, by the maximum information entropy principle. Let us explain in detail what we mean by this statement.

In the general case, the maximum information entropy principle can produce a Gaussian density of the form given in equation (2.5.6), *i.e.* something of the form:

$$\rho(\boldsymbol{x}) = C\exp\left(-\frac{1}{2}\langle x|\boldsymbol{a}|x\rangle - \langle b|x\rangle\right) \tag{2.5.19}$$

Here the matrix $\boldsymbol{a}$, the vector $|b\rangle$ and the constant C are undetermined parameters naturally entering into the formulation of the problem of extremum of the informational entropy, as we have shown above. Given a set of mixed constraints as in equation (2.5.2), we can reduce this Gaussian to the simple form (2.5.15). This means that, *incidentally*, we can have at our disposal a distribution function of the positions in space, that can be used as a research tool in experiments, for instance after the example of Fresnel's light in optics. Therefore, we can write the exponent of (2.5.19) with respect to an average over a finite space distribution of positions, whose extension is evaluated by a given matrix $\boldsymbol{a}$, and to which we can associate a given position $|m\rangle$, using the above procedure of maximum information entropy. In order to use this Gaussian as a research tool at some space scale *in any position a priori*, the exponent needs to be extended to a form like that from equation (2.5.7), showing that the distribution is centered in some *other position* at the same scale, *i.e.* it should be of the form

$$F(\boldsymbol{x}\,|\,\boldsymbol{m},\boldsymbol{a}) = \langle x-m|\boldsymbol{a}|x-m\rangle + 2\langle b|x\rangle \tag{2.5.20}$$

Now, proceed with this quadratic form as before, in reducing it to another 'center' in order to cope with the presence of new information contained in the vector $\boldsymbol{b}$, reflecting the new position of the distribution. We assume therefore a physical possibility of *a priori* choice of a certain 'center' connected to the mean of our research Gaussian. Specifically, assume that we *have chosen* a position, $\boldsymbol{x}_0$ say, such that all of the possible positions of our distribution are written with respect to this in the form of vectors:

$$\boldsymbol{x} = \boldsymbol{x}_0 + \boldsymbol{y}$$

A situation illustrating such a choice is the one giving the center of force with respect to the center of orbit in a Kepler problem. As we have shown above, this is defined by the Runge-Lenz vector, which is connected to the initial conditions of the motion, and are contemporarily manifested as the eccentricity of the Kepler orbit. Another case can be the construction of the virial as a statistics for an ideal gas, involving the position of one arbitrary molecule as a reference. In such cases, the equation of the quadric (2.5.20) becomes:

$$F(\boldsymbol{x}\,|\,\boldsymbol{m},\boldsymbol{a}) = \langle y|\boldsymbol{a}|y\rangle + 2\langle x_0-m|\boldsymbol{a}|y\rangle + 2\langle b|y\rangle + \langle x_0-m|\boldsymbol{a}|x_0-m\rangle + 2\langle b|x_0\rangle$$

If $\boldsymbol{x}_0$ is chosen such that $\boldsymbol{y} = \boldsymbol{0}$ is the center of the quadric (2.5.20) then we must have

$$|x_0-m\rangle = -\boldsymbol{a}^{-1}|b\rangle \tag{2.5.21}$$

so that (2.5.20) becomes

$$F(\boldsymbol{x}\,|\,\boldsymbol{m},\boldsymbol{a}) = \langle y|\boldsymbol{a}|y\rangle - \langle b|\boldsymbol{a}^{-1}|b\rangle + 2\langle b|m\rangle \tag{2.5.22}$$

Now, the matrix $\boldsymbol{a}$ should be positively defined, *for it is a matrix of covariances*. However, for $|b\rangle$ real and arbitrary, the quadratic form (2.5.22) may be negatively defined, which would imply a positive exponent in the Gaussian of the maximum information entropy principle. This would mean that when, in order to calculate probabilities from it, we need to normalize the density for all *a priori* possible positions $|b\rangle$ in the world of our reach, we are not free to do this unconditionally. We can do it, indeed, but only on some finite ranges of this vector. This implies results always dependent of such ranges, therefore results without universal validity *a priori* in a given ordinary space. It is only if we accept that the vectors $|b\rangle$, which are impossible to specify through a given average only, *are purely imaginary*:

$$\left|b\right\rangle \equiv i\left|p\right\rangle; \quad \left|p\right\rangle \in R^3 \tag{2.5.23}$$

that everything comes to normal, albeit the quadratic form $F(\boldsymbol{x};\boldsymbol{p})$ becomes 'essentially complex', and has the expression:

$$F(\boldsymbol{x}\,|\,\boldsymbol{p},\boldsymbol{a}) = \left\langle y\middle|\boldsymbol{a}\middle|y\right\rangle + \left\langle p\middle|\boldsymbol{a}^{-1}\middle|p\right\rangle + 2i\left\langle p\middle|m\right\rangle \tag{2.5.24}$$

In this case, the Gaussian of maximum information entropy principle can even be even taken as a wave function – a *Gausson* in the expression of Iwo Białynicki-Birula and his collaborators [(Białynicki-Birula, 1986); however, see especially (Białynicki-Birula & Mycielski, 1976)] – and can be written as:

$$\rho(\boldsymbol{y}\,|\,\boldsymbol{m},\boldsymbol{a}) = \sqrt{\frac{\det\boldsymbol{a}}{(2\pi)^3}}\,e^{-\frac{1}{2}\langle y|\boldsymbol{a}|y\rangle}\,e^{-\frac{1}{2}\langle p|\boldsymbol{a}^{-1}|p\rangle - i\langle p|m\rangle} \tag{2.5.25}$$

The previous results now appear only 'second-hand', so to speak: integral over all $|p\rangle$ can be performed with the result [(Cramer, 1962); §11.12]

$$\iiint_{p-Space} e^{-\frac{1}{2}\langle p|\boldsymbol{a}^{-1}|p\rangle - i\langle p|m\rangle}\,d^3\boldsymbol{p} = \sqrt{(2\pi)^3\det\boldsymbol{a}}\cdot e^{-\frac{1}{2}\langle m|\boldsymbol{a}|m\rangle} \tag{2.5.26}$$

so that (2.5.25) becomes:

$$\rho(\boldsymbol{y}\,|\,\boldsymbol{m},\boldsymbol{a}) = (\det\boldsymbol{a})\cdot e^{-\frac{1}{2}(\langle y|\boldsymbol{a}|y\rangle + \langle m|\boldsymbol{a}|m\rangle)} \tag{2.5.27}$$

This is not a normalized density, but it can be normalized by adjusting the density with a constant factor, and then asking that its the integral over the continuum of all *a priori* positions represented by the vector $|m\rangle$ be finite, specifically *1*. Using the formula (2.5.26) with a special estimation $|m\rangle \equiv |0\rangle$, and with $\boldsymbol{a}$ instead of $\boldsymbol{a}^{-1}$, we get the following final result for the Gaussian density function, after integrating over $\boldsymbol{m}$:

$$\rho(\boldsymbol{y}\,|\,\boldsymbol{a}) = \sqrt{\frac{\det\boldsymbol{a}}{(2\pi)^3}}\,e^{-\frac{1}{2}\langle y|\boldsymbol{a}|y\rangle} \tag{2.5.28}$$

which is normalized, and independent of the *a priori* positions estimations $|m\rangle$.

Now, as long as we are on this issue, we can set the problem even some other way by asking: what should be the estimation $|m\rangle$ to be used in equation (2.5.27) in order that the function thus obtained should be a probability density, but *directly*? This means no further necessary calculations, like the calculation we just have done, no 'second-hand' considerations, as it were. Using the same algebraical argument, we could conveniently have based our calculations on equation (2.5.27), asking, for instance

$$\iiint_{Space} \rho(\boldsymbol{y}\,|\,\boldsymbol{m},\boldsymbol{a})\cdot d^3\boldsymbol{y} = (2\pi)^3 \quad \therefore \quad \left\langle m\middle|\boldsymbol{a}\middle|m\right\rangle = \ln(\det\boldsymbol{a}) \tag{2.5.29}$$

where ‘Space’ in the integration sign here means a ‘coordinate space’ in the terminology of Darwin. This last equation can be taken as a connection between the *ordinary space* – which, in this case is the *space of mean positions* – and the *coordinate space* – whose volume extension is given by *the determinant of the matrix of variances*. The physics knows of one such case, whereby the metric is decided by an ‘index of anisotropy’ representing the variance of the Hubble expansion parameter [(Misner, 1968); see, for mathematical details, (Mazilu, Agop, & Mercheș, 2019)].

The argument just presented above allows us to describe a ‘subjective’ version, as it were, for the procedure giving the application function of Louis de Broglie. It leads to the complex function

$$e^{-\frac{1}{2}\langle p|a^{-1}|p\rangle - i\langle p|m\rangle} \tag{2.5.30}$$

that determines the universal result (2.5.28) in the case of maintaining the scale of space measurements. This complex factor remains however *as such* within any physical theory, in cases where the space scale transitions are involved, as the Fresnel example of quadric of elasticities shows, *i.e.* in case the ordinary space goes over into a coordinate space. This is, for instance, the case of measurements of light which also involve the light we receive from distant stars, as we mentioned above; and again, in fact, the science of optics essentially evolved by incentives that came from astrophysics, which means applications of the kind we have just described. However, for a scale transition we cannot dispense with the complex ‘application function’, without properly taking care of the procedure.

Chapter 3 The Space Physics in a World of Charge

The physical features defining a classical material point – mass and charge – can hardly be considered among those *measurable* physical features allowed by the wave mechanics: they are supposed to be constants, independent of any position that the classical material point may happen to have, in fact, independent of any space geometry. Indeed, according to Schrödinger, to take a well-known stance in this connection, the measurable quantities of the wave mechanics must be considered as operators acting on the wave functions, which depends, naturally we should say, on position and time. This definition raised a serious question in its time, namely if the wave function is able to describe a *physical reality* altogether. Our implicit answer, as it appears from the progress of this work thus far, is clearly in the negative. To make it even clearer from the very beginning of the present topic of our discussion, we can say that the wave function can describe a *physical reality* only in cases where the interpretative ensemble of material points would refer to that physical reality, which is clearly not the case, at least not always. This observation has been raised a long time ago, by Albert Einstein, Boris Podolsky and Nathan Rosen in a short but epoch-making article (going usually by the abbreviation EPR in the physical literature). They take as the condition of reality of a certain physical feature, not a physical structure characterized by this feature, but *its possibility of being predicted with certainty*, *i.e.* with probability one. The wave function does not satisfy this reality criterion, for, according to wave-mechanical rules, it does not exhaustively describe a classical material point. Quoting the conclusion of that classical EPR article:

> One could object to this conclusion on the grounds that our criterion of reality is not sufficiently restrictive. Indeed, one would not arrive at our conclusion if one insisted that two or more *physical quantities can be regarded as simultaneous* elements of reality *only when they can be simultaneously measured or predicted* (*original Italics in this phrase, n/a*). On this point of view, since either one or the other, but not both simultaneously, of the quantities *P* and *Q* can be predicted, they are not simultaneously real. This makes the reality of *P* and *Q* depend upon the process of measurement carried out on the first system, which does not disturb the second system in any way. *No reasonable definition of reality could be expected to permit this*.
>
> While we have thus shown that the wave function does not provide a complete description of the physical reality, *we left open the question of whether or not such a description exists*. We believe, however, that such a theory is possible. [(Einstein, Podolsky, & Rosen, 1935); *our emphasis*, *except as indicated*, *n/a*]

In other words, if the measurement is part and parcel of a physical reality, the wave mechanics does not provide a complete description of it. However, the wave function has always described a reality we could think of, in terms of an interpretation. This reality is not physical, inasmuch as experiments are not possible in it. However, this is a general trait of our knowledge and therefore the wave mechanics makes no exception. All we have to do is to accept this reality and go along with it.

From the natural philosophical point of view, the mass and charges are physical quantities that "can be *regarded as simultaneous* elements of reality": the whole theory of elementary particles is built upon idea that the physical particles possess at least mass and charge, if not a lot more physical characteristics, which are 'measurable'. However, such characteristics can hardly be considered as measurable in the sense taken by Einstein, Podolsky and Rosen, *i.e.* with reference to an explicit condition of simultaneity: that of a certain scale of time. And as we have seen thus far, a description as that implied by these authors exists when one considers those characteristics as measurable, indeed, but *not for a physical reality*. All our work thus far is based on a fictitious fluid structure – therefore, *not a physical structure* – whereby the material points of the ensemble serving for interpretation should possess at least two concurrent physical features. These are represented, for instance, by the quantities of inertial mass, and electric charge. While any one of them can be 'predicted with certainty' when the other is known, this fact cannot be taken as a measurement in the sense devised by the wave mechanics, the way it was updated using the quantal precepts. Inasmuch as the view incriminated by the Einstein, Podolsky and Rosen's article was prevailing at the time, no one could think of a way around it. This would involve the idea of application of matter on space, or vice versa. However, as we have seen, just about the time of the article of Einstein, Podolsky, and Rosen, the application of matter over space took a positive turn with Louis de Broglie, (de Broglie, 1935). He devised a way of application between particle's amplitude and the wave amplitude, which, properly extended, based on the properties of the Lorentz transformation, would lead to even more profound characterization of the *concept of application*. That way devised by de Broglie was never taken by physics, for the EPR view dominated the natural philosophy. However, one case that apparently needed such an approach, occurred at just about the same time with EPR and de Broglie's article just cited, revealing the necessity of discerning between a reality in general, and a physical reality, where the experiments are possible.

We owe to Richard Feynman an important observation on the historical beginnings of the wave mechanics, which he has taken as inception point of one of the most beautiful and intuitive explanations of the phenomenon of *supraconductibility* [(Feynman, Leighton, & Sands, 1977), Volume III, §21–5]. In hindsight it may be quite significant that this phenomenon started being physically explained *as an interpretation*, according to the theory of fluids (London & London, 1935). And the material points entering the fluid ensembles that may serve for this interpretation, are described by physical quantities having those characteristics of certainty invoked by the EPR criterion of reality, without, nevertheless, recurring to the concept of measurement, as defined by the rules of either wave mechanics or quantum mechanics. Quoting, therefore:

> The wave function $\psi(\boldsymbol{r})$ for an electron in an atom, does not, then, describe a smeared-out electron with a smooth charge density. *The electron is either here, or there, or somewhere else, but wherever it is, it is a point charge* (*our emphasis here, n/a*). On the other hand, think of a situation in which there are an enormous number of particles in exactly the same state, a very large number of them with exactly the same wave function. Then what? One of them is here and one of them is there, and the probability of finding one of them at a given place is proportional to $\psi\psi^*$. But since there are so many particles, if I look in any volume *dxdydz* I will generally find a number close to $\psi\psi^* dxdydz$. So in a situation in which ψ is the wave function for each of an enormous number of particles which are all in the same state, $\psi\psi^*$ *can* be interpreted as the density of particles. If, under these circumstances, each particle carries the same charge q, we can, in fact, go further and interpret $\psi^*\psi$ as the density of *electricity*. Normally, $\psi\psi^*$ is given the dimensions of a probability density, then ψ should be multiplied by q to give the dimensions of a charge density. For our present purposes we can put this constant factor into ψ, and take $\psi\psi^*$ itself as the electric charge density. With this understanding, $\boldsymbol{J}$ (the current of density of probability I have calculated) becomes directly the electric current density.
>
> So in the situation in which we can have very many particles *in exactly the same state,* there is possible *a new physical interpretation of the wave functions. The charge density and the electric current can be calculated directly from the wave functions and the wave functions take on a physical meaning which extends into classical, macroscopic situations* (*our emphasis here, n/a*). [(Feynman, Leighton, & Sands, 1977), Volume III, §21–4]

Continuing on along this line of thought, Feynman comes to the conclusion that 'the wave function turns out to be the vector potential $\boldsymbol{A}$'. Now, such a situation would mean that in the case where the electron is 'either here or there' the EPR conditions of reality may be incidentally satisfied, as we have three scalar wave functions: the components of the potential vector. The fact that they are the components of the same vector can be taken, indeed, as incidental here, and, if one wave function is not enough, three may suffice for an exhaustive description of this world. However, the key point for such a conclusion, at least as we see it, would be to "think of a situation in which there are *an enormous number of particles in exactly the same state*". Our contention is that there is such a possibility, in fact, a classical possibility, once it is offered by the idea of classical material point endowed with mass and charge: we can think of an interpretation by such material points. Indeed, an ensemble of identical material points having gravitational mass and charges – electric and even magnetic ones – can, according to the Newtonian description of the action at a distance, exist in static equilibrium, inasmuch as in such a situation the electric and magnetic forces are repulsive, and the gravitational forces are attractive, and for a material point the third principle of dynamics should be *a priori* valid, so to speak, inasmuch as it has no space extension. Therefore, the Newtonian forces in such an ensemble, acting on a material point thus defined are in equilibrium in any direction at any distance, and the whole ensemble of such classical material points can be imagined as being in equilibrium. This is a sufficient condition of reality of each one of the three quantities, however, not a physical reality as we can perceive it.

Obviously, indeed, in the real world, *i.e*. in the world presented to us by our senses, such a situation does not exist: to use the expression of Eddington, it 'exists only in our minds'. Fact is, however, that *we are logically free to think of it*, and even describe it mathematically (Mazilu, Agop, & Mercheș, 2020). The only realities we know of, connected to such a description – nonetheless, still very much corrupted by our thinking – are those regarding the cosmology of the universe at large, where the gravitation seems to dominate [(Weyl, 1923), §39], and the microscopic world of atoms, where, just like in the world presented to us by our senses, the electric forces seem to dominate over gravitation. What is more, we have a positive means to describe this dominance with important consequences. Incidentally, it would seem that, according to these statements, there is no difference between the microscopic world and the world of our senses, which is clearly not the case. This compels us to a more specific definition of the very concept of 'world', which will be made clear in what follows.

First of all, in order to describe the fundamental units of matter in the same way in both cases – cosmos and daily life – our experience shows that the mass should be taken as *gravitational mass*. It is only in this circumstance that one can declare that for fundamental constitutive units – classical material points – possessing just gravitational mass and charge, the Newtonian force due to gravitational mass prevails quantitatively over that due to electric charge in the case of 'cosmos'. And only in this case can we add that, for the 'daily life', the Newtonian force due to electric charge – the Coulombian force, as it is usually called – prevails over the force due to their gravitational mass, in order to be experimentally noticeable in a gravitational background, as it was at the times of Charles Coulomb. Therefore, if we are to describe *physically* a certain universe, the material points should exercise two kinds of Newtonian forces acting simultaneously *in any direction* in space, *at any distance*: gravitational and electric. Then the world in a *universe of Newtonian forces* is that decided by the space scale of the human beings: this is the finite space scale in the sense of Nicholas Georgescu-Roegen, and it depends on our size and senses (Georgescu-Roegen, 1971). This is the world of reality in the universe we inhabit. Let us make more precise what this reality asks for, in terms of Newtonian forces.

3.1 Application of Charge to a Coordinate Space: Richard Feynman

Therefore, in the case of a classical approach of a universe dominated by forces, this universe can be *interpreted* by an ensemble of identical material points, randomly distributed in space, the way Richard Feynman suggestively describes it. There are well-known classical examples of such interpretation, like the molecules of an ideal gas, or those of an ideal fluid, with forces acting between them: it should be significant that these examples paved the way to the wave function, enabling the appearance of the wave mechanics and quantum mechanics. The magnitude of the whole Newtonian force, acting between any two such identical fundamental physical units on a certain direction, at any distance, in a universe described in this way, can be written as

$$G\frac{m^2}{r^2}-\frac{1}{4\pi\varepsilon}\frac{e^2}{r^2} \tag{3.1.1}$$

with r denoting the distance between them. As we said, it is assumed here that the *material points of this universe are all identical*, having the gravitational mass m and the electric charge e, and the universe is imagined as existing in a certain space described by the gravitational constant G and electric permittivity ε. The equation (3.1.1) also expresses the fact that, according to our experience, the two Newtonian act differently along one and the same direction forces – the gravitational force is *attraction* and the electric force is *repulsion* –: the different signs of the monomials in (3.1.1) represent an algebraic writing with respect to the orientation on the direction along which the action is exerted. Thus, it is only in this interpretation – for it cannot possibly be but an interpretation as defined by Charles Galton Darwin – that the universe at the cosmic scale can be characterized by the strong inequality

$$Gm^2 - \frac{e^2}{4\pi\varepsilon} \gg 0 \tag{3.1.2}$$

so that the 'electric force can be dismissed', as Hermann Weyl would say, *at any distance and in any direction*, while the microscopic universe – or the universe at the daily scale for that matter – can be characterized by

$$Gm^2 - \frac{e^2}{4\pi\varepsilon} \ll 0 \tag{3.1.3}$$

so that here the 'gravitational force can be dismissed' *at any distance in any direction*, as *proved* by Coulomb type experiments. Certainly the 'distance' should have different quantitative meaning in the two cases, and perhaps the 'direction' too. Whence, in our opinion, the possibility of describing *concurrent universes*, even at different space scales in the same world, to which we suggest now a positive possibility of approach, offered by the ideas of the absolute – or Cayleyan – geometry, and based on the concept of Newtonian forces.

Start with the observation that the structure of a universe is always a hypothesis, so that the problem occurs: can we think of an ideal *static structure* of the universe, *formally the same at any scale*, that would be able to describe even the structure of a fundamental physical unit of the universe? Speaking of the 'fundamental physical unit' the Newtonian central forces enter the stage just naturally: it is the very concept that started the idea of interpretation is the wave function. This concept was, indeed, introduced for the planetary model of atom, whose dynamics involves Newtonian forces, the kind of forces that, in fact, it helped historically define. Then, mathematically speaking we can always think of a static ensemble of identical material points, endowed with electric charge and gravitational mass, for the very reason Newton once would think that the system of stars is a static system. Quoting:

> COR. I. The fixed stars are *immovable,* seeing they keep the same position to the aphelions and nodes of the planets.
>
> COR. II. And since these stars are liable to no sensible parallax from the annual motion of the earth, they can have no force, *because of their immense distance*, to produce any sensible effect in our system. *Not to mention that the fixed stars, everywhere promiscuously dispersed in the heavens, by their contrary attractions destroy their mutual actions*, by Prop. LXX, Book I. [(Newton, 1974), Volume II, Book III: The System of the World, Prop XIV, Theorem XIV; *our Italics*]

The only forces Newton knew of, were the forces he invented, and those were forces of attraction. Then the stars would be 'immovable', as the astronomical data shows – "the same positions to the aphelions and nodes of the planets" – 'because of their immense distance', for in that case the forces between them would fade away. However, with forces between material points in equilibrium at any distance in any direction, we just have to choose the other reason, mentioned in the above excerpt as only subsidiary, but emphasized by us, for it becomes essential: if the material points are 'everywhere promiscuously dispersed in… a space, their contrary attractions…, and repulsions, destroy their mutual actions', so that the material points are… 'immovable'. Thus, like the old stars of Newton, we think of our material points as being able to form an ensemble, liable for serving to interpretation without any strings attached: no need to consider the distant matter that creates inertia, no need to consider a motion with respect to an absolute space, nothing like that. The interpretation is simply valid in a certain coordinate space, as Darwin would say!

Therefore, a *static equilibrium ensemble* can only exist… in our mind, just as Richard Feynman wanted, in order to describe the phenomenon of supraconductibility according to Londons' theory, which is, first and foremost just an interpretation, if any material point is endowed with two physical features connected to the known innate forces of our world. The only problem is if this interpretation is independent of space scale, *i.e.* if it is the same in the universe at large, as well as in the microcosmos. As we shall see here soon, this should be the case indeed, so that we can think of a 'world of Feynman' as it were, wherein the relation between gravitational mass and electric charge:

$$Gm^2 - \frac{e^2}{4\pi\varepsilon} = 0 \tag{3.1.4}$$

is valid regardless of distance and direction. Stretching a little our imagination, in such a world, every material point is virtually free: at least from a classical point of view it is legitimate to think like this. In other words, in a realm like this, the interpretation is made in the very same terms Schrödinger's equation (1.1.6) describes the ensemble: *by free particles*.

As it is referring to a homogeneous quadratic, the equation (3.1.4) can have a fundamental geometrical connotation: we can build an *absolute geometry* of fundamental physical features of the material points serving for interpretation, based on the static equilibrium of forces, using it as the equation of *an absolute* in a geometry of the *physical attributes* describing the material points. First of all, contemplating some simplicity of the mathematics involved here, let us arrange a uniform notation based on equation (3.1.4), in order to simplify the algebra that follows immediately. The terms in equation (3.1.4) are physically homogeneous, and both of them have the same composed unit ($kg \cdot m^3 \cdot s^{-2}$). So, in order to make the notation uniform, we include the constants describing the space of residence of matter in the definition of the physical properties of the material particles, by the following transcriptions:

$$m \;\; instead \;\; of \quad m\sqrt{G}, \quad e \;\; instead \;\; of \quad \frac{e}{\sqrt{4\pi\varepsilon}}$$

These notations are intended to suggest that, regardless of the attributes of the very host space of matter, the first term in (3.1.4) is referring only to gravitational mass (m), up to a multiplicative factor, while the second is referring only to the electric charge (e) up to a multiplicative factor. These are those features that can be considered as connected with the Newtonian central and conservative forces, having the intensity inversely proportional with the square distance between the material points, that established the modern physics as it is today. Therefore, limiting our considerations to just these two of the possible physical attributes of a classical material point serving for interpretation, the condition to be satisfied by an equilibrium ensemble of identical material points can be transcribed, with the convention above, in the form:

$$Q(m,e) \equiv m^2 - e^2 = 0 \tag{3.1.5}$$

The left hand side of this equation symbolizes a homogeneous *quadratic*, *i.e.* a homogeneous polynomial of second degree in its variables; this is why we chose the suggestive symbol Q in the first place.

Taken as absolute of a Cayleyan *geometry of the two-dimensional physical quantities* – gravitational mass and electric charge – the equation (3.1.5) divides the plane of these physical attributes into two parts: the 'inside' part, for which we assume that the quadratic form is positive, and the 'outside' part, for which the quadratic form should be, obviously, negative. This is just a convention, adopted by us in order to make the 'cosmos', characterized by equation (3.1.2), occupy a portion *inside the absolute* very close to its center with respect to the charge, while the 'microcosmos', described by equation (3.1.3), should occupy a region *outside the absolute*, very far away with respect to the gravitational mass. There may be also some other advantages of this convention, to be made obvious as we go along with our presentation. The true measure of these degrees of 'closeness' is, nevertheless, offered by the value of the quadratic Q defined in equation (3.1.5). The positive part of this construction is that, once we give a metric for this geometry, we are always able to find space distributions of the quantities of gravitational mass and charge, using, for instance, the *harmonic mappings*. The actual construction of these distributions may not be quite as easy as it appears, but in some cases it may be indicative of the right path to be followed by physics. Let us present here the starting point of this path.

The absolute metric of the geometry based on equation (3.1.5) can be offered by a formula for the elementary arclength, due to Dan Barbilian, which is always fit for such occasions, inasmuch as it is valid for a homogeneous polynomial of arbitrary degree [(Barbilian, 1937); see also (Mazilu, Agop, & Mercheș, 2019), equation (5.13)]. This formula offers, in the present case, a Cayleyan metric of the plane of the *two* physical attributes, in the form

$$(ds)^2 = \left(\frac{mdm - ede}{m^2 - e^2}\right)^2 - \frac{(dm)^2 - (de)^2}{m^2 - e^2}$$

up to an arbitrary constant coefficient. The reason for this choice of sign for the metric becomes obvious by noticing that, after calculations, it shows up as a perfect square:

$$(ds)^2 = \left(\frac{edm - mde}{m^2 - e^2}\right)^2 \tag{3.1.6}$$

and the square of a real quantity should be always positive. Thus the interior of the absolute is characterized by a proper hyperbolic angle, ϕ say, whose variation turns out to be our metric. Indeed we have:

$$(ds)^2 = (d\phi)^2; \quad \tanh\phi \equiv \frac{e}{m}$$

The metric of physical attributes of the universe thus described depends on the ratio between charge and mass, a case well known in the history of physics. Only, we have to notice that here the mass is gravitational, while in the historical case the mass was inertial, as a consequence of dynamics used in describing the electron.

A problem surfaces when the charge 'splits', as it were, *i.e.* there are Newtonian forces of *electric nature* and also Newtonian forces of *magnetic nature* among the material points. This could be the case if the material points serving for interpretation have *a third physical attribute*, *viz*. a *magnetic charge* and, as a result of this, a magnetic force among them, which is thought to be still Newtonian in character [see, for instance, (Maxwell, 1873), Volume

II, Part II; §§ 371–376]. As our experience shows, the magnetic poles of the same name behave exactly like the electric poles of the same charge. Therefore, a static universe interpreted by ensembles of such identical material points with three physical characteristics, not just only two, will be described by a quadratic quantity reflecting the presence of both electric and magnetic charges. According to the very same theory of the Newtonian forces, instead of (3.1.5), this quantity must be

$$Q(m,q_E,q_M) = m^2 - q_E^2 - q_M^2 \tag{3.1.7}$$

Indeed, the vanishing of this quantity, with the electric charge q_E and the magnetic charge q_M appropriately defined, describes a static ensemble of identical material points in equilibrium under the *three* Newtonian forces prompted by the three physical properties of a material point.

In this case, according to our metrization procedure of the physical attributes of what we think are the fundamental interpretative elements of the matter, the same rules apply for calculating the absolute metric as in the case of two physical attributes of those interpretative elements. To wit, instead of (3.1.6), we will have, with (3.1.7), the absolute metric

$$\left(\frac{mdq_E - q_E dm}{m^2 - q_E^2 - q_M^2}\right)^2 + \left(\frac{mdq_M - q_M dm}{m^2 - q_E^2 - q_M^2}\right)^2 - \left(\frac{q_M dq_E - q_E dq_M}{m^2 - q_E^2 - q_M^2}\right)^2 \tag{3.1.8}$$

In spite of its appearance, this quadratic form is a kind of *surface property*, insofar as it can be expressed in just *two* variables: $x \equiv q_E/m$ and $y \equiv q_M/m$ in the form:

$$\left(\frac{dx}{1-x^2-y^2}\right)^2 + \left(\frac{dy}{1-x^2-y^2}\right)^2 - \left(\frac{ydx - xdy}{1-x^2-y^2}\right)^2$$

However, this property of the expression can be better seen directly through a transformation suggested by the quadratic form from equation (3.1.7), namely:

$$m = q\cosh\phi, \quad e = q\sinh\phi, \quad \begin{matrix} q_E = q\sinh\phi\cos\theta \\ q_M = q\sinh\phi\sin\theta \end{matrix} \tag{3.1.9}$$

Inserting these replacements into (3.1.8) results in a well-known form of the metric in ϕ and θ:

$$(ds)^2 = (d\phi)^2 + \sinh^2\phi\cdot(d\theta)^2 \tag{3.1.10}$$

This is manifestly a metric of negative curvature, that can also be revealed for the relativistic velocity space, for instance [see (Fock, 1959); see also (Mazilu, Agop, & Mercheș, 2019), for the connection of this metric with the classical Kepler motion]. Obviously, the quantity q can be calculated from equation (3.1.9) and amounts to

$$q^2 = m^2 - q_E^2 - q_M^2 \tag{3.1.11}$$

but the absolute metric does not depend on it explicitly. In other words, the absolute metric, in the space of physical attributes of matter, like everywhere in fact, is a two-dimensional – *surfacelike* – quantity, as we said. From the concept of interpretation point of view, the condition $q \neq 0$ is a *nonequilibrium condition*, referring to the very ensemble serving to interpretation. There are such ensembles of 'luxons' or 'photons' or any other '…ons' for that matter, both for the interior of the absolute, as well as for the exterior of the absolute.

As *per our convention*, the interior of absolute describes the matter at the cosmic scale, and in this case we have a quite well known case indicating the nature of q. One notices, indeed, that, towards the center of the absolute, q approaches m, and this fact entitles us in considering it as a measure of the *inertial mass*. In this definition, however, the inertial mass, which is not exactly equal to the gravitational mass, has a precise algebraical, structure due to the contribution of the two kinds of charges to the Newtonian forces of the interpretative ensemble of equilibrium. Rather, in keeping with the previous definition of the ratio between charge and gravitational mass, we can define two such ratios, so that the equation (3.1.11) can be rewritten as:

$$\left(\frac{q}{m}\right)^2 = 1 - x^2 - y^2 \tag{3.1.12}$$

showing *positively* that in building the general relativity, Einstein was, indeed, entitled to 'dismiss' the difference between the gravitational and inertial mass (Einstein, 2004), because, as Hermann Weyl would say, we are always entitled to 'dismiss the electricity in the economy of the universe' [(Weyl, 1923), §39].

The equation (3.1.7) then suggests an important identity which is, we should say, in the natural order of things we perceive in this universe. To wit, we can *assume* the relation:

$$e^2 = q_E^2 + q_M^2 \tag{3.1.13}$$

defining the 'perceived charge', as it were. This mathematical expression further suggests the idea that we cannot *experimentally* decide how much of the perceived charge e is electric and how much of it is magnetic (Katz, 1965). This idea leads to what we believe as one of the most striking pages of the modern natural philosophy, even if quite singular in its way. Obviously, it is also sort of unique of its kind too! This is why we think it deserves a little more elaboration from our part, to say nothing of the positive fact that, as a matter of fact, it turns out to be quite useful in developing a theory of some universes based *not* on the idea of mechanical inertia, involving some kind of Mach principle, but can also be useful in defining some *static universes*, based on the electric and magnetic 'inertia', if we may say so, as in the case of brain and heart (Mazilu, 2019, 2020).

The essential physical trait of any model universe nowadays is its *uniqueness*. This means that the universe is conceived as the world we live in, and, no matter of their space scale, all things material in this universe are to be described as simultaneously existing in such a world, and having the same *fundamental* physical structure. For instance all of them have the same property of inertia, all of them have to be described by the rules of dynamics *etc*.: there is no discrimination in the dominance of the physical attributes, of the kind we have shown above. This philosophy has always promoted the synthetic approach of the whole, usually represented as a set of parts put together by *external connections*. Such an approach may serve, indeed, the social life to a certain extent, by promoting the technology, but surely it does not touch the fundamental laws of existence. The reason is that the external connections in question are, as a rule, only those controllable by human means, and these are quite limited in number, while a part, in its capacity as organ, for instance, is itself a universe, with an unlimited number of connections, as it were, internal as well as external.

Perhaps these ideas are not quite so clear for everybody, but the previous manner of constructing of a physical universe, surely can display the essential points as clearly as possible. First, the matter is there present by a density related to its three attributes – the mass and the two charges – concurrently. The *usual* approach in physics is a 'one-by-one', so to speak: the mechanics deals in the density of mass, the electrostatics deals in the density of charge, and so on. But this is not all of it: in the construction of a model universe, one usually assumes that the *mass is dominant at any scale*, be it microcosm, daily life or macrocosm, and in the very same way, mathematically expressed by the positive sign of the expression (3.1.11). The Einstein's conclusion, formally sanctioning the results of the Eötvös experiments, according to which the inertial mass can be safely considered as identical to gravitational mass, is always presented as independent of space scale where the universe is contemplated. In other words, the *universe is unique*, as we said before, and the physics of its parts should be the same, for it is independent of the space scale. With the model universe presented by an absolute geometry of the physical attributes, the things go along an essentially different direction of thinking.

To wit, that model was constructed based on the explicit *quantitative dominance* of the gravitational mass in each and every one of its identical constituents serving for interpretation, *i.e.* the material points. Thus, the whole metric geometry of the universe is presented based on an inequality:

$$q^2 \equiv m^2 - q_E^2 - q_M^2 > 0 \tag{3.1.14}$$

assumed to be valid in this universe for every *real* material particle of it. We called the quantity *q inertial mass*, and if we think of it in classical terms, this universe is filled with matter whose constitutive formations all satisfy (3.1.14) in a way or another. Dealing in inertial mass, as it were, the universe 'extends' *a priori* until the condition $q^2 = 0$ becomes effective in time and space, which represents the infinity in its definition: *the absolute*. Physically, the absolute represents an ideal world of particles having null inertial mass, just like de Broglie's photons in the case of light, or the Coll's luxons (Coll, 1985), or anything like that. But here the analogy stops: these 'null mass' particles – which we generically call classical material points for now – are, unlike the old photons, *static fictitious particles*, serving merely for the wave-mechanical *interpretation* of matter.

Now, this is only what the mathematics would tell us. Physically, the inequality (3.1.14) is nowadays taken as characterizing a universe in its entirety. The inertial mass is thereby controlled by the matter located beyond the spatial possibilities of human accessibility, whereby that 'beyond' is judged quantitatively: it is the *spatial infinity*, as claimed by Mach's principle. Accordingly, the equality between inertial and gravitational mass should mean that such matter as that creating the inertia, prevails quantitatively regardless of scale. This further means that, regardless of how far can we extend the limits of our observations in space, the matter outside those limits is still quantitatively dominant. In other words, we cannot access but an infinitesimal amount of matter in any finite space, and regardless of how much we extend our capability of knowing the universe, there will always be a substantial amount of 'missing mass', if it is to use the guise of modern theoretical astrophysics.

However, this does not seem to be the direction of the modern cosmology, at least as it was established based on Einstein's ideas. A detailed analysis of that gnoseological moment of natural philosophy seems to indicate that its consensus points towards a universe which is only relatively unique [see, for this, (Mazilu, Agop, & Mercheș, 2020), especially Chapter 1], in the sense that its physical existence is predicated on the fulfilment of condition (3.1.14). According to this condition, the inertia is manifested naturally in the very existence of forces as Newton asserted it, and the lack of inertia is a characteristic of the static ensemble of classical material points serving to interpretation, which represents the absolute of the geometry of physical attributes of those material points. If either one of these ensembles has something to do with the spatial infinity is a problem of further mathematical details, as we shall show here.

The 'physical' character of (3.1.14) seems to prohibit the opposed condition

$$q^2 \equiv m^2 - q_E^2 - q_M^2 < 0 \tag{3.1.15}$$

as being 'physically inadmissible', so to speak, inasmuch as this condition would represent a *negative inertia*, of which we cannot talk quite freely according to our experience. However, we can handle it mathematically: from the point of view of an absolute geometry based on (3.1.14), this condition represents points *outside* the absolute, and these are describable by an *angular metric*, as they call it, inasmuch as the square root of this quantity is purely imaginary. In other words, these points are prone to representing *phases*, like the de Broglie's, or even

Madelung's phases, which can be described as such with respect to the speed of the particles serving to interpretation. They characterize, indeed, the argument of the waves associated with particles in the usual physical cosmology, and these waves propagate with a speed higher than that of light, as in the theory of phase waves of Louis de Broglie.

Still, from a general physical point of view and by the very same token, we can have a real world where the *charge prevails over the gravitational mass*, and this world is certainly real, according to our very experience: it can be either a world of our daily life, like the brain or the heart, which can be considered universes by themselves, in which the charge dominates over gravitational mass (Mazilu, 2019, 2020), or, even better, at another scale, the microcosmos of the physical particles constituting the matter. These satisfy to the inequalities:

$$q^2 \equiv q_E^2 + q_M^2 - m^2 > 0 \tag{3.1.16}$$

Incidentally, one might thus have a logical explanation of why the wave mechanics made its mark especially in the microcosmos, and why by the intermediary of a phase: in a regular cosmology, the condition (3.1.16) is to be read like (3.1.15) and this means phase, as we just said. However, the general idea is that, mathematically and physically, a universe proper cannot exist without its two worlds – of particles and waves – at different scales, representing the structure of matter contained in it, and the conditions (3.1.15) and (3.1.16) should remain as they are, *i.e*. understood relative to the universes they define. Taken as such, the universes can be multiple with no problems: they can be described by the position of their absolutes with respect to each other. However, as this description can become awfully involved, for the moment we shall describe a *unique universe* of the microscopic world based on the quantitative inequality (3.1.16), just like we did before for (3.1.14). This represents a world where *the charge prevails over the gravitational mass*. At the daily spatial scale of things, this would be, as we said, the world of human brain, for instance, or even of world of human heart, while for the microscopical scale we have the universe of fundamental physical particles.

The conditions (3.1.14) and (3.1.16), each aptly defines a real world, which is thus a matter of convention, along with the concept we make of the finite space scale (Georgescu-Roegen, 1971). Thus, in a real world defined by (3.1.14), which also includes the world of our senses, the microscopic world – as well as the world of human brain and heart, for that matter – is defined by the exterior of the absolute, through relation (3.1.15). According to the precepts of the Cayleyan geometry, those points are dual to the points inside, and the metric represents for them angular variations. On the other hand, if the real world is defined by equation (3.1.16), then the opposite inequality represents dual points, outside of the absolute, and this time their metric is the angular metric. In other words in a microscopic world – the world of material point satisfying condition (3.1.16) – it is the gravitation that determines such angles, according to the Cayleyan geometry. This is apparently a point of view that, within an EPR philosophy of reality, was introduced to the physics of latter times under the concept of 'protective measurement' (Aharonov, Anandan, & Vaidman, 1993): the gravitation, therefore the corresponding mass, just 'arranges' things in such a way that the wave function does not collapse. Now, by the same token, we can state that the charge 'arranges' things in the world of our senses, so that the wave function does not collapse. This extension of the statement may have important connotation from physiological point of view, if we take notice of the fact that the brain and heart *belong to charge-dominated universes*. The bottom line though, is that, if the reality is a matter of convention, the Cayleyan geometry provides us with mathematical means to deal with a 'virtual reality', which is defined with respect to this convention.

In order to elucidate what that 'microscopic world' means, with respect to the previous cosmological scale, we need the absolute metric of a cosmology based on condition (3.1.16). Then, if it is to extend the analogy to details, the analogous of equation (3.1.10) is the best place to start, in fact, the only place to start. The metric still has the geometrical expression (3.1.8) up to its sign, but with mass and charges from equation (3.1.9) given by a transformation satisfying naturally the defining condition (3.1.16) for this universe, namely:

$$m = q\sinh\phi, \quad e = q\cosh\phi, \quad \begin{array}{l} q_E = q\cosh\phi\cos\theta \\ q_E = q\cosh\phi\sin\theta \end{array} \tag{3.1.17}$$

Thus, instead of equation (3.1.10), we have now for the metric of this universe

$$(ds)^2 = (d\phi)^2 - \cosh^2\phi(d\theta)^2 \tag{3.1.18}$$

but the analogy between this universe and the customary one seems to stop here, short of any conclusion. This is due to the fact that we have no idea what to do with such a metric: we do not have as our disposal the Einsteinian guiding principles of general relativity, which places its stakes on the metric in order to describe the motion. But we have mentioned before the world of brain, and the motion in such a world can only be a metaphor, at the most: the charge is here 'transmitted', it is not 'moving' in the usual sense we assign to this word. This impasse should, nevertheless, not be taken seriously, for the Einsteinian doctrine left us still another important dowry, above and beyond an idea of motion.

In view of the meaning of the parameters ϕ and θ, as this meaning comes out of the equations (3.1.9) or (3.1.17), we may be interested in knowing how these parameters behave in the host space of matter, rather than with respect to a motion that may very well be nonexistent. Indeed, the motion is referring to a material point, while the distribution of mass and charge can be a collective property of ensembles of such material points. According to the precepts of general relativity and wave mechanics, the distribution of charge and mass in the host space would then involve the concept of field. And, in this respect, we have to recall that Frederick Ernst has shown that knowing the properties of the host space of matter, the Einstein's field equations are reducible to a variational principle (Ernst, 1968, 1971), namely a Dirichlet-type principle defining some so-called *harmonic applications* of the very kind used by Erwin Schrödinger on the occasion of the introduction of his wave function to the physical knowledge [see also (Misner, 1978), for an appropriate – and quite important, we should say – account regarding the use of harmonic applications in physics]. The general Dirichlet's principle can be presented by an equally general mathematical formalism, particularly important for the message of the present work as we shall see in due time, along the following lines (Eells & Sampson, 1964).

An application of the physical attributes onto the host space should have, in general, two components, if it is to consider the metrics (3.1.10) and (3.1.18) of the manifolds of these physical attributes. As we see it, such an application can be taken as defining a surface. The dimension of the manifold of physical attributes is one, the previous variable ϕ, only in the case of two such attributes: electric charge and gravitational mass. Such an application can be represented therefore as having two components, functions of position in the host space. We shall settle here for a uniform notation, that is:

$$y^\alpha(\boldsymbol{x}) = f^\alpha(x^1, x^2, x^3), \quad y^1 \equiv \phi, \qquad y^2 \equiv \theta \tag{3.1.19}$$

If the host space is characterized by the metric tensor $\boldsymbol{h}$, and the metric tensor of the manifold of physical attributes – let us call it a *surface* for now, in order to highlight its maximal two-dimensional character – is $\boldsymbol{g}$ say, then by the application (3.1.19) the matter induces a deformation in space, described by the following variation, $\delta\boldsymbol{h}$ say, of its metric tensor:

$$\delta\boldsymbol{h} \equiv \boldsymbol{f}^*\boldsymbol{g}; \quad \delta h_{ij} = f_i^\alpha f_j^\beta g_{\alpha\beta} \quad f_i^\alpha \equiv \partial y^\alpha / \partial x^i = (\nabla\boldsymbol{f})_i^\alpha \tag{3.1.20}$$

Thus, we can define a one-parameter family of metric tensors, 'updated by matter' as it were:

$$\boldsymbol{h}_\lambda = \boldsymbol{h} + \lambda\cdot\delta\boldsymbol{h} \tag{3.1.21}$$

where λ is the parameter of this family. The volume of space filled with matter – say H, in order to suggest some accordance with the notation for the metric tensor – can be calculated with the benefit of the deformed metric, the way such a volume is usually calculated, *i.e.* by an integral:

$$V_\lambda(H) \equiv \iiint_H \sqrt{\det(\boldsymbol{h}_\lambda)}(dx^1 \wedge dx^2 \wedge dx^3) = \iiint_H \sqrt{\det(\boldsymbol{h}+\lambda\cdot\delta\boldsymbol{h})}(dx^1 \wedge dx^2 \wedge dx^3) \tag{3.1.22}$$

Then, with this functional, we can define the *energy of application* (3.1.19) as:

$$E(\boldsymbol{f}) \equiv \left.\frac{d}{d\lambda}\right|_{\lambda=0} V_\lambda(H) = (1/2)\iiint_H [tr(\boldsymbol{h}^{-1}\delta\boldsymbol{h})]\sqrt{h}(dx^1 \wedge dx^2 \wedge dx^3) \tag{3.1.23}$$

where h is the determinant of the extant host space metric $\boldsymbol{h}$. The scalar integrand:

$$e(\boldsymbol{f}) \equiv (1/2)[tr(\boldsymbol{h}^{-1}\delta\boldsymbol{h})] = (1/2)h^{ij} f_i^\alpha f_j^\beta g_{\alpha\beta} \tag{3.1.24}$$

counts as a *density of energy* of the application $\boldsymbol{f}(\boldsymbol{x})$. Considering the functional $E(\boldsymbol{f})$ stationary, a regular variational principle produces the system of partial differential equations

$$h^{-1/2}\partial_i(\sqrt{h}h^{ij} f_j^\alpha) + \Gamma_{\beta\gamma}^\alpha(\boldsymbol{g}) f_i^\beta f_j^\gamma h^{ij} = 0 \tag{3.1.25}$$

where we used some traditional notations for the partial derivatives and the Christoffel's symbols of second kind of the surface metrics. Ernst's important discovery is that the Einstein gravitational equations for the vacuum and electrovacuum fields can be reduced to such an equation for a variant of the metric (3.1.10) involving a *complex potential* (Ernst, 1968).

Thus the analogy between the customary universe of physics, as described by a general-relativistic philosophy, and the brain universe, for instance, or the universe of microparticles for that matter, does not really stop: it just needs to be properly conducted further. For convenience, we illustrate now the Ernst theory in the brain universe, or in the regular physical universe, anyway, with reference to the metric (3.1.18), *assuming* that the host space of this matter is Euclidean. The principle of harmonic applications will then produce the system (3.1.25) in the form

$$\nabla^2\phi + \sinh\phi\cosh\phi\cdot(\nabla\theta)^2 = 0; \quad \nabla[\cosh^2\phi\cdot(\nabla\theta)] = 0 \tag{3.1.26}$$

which shall be used to describe the location of charge in space and the characteristics of its distribution. One can see that in the case of just electric charge and gravitational mass, the only equation remaining here is the *Laplace equation* for the ratio between charge and gravitational mass [see equation (3.1.6)]. However, we are really interested in a general manifold of physical attributes, having therefore two dimensions.

Indeed, speaking of a the charge at least, it is well known that it always realizes stationary distributions on surfaces, in case the charge can move freely. Again, a manifold coordinated by the parameters ϕ and θ as functions of position in space can also be organized as a surface in space. So, we can try to solve the system (3.1.26) with respect to a parameter representing the distance from a plane in space, which, incidentally, can be the tangent plane of such a surface. That dependence can be realized *via* a linear form in the position coordinates, *i.e.* through a dot product of the form $\xi \equiv \boldsymbol{a}\cdot\boldsymbol{x}$. In this case, the system (3.1.26) can provide solitonic-type solution: the distribution described by the parameters ϕ and θ as functions of the distance from a plane in space. Equation (3.1.26) can be written as a system of two second order ordinary differential equations:

$$\phi'' + \sinh\phi\cosh\phi(\theta')^2 = 0; \quad [\cosh^2\phi\cdot(\theta')]' = 0 \tag{3.1.27}$$

with a prime denoting the derivative with respect to ξ. The second of these equations can be integrated right away, and gives the result

$$\theta' = \frac{C}{\cosh^2\phi} \tag{3.1.28}$$

where C is a constant of integration, which we take as a real number. Using (3.1.28) in the first of equations from (3.1.27) gives

$$\phi'' + C^2\frac{\sinh\phi}{\cosh^3\phi} = 0$$

This can again be integrated directly, multiplying it by $2\phi'$, which leads to a first implicit integral

$$(\phi')^2 = C_1^2 + \frac{C^2}{\cosh^2\phi}$$

with C_1 a new real constant of integration. There is no sign problem in the right hand side here: all things considered real, the expression is positive, and the square root does not involve any complex numbers. After some routine calculations we can arrange here a total differential:

$$\frac{d(\sinh\phi)}{\sqrt{\sinh^2\phi + \cosh^2\phi_0}} = C_0\cdot(d\xi)$$

and therefore a direct integration to a closed form solution:

$$\sinh\phi = \cosh\phi_0\cdot\sinh[C_0(\xi-\xi_0)] \tag{3.1.29}$$

where C_0, ϕ_0 and ξ_0 are real constants of integration. Using this last result in equation (3.1.28) we get the exact differential

$$d\theta = C_0^2\tanh^2\phi_0\frac{d\xi}{\sinh^2[C_0(\xi-\xi_0)]}$$

that can be integrated right away giving the closed form result:

$$\theta = \theta_0 - C_0\tanh^2\phi_0\frac{\cosh[C_0(\xi-\xi_0)]}{\sinh[C_0(\xi-\xi_0)]} \tag{3.1.30}$$

θ_0 being still another integration constant. In getting these results one can use tabulated formulas in order to avoid too much calculational effort [see (Gradshteyn & Ryzhik, 2007), §§ 2.4–2.6]. The equations (3.1.29) and (3.1.30) provide our solution to differential system (3.1.27). It depends on four real parameters: θ_0, ξ_0, ϕ_0 and C_0, which can be fixed by some 'boundary conditions', appropriate to the problem. This may turn out to be routine, but it does not touch the essentials of the physical argument. However, that argument is strongly influenced by the fact that the vector $\boldsymbol{a}$ is *completely arbitrary* in the definition of the parameter ξ.

In spite of the fact that the usual concept of inertia has to be 'dismissed', if we may say so, in a universe of the charge-dominated world, the physical basis of operating of such a world can still be understood by a sheer analogy with the usual 'inertial universe' of physics. This statement has, nevertheless, a precise meaning here: to wit, the charge q, in a charge-dominated universe, just like the inertial mass in the regular Newtonian universe, should be somehow… *induced*. Indeed, in the case of a regular universe, one can think of an 'induction of inertial mass', sanctioned, as it were, by the Mach's principle. This principle can be rephrased to say that the 'inertia of matter is induced by the matter outside the possibilities of human access'. Only, in the case of charge, we have two major complications from which, in fact, we have to learn, and thus correctly rephrase the Mach principle as a universal statement. First, as there are two kinds of possible charges, things become a little more involved than in the case of inertial mass: the charge can be *a priori randomly induced* as electric, as well as magnetic charge. Due to the equation (3.1.13) a phase proper is involved in this description. Therefore, according to Feynman's way of interpretation, the charge may be described, indeed, as Schrödinger intended to do, when he introduced his wave function. What is certainly sure, and we shall prove here, is that a de Broglie-type wave can be associated with the constitutive particles of this universe. Secondly, the right rephrasing of the Mach's principle, may come with a proper interpretation of the mathematical description of the charge association with a classical material point: *the charge is randomly associated in experiments*. For once, this makes those 'possibilities of human access' a little more precise: it is not any kind of human possibility of access, but the *one by specific experiments*. Let us see how these conclusions surface from the previous equations.

First, if we *assume* the identity (3.1.13), the charge e 'splits Euclidean-wise', so to speak, into an electric charge q_E and a magnetic charge q_M. In today's theoretical physics framework, this speaks out of a specific invariance of electromagnetic theory: the invariance with respect to what is generally known as the *duality rotation* in electromagnetics. Expressed simply, this rotation is just an Euclidean rotation that leaves the *experimental* electric charge e of the 'split'suggested in equation (3.1.13) – and used effectively in the equations (3.1.9) and (3.1.17) – invariant:

$$q_E = e\cos\theta, \quad q_M = e\sin\theta$$

Here, however, the Maxwellian doctrine helped building a realistic view of the situation: θ is an angle variable describing the 'split' among the *possible experiments with the charge*. Some of these experiments involve the 'non-gravitational force' as it were, in its 'electric instance', some in its 'magnetic instance', and that assignment is unrecognizable in the experiments with charges, therefore it is humanly uncontrollable, just like the induction of inertial mass.

The argument then goes on to declare that the Maxwellian theory – with its partiality towards the electric charge monopoles, going as far as to eliminate the magnetic charge monopoles from the overall picture of the world – represents just one of the possible choices of the split angle – to wit, $\theta = 0$ – among infinitely many others. The magnetic pole case would be represented here by another special choice, namely $\theta = \pi/2$ for the class of experiments involving charges. Theoretically, the extra degree of freedom, represented here in physical terms by the existence of a *duality rotation* with respect to which the whole charge e behaves invariantly, can be allotted to the known possibility of transition between the *field description* of electrodynamics – by electric and magnetic intensities – and the *Maxwell stresses* description (Katz, 1965). Regarding the *natural philosophical* reason of such a possibility of transition, it is, indeed, quite remarkable! Quoting:

> It is frequently pointed out that the crucial difference between electric and magnetic phenomena, which underlies this dissimilarity (*between Maxwell equation for the electric field and Maxwell equation for the magnetic field, n/a*), is that electric charges *occur in nature* as monopoles whereas magnetic charges do not so occur, *but only as dipoles*, and higher poles. This is demonstrated, for example, by breaking a permanent magnet in two. In so doing one *does not obtain free north and south poles*: each piece has again both polarities of equal magnitude. The

> mathematical formulation of this situation leads then to the equation div$\mathbf{B}$ = 0. On the other hand, electric charges can be obtained free, it is said.
>
> This reasoning is incomplete and deceptive. It is true that a permanent magnet *has equal and opposite magnetic charges near its ends*, and that by breaking the magnet in two and separating the parts and *inserting a chunk of empty space between them* new poles will appear on the new surfaces. But it is equally true that *electric charges occur only in equal pairs of opposite sign at opposite ends of a chunk of vacuum*, for example, by rubbing a rubber rod with a catskin and then separating the two. The vacuum between the rod and the catskin is analogous to the permanent magnet in that it has charges of equal magnitude and opposite sign at opposite ends. If we now *break the vacuum space between the two ends in two*, by *inserting, for example, an isolated conductor between them*, then charges are induced on the metal-vacuum interfaces such that each of the two chunks of vacuum carries again zero total charges at its ends. A well-known variation of this procedure is the so called ice-pail experiment of Faraday. One can pursue this reasoning further. The conclusion is that also electric charges occur only in pairs which can be looked at as the result of polarization. The only difference is that *magnetic poles appear as a result of polarization of a region of space filled with matter* (and so far no region of space filled with vacuum has yielded to polarization of this kind), whereas electric charges appear as *a result of polarization of a region of space filled with vacuum as well as at one filled with matter*.
>
> Logically and formally it is therefore possible to treat electricity and magnetism completely similarly, as long as *one is willing to treat a region of space filled with vacuum on the same footing as a region of space filled with matter*. [(Katz, 1965), *Italics ours*, *a/n*]

This last sentence is, in our opinion, of considerable importance for a natural philosophy of any charge-dominated world: rarely, if ever, is one willing to recognize in physics that while treating, for instance, the vacuum *unreservedly* as a material from theoretical point of view, one has also the obligation to *think of it* as of a material of the daily life, as of a 'chunk' in the expression of Katz. It seems to us that the electromagnetic theory has more in store for showing up than it appears at the first sight, and we will avail here of this possibility. To wit, not only the electromagnetism – in the Maxwellian take, of course – enforced the relativity upon physics at the time it did (Einstein, 1905a), an instance that turned the physics upside down, as it were, but it also able to tell us, in general, how to turn our very intuition into concept, and that even in a comprehensively right way. As far as we are concerned, the message of the previous excerpt is quite clear: the existence or non-existence of the singular magnetic poles – the magnetic monopoles of today's physics – is pending, according to Katz's theory, on the necessity of *describing the electromagnetic field by Maxwell stresses*. This requires, indeed, more than the wave-mechanical idea of interpretation – it calls for a kind of *reverse interpretation* – to which we shall return later on.

For now, it is sufficient to recognize that Katz's case is not singular from the point of view of the interpretation concept. As we have shown in Chapter 2, the Louis de Broglie's interpretation, fills in to completion a historical line of development of the phenomenology of light, culminating with the phenomenon of holography. This phenomenon asks, in a de Broglie interpretation, for a right concept of obstacle. The conclusion was that such a concept should have two differentiae: the *absence* of matter, as in optical devices proper, and the *presence* of matter, as in the de Broglie's optics of electronic devices. The excerpt above from Katz, shows that the phenomenology of charge replicates, by duality – even by a mathematical duality we may, as well, add – this definition of the concept of obstacle in the de Broglie's interpretation of light. No wonder then, that the light became, at a certain point in history, just an electromagnetic phenomenon. However, the introduction of the concept of interpretation for answering to the necessities of wave mechanics, shows that the light phenomenon is by far much more extended than a purely electromagnetic phenomenon of electrodynamics. Let us show how this case can be made for the theory of electricity, by using *perceptibly* what we have learned thus far from de Broglie's particular way of realizing the interpretation.

3.2 Lorentz's Electric Matter

In obtaining the regular classical meaning of the phase – based on expression like that from equations (2.1.2) and (2.1.3) – Louis de Broglie used a condition – which we have called 'strange' at some point in our story – on the amplitude of the wave associated with particles as in equation (2.1.3): the ratio of that amplitude to its derivative along the ray vanishes at the position of particle. This, as de Broglie took it initially, has as a consequence the fact that 'as one approaches the moving particle at constant time' along the ray, the amplitude associated to it varies inversely with the distance to particle. A specification like this is the most general condition of definition of an ensemble of 'contemporary' particles, necessary to a process of interpretation, particularly an *ensemble of simultaneous positions*. However, there is here a subtle reference to the fact that the definition depends on the physical characteristics of the element of this ensemble. If this element is participating in different waves, propagating in different directions in space, the de Broglie's condition is directional, as in the phenomenon of holography: we need to know with respect to which direction that 'inverse proportionality' is actually considered. In the case of a charge-dominated universe, this situation can only be assessed with respect to the tangent plane of the wave touching the point at a certain moment of time, and that distance is measured, in the case of charges, by the parameter ξ of the solitonic theory of charges for instance, introduced by us right above. Then, again, the reference to charge, points out to a historical situation, showing that, in fact, nothing is new in the… knowledge!

Prior to his comprehensive work about the electron (Poincaré, 1906), the great natural philosopher Henri Poincaré critically examined the contribution of every major contemporary work on the electric theory of matter, genuinely highlighting the essential *pros* and *cons* in recommending them as *theories of matter*. Among these theories, the one proposed by Hendrik Antoon Lorentz plays a special part in his presentation [see (Poincaré, 1895), §7], with his recent approach of *electrodynamics of moving matter* (Lorentz, 1892), which brings us to the present issue connected to de Broglie's interpretation. For this special occasion of description of matter – the 'electrodynamical occasion' we should say, but using a particular electrodynamics, specific to the ending of the 19$^{\text{th}}$ century, *i.e.* saturated, as it were, with the Maxwellian doctrine – the analysis of Poincaré adopted three criteria to be satisfied by any of the theories aspiring to what he thought of as being a 'sound' physical theory of electric matter. These criteria reflect, more or less, the position of light phenomenon in the physical thinking at the time. Quoting:

> 1º It must account for the experiments of Mr. Fizeau, i.e. for the *partial* (*original Italics here, a/n*) dragging of the light waves, or, *what comes to the same thing*, of the *transversal electromagnetic waves*;
>
> 2º It must conform the principle of *conservation of electricity and of magnetism*;

> 3º It must be compatible with the principle of *equality of action and reaction*. [(Poincaré, 1895), §9; *our translation and Italics, except as mentioned*]

As one can see, at the time of this work of the great master of modern theoretical physics, the electromagnetic nature of light would have been taken already for granted, to the extent that the electromagnetic light even counted as a model. Now, every major theory to date, analyzed by Poincaré in that year 1895, fails on at least one account of these three and, of course, Lorentz's theory cited above, makes no exception. Poincaré's attitude was one of giving up any hope for new in the world of charge, and thus proceed by 'recognizing' and 'accepting the reality', critically though, by trying to adapt and update that theory that shows the best potential after passing through the fire of critique. In fact, he was very specific:

> One must renounce to *elaborating a perfectly satisfactory theory*, and hold provisionally on the *least flawed of all,* which *seemed to be that of Lorentz*. This suffices for my task, which is *to deepen the discussion of the Larmor's ideas*. [(Poincaré, 1895), §11; *our translation and Italics*]

As he suggested later on (Poincaré, 1900), the criterion of his choice is semi-subjective at best, if we may say so: «*Les bonnes théories sont souples*.» (Good theories are flexible; *original Italics, a/n*). However we uphold the idea that Poincaré's choice was, in fact, *purely objective*, imposed, as it were, by the very nature of the physical problem at hand. On that note, we can very well take the Poincaré's own criterion as having a precise *objective* connotation: a good physical theory is not 'souple' *just by chance*!

First of all, let us see what is the content of the theory of Lorentz, and by what is it 'flawed'. It is an *interpretation* indeed – this is why we considered it as important here, to start with – in the very sense of the later Darwin's definition of this concept. However – and this is a very significant point for us here – *it needs a necessary continuum to be interpreted*. More precisely, it starts directly with the idea of an ensemble of particles, without assuming that these would be somehow 'detached' from a continuum: they are simply existing as the molecules of a classical gas. The theory even contains a suggestion of *confinement* of this ensemble of particles, by referring them to a *hollow space*, that can very well be taken as a coordinate space in the sense of Darwin. However, in this space *there are no static particles*, as the Feynman interpretation above would suggest, but only particles in motion, with no hint whatsoever of how this motion can occur. Quoting again, the structure of electromagnetic *matter* is conceived by Lorentz as a physical structure, based on the idea that…

> A *very large number* of little particles *carying electric charges invariably attached to them*, are dispersed in the *volume of conductors and dielectrics*. They *cross the conductors in all directions* with *very high speeds*. In the *dielectrics*, on the contrary, they cannot experience *but small displacements*, and the *actions of the neighboring particles* tend to reduce them to their *equilibrium positions* from which they are swerving. [(Poincaré, 1895), §7; *our translation and emphasis*]

One can see that in this description the dielectrics and conductors are, indeed, contemplated as *physical structures*, insofar as they have an internal space – a coordinate space in the connotation of Darwin – in the form of a volume: 'electric charges are dispersed in the volume', where they can 'cross in all directions', *i.e.* they can move. Incidentally, let us notice that one can even *imagine experiments* in this volume, capable to account for such a physical structure, as, in fact, historically has happened later on. It is to this kind of fictitious experiments that we have to apply the Katz-type natural philosophy as presented above: any one of such experiments is characterized by an arbitrary charge splitting angle, which thus becomes a statistical variable. Fact is, though, that the randomness of such a physical structure was thought as being characterized by chaotic flows of charges in space, and so it remained in the history of physics. The positive connotation of this approach is that we may conclude that the flows in question can be described as *fluid structures*. From this point of view, the main flaw of a theory built along Lorentz's line can be revealed right away, even in simple words at that: it *does not account for the principle of action and reaction* – the third requirement from the list above. In the words of Poincaré himself:

> … Consider a small conductor A, charged positively and *surrounded by ether*. Assume that *the ether is swept by an electromagnetic wave*, and that at a certain moment of time this wave touches A; the *electric force* due to perturbation acting upon the charge of A *will produce a ponderomotive force acting on the body* A. This ponderomotive force *will not be counterbalanced* from the point of view of the principle of action and reaction, by another force acting upon the ponderable matter. For, all of the ponderable bodies may be supposed as very remote and outside of the perturbed range of ether.
>
> One can get away with this by saying that *there is a reaction of the body* A *upon ether*; it is not less true (though) that we could, if not realize, at least conceive an experiment in which the reaction principle would seem in default, for *the experimenter cannot operate but on the ponderable bodies and does not know how to do it on the ether*. This conclusion would seem therefore hard to accept. [(Poincaré, 1895); *our translation and Italics*]

We need to emphasize again the conclusion here, in order to consider it closer later on: we can *assume an action* of the body upon ether, but experimentally *we do not know* how to accomplish it! This is quite a temperate expression of the general observation that the ponderous bodies *do not act upon ether*: the simple phenomenon of the motion of bodies through vacuum is enough proof for this statement. In the excerpt above, we have however to deal with an electromagnetic wave, and this partially explains the phrase: "does not know how to do it". For, if the ether is conceived as a support for the electromagnetic waves, this is the least we can state, leaving in reserve the fact that, someday, we may come to realize an electromagnetic action upon ether after all.

In the original work from 1892, Lorentz uses d'Alembert principle, then – as today for that matter! – a routine in the mechanics of continua (Lanczos, 1952). However, according to the human experience toward the end of 19^{th} century, he is forced to take special precautions, carried out by an equally special assumption boding the future de Broglie 'strange' condition, in order to cover some stringent necessities. In hindsight we have to recognize one of the great merits of the Lorentz's theory of electric matter, that makes a methodological guide out of it: an interpretation of the electric world has to be, obviously, correlated with a *transport theory*. It was clear by then, that a classical fluid, an ideal fluid to be more precise, at least from the theoretical point of view, should be, in this instance, a model for electric matter. Only, in coping with the transport theory, the d'Alembert principle for this fluid has to have satisfied one important provision, which, we should say, is apparently not properly satisfied even today. To wit, one has to assume that

> If, after *arbitrary movements*, the matter is reduced to its *primitive configuration*, and if, during these movements, *every element of a surface* which is *steadfastly attached to the matter* was

> traversed by *equal quantities of electricity in opposite directions*, all of the points of system will be found in their *primitive positions* [(Lorentz, 1892), §57; *our translation and emphasis*]

Notice that if one takes the 'element of surface steadfastly attached to matter' as referring to an infinitesimal portion of a 'wave surface', the situation indicated by Lorentz in this excerpt is the one indeed envisioned by Louis de Broglie in his condition that we found 'strange' before, in this very work. In view of this, we venture to assume that 'configuration' means here an ensemble of classical material points, so that when Lorentz says that an 'element of surface is attached to matter', we have o understand that this element of surface is determined by the positions of some material points. Lorentz himself finds that this assumption *is not always satisfied*, and one can even tell why, according to his own findings: there is a discrepancy between the *time derivative* and the *substantial derivative* involved in the transport of energy (*loc. cit.*, p. 424). However, Lorentz does not see in this nonconformity a reason not to go any further with the model, and this shows to what extent was he prepared to going himself with the very electric fluid as a model: whatever cannot be conceived for a *classical fluid*, in general, cannot be applied to electric matter either. The classical fluid is thus taken as a reference concept for what we can possibly ever conceive as a fluid, no matter of the fact that the constitutive particles are generating forces or not. This is a big point of the classical theories, for, as we shall see lster, they need here a gauging. Quoting, then:

> If this hypothesis *cannot be admitted in the case of an ordinary fluid*, it *could not be applied to the electric fluid either*. However, this fact does not prevent our equations of motion from being accurate. Indeed, *the mass of this last fluid was supposed to be negligible*, and in calculating the variation δT (*kinetic energy*, *n/a*) only that kinetic energy was considered *which is specific to the electromagnetic movements*; it will suffice therefore that *the material points liable of these motions*, and which *are not to be confused with the electricity itself*, enjoy the property of returning to the same positions *if for each surface element the algebraic sum of the quantities of electricity* by which it has been crossed, is *0*.
>
> Now, one is entirely free to try on the mechanism that produces the electromagnetic phenomena *any convenient assumption*, and while recognizing the difficulty of *imagining a mechanism that possesses the desired property*, it seems to me that we do not have the right to deny its possibility. [(Lorentz, 1892), §67; *our translation and Italics*]

Notice, incidentally, the careful observation that the material points – the classical material points of dynamics – 'liable of motion' are 'not to be confused with electricity itself', a distinction which, we may say, brings forward the very observation of Poincaré about the impossibility of action upon ether. This is a very important observation, important to the point that we have to account for it in the definition of the classical material point, as presented above in this very work. Classically, it was accounted for by the difference between the inertia and the acting forces, in the second principle of dinamics. Also notice that the Lorentz's matter thus defined, belongs to a world of a physical universe at large, but nevertheless a charge-dominated universe, inasmuch as the roles of mass and charge are swapped in the terms of the classical mechanics (Weyl, 1923): 'the mass of *electric* fluid' is here assumed to be negligible, not the charge, as in the regular mechanical universe, and thus the charge becomes now 'dominant'. This speaks of a 'Lorentz universe' where the charge is dominant indeed, however we cannot 'dismiss the mass', as Weyl would say about charge in the regular cosmology, at least not the way we do it in the case of physical universe as we imagine it today.

As for the rest, we realize that a coordinate space is here generated by an ‘element of surface steadfastly connected with the matter’, exactly like in the case of de Broglie’s interpretation, however, with an ‘extra-hint’, if we may say so. Indeed, the transport properties referred to by Lorentz in the excerpt above, which relegate the issues to a transport theory *per se*, can be taken as pointing towards the quotidian geometry analogous to that of a coordinate space referred to the Earth’s crust surface. Related to this, there are two points of interest from which we can draw positive conclusions for further developments: first we have the Galilei kinematics that will offer us a possibility of a gauging having a precise theoretical meaning, as we shall see later. Secondly, we have the very transport theories to be established in this coordinate space: the deformation of Earth’s surface, for once, that changes everything in the coordinate space engendered by a portion of it, and then the transport properties of the atmosphere, from which we have to learn how the charge is properly transported in such a coordinate space.

We cannot leave this presentation of one of the essential aspects of the modern natural philosophy, without taking due notice of the most significant physical point in the excerpt above. Namely, we have today even a technology involving ‘that mechanism’ to be ‘imagined’, which, in producing the electromagnetic phenomena, is ‘possessing the desired property’, from the last sentence of the excerpt above. Just about the time of appearance of the important work of Lorentz from which we excerpted the above fragment, Heinrich Hertz demonstrated that the mechanism in question was not to be ‘imagined’ anymore: it was *physically accomplished in the form of the field generated via a periodic charge in motion* [see the English translations collected in (Hertz, 1893), for the fundamental works which instituted the modern theory of electromagnetic field]. And so, the electromagnetic theory today, does not need any effort of imagination from our part! Or, doesn’t it, really…

3.3 A Coordinate Space and the Statistics it Involves

In view of the fact that, by the geometrical theory of fundamental attributes of the classical material points, as we presented in the §3.1 right above, the matter can be always described interpreting it through a metric geometry, a coordinate space may represent the matter just as well as it represents the space *per se*. The construction of a coordinate space based on the local properties of a surface, as one needs for the Lorentz matter or, generally, for a de Broglie interpretation, proceeds the very same way for the case of the Earth’s surface, as for the case of a de Broglie light ray, or for the case of a physical elementary particle, like the electron of the planetary model, or the very nucleus of such a model. This section builds on the classical differential geometry of surfaces for the presentation of the coordinate space as a $\mathfrak{sl}(2,R)$ metric geometry [see, for full conformity with what we have to say here, (Mazilu, Agop, & Mercheș, 2019, 2020)]. We avail ourselves of this occasion, in order to remind the reader some essential geometrical results in connection with the physics involved in the idea of surface.

The $\mathfrak{sl}(2,R)$ infinitesimal action preserving origin of such a surface-related geometry is given by the three base vectors of this algebra, realizing, at the finite scale, an action in two variables, represented by the corresponding linear homogeneous transformations group. These vectors are:

$$\boldsymbol{X}_1 = s^2\frac{\partial}{\partial s^1},\quad \boldsymbol{X}_2 = \frac{1}{2}\left(s^1\frac{\partial}{\partial s^1} - s^2\frac{\partial}{\partial s^2}\right),\quad \boldsymbol{X}_3 = -s^1\frac{\partial}{\partial s^2} \tag{3.3.1}$$

The corresponding action, in the space of curvature parameters ‘locating’, as it were, the element of surface, is realized by the vectors:

$$A_1 = -\alpha\frac{\partial}{\partial\beta} - 2\beta\frac{\partial}{\partial\gamma}, \quad A_2 = -\alpha\frac{\partial}{\partial\alpha} + \gamma\frac{\partial}{\partial\gamma}, \quad A_3 = 2\beta\frac{\partial}{\partial\alpha} + \gamma\frac{\partial}{\partial\beta} \tag{3.3.2}$$

Here (s^1, s^2) are the coordinates in a tangent plane of the surface at a certain point, and (α, β, γ), are what we just called, and would further like to call the *curvature parameters*, gauging the local geometry in the tangent plane of the surface. The realization given by equations (3.3.2) describes an intransitive finite action in the space of these curvature parameters, which allows transitivity only along specific manifolds, given by constant discriminant of the second fundamental form. Therefore, in a classical geometry, the action realized by operators (3.3.2) is transitive only for constant Gaussian curvature. Let us make this purely geometrical gauging a little more precise, while giving an essential result that together with Cartan's auxiliary algebraical lemmas, helps introducing the statistics, and therefore the physics, in the theory of surfaces.

The functions of geometrical interest, and so much the more those of physical interest, can be presented here as *joint invariants* of the two actions given by equations (3.3.1) and (3.3.2), with the help of *Stoka theorem* (Stoka, 1968). According to this theorem, any joint invariant of the two actions is an arbitrary continuous function of the two algebraic formations

$$\alpha(s^1)^2 + 2\beta s^1 s^2 + \gamma(s^2)^2, \quad \alpha\gamma - \beta^2 \tag{3.3.3}$$

solutions of the system of partial differential system $(X_k+A_k)f(s^1,s^2;\ \alpha, \beta, \gamma) = 0,\ k = 1, 2, 3$. Geometrically, these two algebraical expressions are well known: the first one is the second fundamental form of the surface, which serves to gauge the surface by Dupin's indicatrix, offering the 'level contours', as it were (Struik, 1988); as to the second expression, it is the Gaussian, or absolute curvature of the surface described locally by this Dupin indicatrix. Obviously, if the surface is 'equally' distributed around a local reference level – as explicitly asked, for instance, by the condition defining the electricity in a 'Lorentz world', or by the Louis de Broglie's condition we have called 'strange' – the Gaussian probability density can measure this probability. While geometrically this may be hardly relevant, in physics this instance may, however, prove essential in the cases of Lorentz matter and Louis de Broglie's interpretation. Both these cases ask, in the first place, for a distribution of the distance from a surface in a certain point. A statistics of this quantity can be constructed as follows. The straight lines through origin $s^1 = s^2 = 0$ of the tangent plane can be presented as joint invariants of two actions realized by operators (3.3.1), while the joint invariants of two actions realized by operators (3.3.2), one in the variables (a, b, c) say, the other in the curvature parameters (α, β, γ), are arbitrary continuous functions of the following three algebraic expressions (Mazilu, 2006):

$$\alpha\gamma - \beta^2, \quad ac - b^2, \quad \alpha a - 2\beta b + \gamma c \tag{3.3.4}$$

These quantities are important in problems transcending between the manifolds of transitivity, of which an example will be given presently.

Before entering the calculational detail, let us notice that such a line of contemplation tips us to ammend the definition of a shape as given, for instance, by Shapere and Wilczek [see the works included in the collection (Shapere & Wilczek, 1989)]. Namely, we consider the instant shape – of a nucleus of the planetary model say, or of the Earth for that matter – first of all as a collection of elementary events, characterized by triplets of real numbers representing the values of curvature parameters. Again, the illuminating examples are classical Fresnel's construction of the wave surface from pieces accessible in diffraction experiments, Lorentz electrical matter as presented above, and de Broglie's interpretation which is the overall subject of the present work. All these examples can be appended as particulars to the following general statistical theory.

According to Stoka theorem, the statistical ensemble of heights of the surface with respect to the tangent plane in a point, taken as reference level, *may be described*, at last in some particular cases to be specified as needed, by a normal probability density

$$p_{XY}(s^1,s^2\,|\,\alpha,\beta,\gamma)\equiv\frac{\sqrt{\alpha\gamma-\beta^2}}{2\pi}\exp\left(-\frac{1}{2}[\alpha(s^1)^2+2\beta s^1 s^2+\gamma(s^2)^2]\right) \tag{3.3.5}$$

Here we have two statistical variables *X* and *Y*, of which we do not know too much for now, other than that they are 'quantified' by the coordinates (s^1, s^2) of position on any one of the spots of the surface, as suggested before. We have, therefore, a way to calculate the statistics of a quadratic variable *Z*(*X*, *Y*), having the generic values:

$$z(s^1,s^2)\equiv\frac{1}{2}[a(s^1)^2+2bs^1s^2+c(s^2)^2] \tag{3.3.6}$$

This can represent, for instance, local perturbations of the second fundamental form, due to some surface deformations. Thus we need to find, first of all, the probability density of this variable, under condition that the reference geometry is characterized by an *a priori* probability density as given by the Gaussian from equation (3.3.5). This is a routine statistical problem, but our point here is that the probability density of *Z* should also satisfy the Stoka theorem, in the precise sense that it must be a function of the algebraic formations from equation (3.3.4). This leaves us with a functionally undecided probability density though, even if we impose some natural constraints in order to construct it. However, once the Stoka theorem satisfied, we can be sure to have a way of physics in the geometry of surfaces, and therefore we are assured of the universality of this statistics.

Therefore, we solve the problem by proceeding directly to statistical calculations, in the usual manner of the statistical theoretical practice. Thus, we have to find, first, the characteristic function of the variable (3.3.6). This is the expectation of the imaginary exponential of *Z*, using (3.3.5) as probability density. Performing this operation directly, we get, with an obvious notation for the average:

$$\langle e^{i\zeta Z}\rangle=\left(2\pi\sqrt{1+(i\zeta)\frac{\alpha c-2\beta b+\gamma a}{ac-b^2}+(i\zeta)^2\frac{\alpha\gamma-\beta^2}{ac-b^2}}\right)^{-1} \tag{3.3.7}$$

In view of (3.3.4), this characteristic function certainly satisfies the Stoka theorem, which thus reveals its right place in a physical theory: it should serve for the selection of the correct physical functions, specifically the probability density, or the characteristic function, as in this case. The sought for probability density, corresponding to this characteristic function, can then be found, as usually in theoretical statistics, by a routine Fourier inversion applied to the right hand side of equation (3.3.7). This can be done based on existing tabulated formulas [see (Gradshteyn & Ryzhik, 2007), especially the examples 3.384(43); 6.611 (40); 9.215(16) & (39)]. The result is:

$$p_Z(z\,|\,\alpha,\beta,\gamma)\equiv\sqrt{A\cdot B}\exp\left(-\frac{A+B}{2}z\right)\cdot I_0\left(\frac{A-B}{2}z\right) \tag{3.3.8}$$

Here I_0 is the modified Bessel function of first kind and order zero, and *A*, *B* are two constants to be calculated from the formulas

$$A+B=-\frac{\alpha c-2\beta b+\gamma a}{ac-b^2},\quad A\cdot B=\frac{\alpha\gamma-\beta^2}{ac-b^2}\quad A>B \tag{3.3.9}$$

Again, this probability density obviously satisfies the Stoka theorem, for it is a function of the joint invariants from equation (3.3.4). And so do the statistics describing the variable Z, *i.e.* its mean and variance, for they can be calculated as

$$\langle Z\rangle \equiv \frac{1}{2}\left(\frac{1}{A}+\frac{1}{B}\right) = -\frac{1}{2}\frac{\alpha c - 2\beta b + \gamma a}{\alpha\gamma - \beta^2};$$
$$\mathrm{var}(Z) \equiv \frac{1}{2}\left(\frac{1}{A^2}+\frac{1}{B^2}\right) = \frac{1}{2}\left(\frac{\alpha c - 2\beta b + \gamma a}{\alpha\gamma - \beta^2}\right)^2 - \frac{ac - b^2}{\alpha\gamma - \beta^2} \tag{3.3.10}$$

We thus have the interesting conclusion that the essential statistics related to variable Z do not depend but on its coefficients and the values of the curvature parameters describing the portion of surface characterized by them.

As noticed before, the previous theory can help us in securing, from a theoretical point of view, a purely statistical connotation in the curvature space itself, and of its variations due to incidental physical reasons. Assume indeed, that a, b and c are *some variations* of the curvature parameters α, β and γ, respectively, over an ensemble of points, locally representing an instantaneous surface *inside* matter, as in the case above of Lorentz electrical matter. This instantaneous surface may be a surface proper separating the matter from space, or even an imaginary surface inside matter itself, which is Lorentz's case actually. It turns out that the statistical variable having its values given by equation (3.3.6) can be taken, in fact, as a variation of the second fundamental form of such a surface, when this variation is controlled only by the variations of its coefficients. Such a situation is particularly important for physical applications. In this case, the equation (3.3.6) gives the values of a statistical variable – let us call dZ in order to show its 'differential' nature, even though it may be 'fractal' in general – which has, according to equation (3.3.10), the *expectation value*, which is a first order symmetric differential in the curvature parameters, given by equation

$$\langle dZ\rangle = \frac{1}{2}\frac{2\beta d\beta - \alpha d\gamma - \gamma d\alpha}{\alpha\gamma - \beta^2} \tag{3.3.11}$$

and the variance, which is the second order symmetric differential in the curvature parameters:

$$\mathrm{var}(dZ) = \frac{1}{2}\left(\frac{2\beta d\beta - \alpha d\gamma - \gamma d\alpha}{\alpha\gamma - \beta^2}\right)^2 - \frac{d\alpha d\gamma - (d\beta)^2}{\alpha\gamma - \beta^2} \tag{3.3.12}$$

These two statistics have a precise *geometrical* meaning in the space of curvature parameters, which may not be obvious by itself to the casual observer. However, if we use them in building another quadratic statistic, to wit:

$$\mathrm{var}(dZ) - \langle Z\rangle^2 = \frac{1}{4}\left(\frac{2\beta d\beta - \alpha d\gamma - \gamma d\alpha}{\alpha\gamma - \beta^2}\right)^2 - \frac{d\alpha d\gamma - (d\beta)^2}{\alpha\gamma - \beta^2} \tag{3.3.13}$$

this statistic has a precise geometrical meaning. First, the right hand side of this formula is the Riemannian metric which can be built by the methods of absolute geometry for the space of the 2×2 symmetric matrices, having the curvature matrices with null Gaussian curvature as points of the absolute quadric (Mazilu & Agop, 2012). This is actually the Klein model of the so-called 'fourth geometry of Poincaré', using some modern terms (Duval & Guieu, 2000). Secondly, one can prove that the quadratic form (3.3.13) is just the Cartan-Killing metric of a homographic action of the 2×2 real symmetric matrices. For, it is, indeed, the quadratic form

$$\left(d\ln\sqrt{\alpha\gamma - \beta^2}\right)^2 + \left(\frac{\omega^2}{2}\right)^2 - \omega^1\omega^3 \tag{3.3.14}$$

where $\omega^{1,2,3}$ are the 1-forms giving the coframe of the $\mathfrak{sl}(2,\mathrm{R})$ algebra:

$$\omega^1 = \frac{\alpha d\beta - \beta d\alpha}{\alpha\gamma - \beta^2}, \quad \omega^2 = \frac{\alpha d\gamma - \gamma d\alpha}{\alpha\gamma - \beta^2}, \quad \omega^3 = \frac{\beta d\gamma - \gamma d\beta}{\alpha\gamma - \beta^2} \tag{3.3.15}$$

and the quadratic (3.3.14) can be written, up to a numerical factor, as $tr[(\boldsymbol{b}^{-1} \cdot d\boldsymbol{b})^2]$, where $\boldsymbol{b}$ is the curvature matrix:

$$\boldsymbol{b} \equiv \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \tag{3.3.16}$$

as one can verify right away by a little more involved calculations.

A few algebraical relations among the differential forms from equation (3.3.15) are in order, insofar as they are necessary in future developments in this very work, and this is a good occasion to gather them together. The differential forms (3.3.15) give a *coframe* for a general $\mathfrak{sl}(2,\mathrm{R})$ algebra. It turns out that, among other things, the space of curvature parameters of a portion of surface in space can be organized as a Riemannian space, but this is not the only instance where this algebra is involved. One important example that needs to be noticed right away is the case of physical features of a classical material point, as in the §3.1: The 'inertial mass' q from equation (3.1.25) or (3.1.13), can be reduced to $(ac - b^2)$ by a simple linear transformation of the charges and gravitational mass: the physical features, just like the curvature parameters of a surface are described by the same algebra. Anyway, from a purely formal point of view, notice first the following differential relations in the space of curvature parameters, which can be proved by a direct calculation:

$$d \wedge \omega^1 = \frac{\alpha}{\sqrt{\alpha\gamma - \beta^2}}\boldsymbol{\Theta}, \quad d \wedge \omega^2 = \frac{2\beta}{\sqrt{\alpha\gamma - \beta^2}}\boldsymbol{\Theta}, \quad d \wedge \omega^3 = \frac{\gamma}{\sqrt{\alpha\gamma - \beta^2}}\boldsymbol{\Theta} \tag{3.3.17}$$

Here Θ is what we would like to call the *differential 2-form of Hannay*, defined by:

$$\boldsymbol{\Theta} \triangleq \frac{\alpha d\beta \wedge d\gamma + \beta d\gamma \wedge d\alpha + \gamma d\alpha \wedge d\beta}{\boldsymbol{\Delta}^{3/2}}; \quad \boldsymbol{\Delta} \equiv \alpha\gamma - \beta^2 \tag{3.3.18}$$

The Hannay's 2-form Θ is closed, because it is the exterior differential of a 1-form:

$$\boldsymbol{\Theta} \equiv d \wedge \chi; \quad \chi \equiv \frac{\alpha + \gamma}{\boldsymbol{\Delta}^{1/2}} d\left(\tan^{-1}\frac{2\beta}{\alpha - \gamma}\right) \tag{3.3.19}$$

which represents the classical *Hannay angle* for the problem of variation of the second fundamental form of a surface. In the present context this 1-form can be written as

$$\chi = \frac{2}{\sinh\xi}(\omega^1 + \omega^3), \quad \xi \triangleq \ln\frac{\alpha + \gamma}{2\sqrt{\boldsymbol{\Delta}}} \tag{3.3.20}$$

including explicitly the ratio between the mean and Gaussian curvatures of the surface. It gives therefore a way of establishing the mathematical procedure for the local problem of surface physics, but certainly has everything in common with the angle originally designated as such [see (Hannay, 1985); see also (Berry, 1985)]. More than this, there is also a general statistic – which has to be considered as fundamental for a gauging theory of scale transitions, as we shall see – involved in this exterior calculus, generalizing the case put forward by Hannay himself, which will be revealed later on along our story.

Meanwhile, continuing on with the pure algebraic relations, we can verify the following 'structural' ones:

$$\omega^1 \wedge \omega^2 = \frac{\alpha}{\sqrt{\Delta}}\Theta, \quad \omega^2 \wedge \omega^3 = \frac{\gamma}{\sqrt{\Delta}}\Theta, \quad \omega^3 \wedge \omega^1 = -\frac{\beta}{\sqrt{\Delta}}\Theta \tag{3.3.21}$$

They are, indeed, structural: from (3.3.17) and (3.3.21) we have the characteristic equations of a $\mathfrak{sl}(2, \mathrm{R})$ coframe structure given by:

$$d\wedge\omega^1-\omega^1\wedge\omega^2=0,\quad d\wedge\omega^2+2(\omega^3\wedge\omega^1)=0,\quad d\wedge\omega^3-\omega^2\wedge\omega^3=0 \tag{3.3.22}$$

Using these relations we can draw an important conclusion, destined to guide our future research in the physical problems connected to this algebra. As we said, the relations above will not be used exclusively for the theory of surfaces, but for a host of other physical problems involving 2×2 matrices: this is why we thought opportune to gather them together, on this occasion, in the first place. One of the physical applications of this algebra, we know for sure, was epoch-making, and we need to reveal it in this connection, for future reference; some other applications, usually thought as having no connection with it, will be presented by us under the same heading, as we go along with this work.

That physically relevant, epoch-making application we are talking about, is connected to the following observation of which we took notice only in passing above: assume that we have a quadratic form with variable coefficients, in a linear space whose dimension is not important for the moment, for, what we have to say is, up to a point, a matter of general linear algebra. Let that quadratic form be, in a modern notation:

$$Q\equiv\langle x|\boldsymbol{m}|x\rangle \tag{3.3.23}$$

with $\boldsymbol{m}$ designating a symmetric 'matrix', as usual. Assume now that any variation of this quadratic form can be expressed as composed of the variations of its *formal* components, $\boldsymbol{m}$ and $|x\rangle$, according to the rules of differential calculus. This means that the variation of Q, can be written as a sum of three formal components:

$$\delta Q=\langle\delta x|\boldsymbol{m}|x\rangle+\langle x|\delta\boldsymbol{m}|x\rangle+\langle x|\boldsymbol{m}|\delta x\rangle \tag{3.3.24}$$

Now, still assume that the variation of the vector $|x\rangle$ is a vector whose components depend linearly on the components of the vector itself; in a word, we assume that

$$|\delta x\rangle=\boldsymbol{G}|x\rangle\quad\therefore\quad\langle\delta x|=\langle x|\boldsymbol{G}^t \tag{3.3.25}$$

where the symbol $\boldsymbol{G}$ is intended to signify 'gauging', and the upper index t signifies 'transposed'. In this special gauging, the equation (3.3.24) can further be written in the form:

$$\delta Q=\langle x|\delta\boldsymbol{m}+\boldsymbol{G}^t\cdot\boldsymbol{m}+\boldsymbol{m}\cdot\boldsymbol{G}|x\rangle \tag{3.3.26}$$

As a benefit of gauging, this is still a quadratic form, like the original one from equation (3.3.23). The existence of such a gauging, allows us to describe in more detail some important incidental cases of variation of a quadratic form, starting with the observation that $Q+\delta Q$ still remains a quadratic form. For instance, if we are allowed to choose the gauging such that the two matrices satisfy the relation

$$\boldsymbol{G}^t\cdot\boldsymbol{m}+\boldsymbol{m}\cdot\boldsymbol{G}=\boldsymbol{0}\quad\therefore\quad\delta Q=\langle x|\delta\boldsymbol{m}|x\rangle \tag{3.3.27}$$

then the variation of the quadratic form in this gauge is simply dictated by the variation of entries of the matrix $\boldsymbol{m}$. If these are constants, the quadratic form is a conserved quantity, otherwise it varies. This is, for instance, the case of the classical Hamiltonian of a harmonic oscillator, based on which we need to insist when describing that epoch-making case we were mentioning above. In order to do this properly, we need to describe yet another gauge, referring, this time, to the matrix $\boldsymbol{m}$ and its variation. So, if we are allowed to gauge the matrix $\boldsymbol{m}$ such that

$$\delta\boldsymbol{m}=-\boldsymbol{m}\cdot\boldsymbol{G}\;\;\therefore\;\;\delta Q=\langle x|\boldsymbol{G}^t\cdot\boldsymbol{m}|x\rangle \tag{3.3.28}$$

The variation of the quadratic form on the vectors $|x\rangle$ gauged according to the equation

$$\left|\delta x\right\rangle = -(\boldsymbol{m}^{-1}\cdot\delta\boldsymbol{m})\left|x\right\rangle \tag{3.3.29}$$

still depends *exclusively* on the variations of entries of matrix $\boldsymbol{m}$, because from equations (3.3.28) and (3.3.29) we get right right away

$$\boldsymbol{G}^{t}\cdot\boldsymbol{m} = -\delta\boldsymbol{m} \;\;\therefore\;\; \delta Q = -\left\langle x\right|\delta\boldsymbol{m}\left|x\right\rangle \tag{3.3.30}$$

Let us apply now these general considerations for the case of a 2×2 symmetric matrix, like the curvature matrix from equation (3.3.16). For such a circumstance, in the notations above, we have

$$\boldsymbol{G}_{-} \equiv -(\boldsymbol{m}^{-1}\cdot\delta\boldsymbol{m}) = (d\eta)\boldsymbol{1} + \boldsymbol{\Omega};\quad \boldsymbol{\Omega} \triangleq \begin{pmatrix} \omega^{2}/2 & \omega^{3} \\ -\omega^{1} & -\omega^{2}/2 \end{pmatrix} \tag{3.3.31}$$

where $\boldsymbol{1}$ is the identity 2×2 matrix, and we used the notation $\eta \equiv -ln(\sqrt{\Delta})$. The index of the gauging matrix is intended to show that the variation of the quadratic form Q is 'negative', as in equation (3.3.30). The previous equations show, therefore, that there are two cases of gauging by an equation like (3.3.25), where the variation of the quadratic form (3.3.23) is essentially dictated by the sole variations of the entries of its matrix. The first case is given by a matrix $\boldsymbol{G}$ solution of the matrix equation (3.3.27), which in cases where $\boldsymbol{m}$ is the curvature matrix $\boldsymbol{b}$ from equation (3.3.16), has the solution

$$\boldsymbol{G}_{+} = (\delta\tau)\cdot\begin{pmatrix} \beta & \gamma \\ -\alpha & -\beta \end{pmatrix} \tag{3.3.32}$$

where the front factor ($\delta\tau$) has to be the variation of a *continuity parameter*, in order that the equations (3.3.25) make sense. The lower index here shows that the variation of quadratic form is 'positive', as in equation (3.3.27). In view of the fact that the gauging equations are not necessarily equations of motion, but equations of global behavior of the vectors, ($\delta\tau$) does not have to be necessarily a time. A connection between the two possibilities of gauging will be able to indicate what ($\delta\tau$) may be, and this is provided by equation (3.3.17). Namely, the two gauging cases is given in the form

$$d\wedge\boldsymbol{G}_{-} = \frac{\Theta}{\delta\tau}\cdot\boldsymbol{G}_{+} \tag{3.3.33}$$

where Θ is the Hannay 2-form (3.3.18). This suggests a connection between the Hannay 2-form and the parameter ($\delta\tau$), that will become more obvious on the occasion of studying some metric properties of the $\mathfrak{sl}(2,\mathrm{R})$ Riemannian space.

3.4 George Yuri Rainich: Transcribing a Coordinate Space into an Ordinary Space

The problems involved in such a job are vast, sometimes seem even insurmountable, and, moreover, multiple. Since we are on the subject of of the local theory of surfaces, we can very well start the task by 'tweaking' a little bit the classical problem of local embedding a surface in an ambient three-dimensional space. It is what in the words of Charles Galton Darwin we can designate as the problem of 'translating a coordinate space onto an ordinary space'. This should be taken as a first step building the physics of an ordinary space. The equations of embedding in an infinitesimal space $s^{k} \equiv dx^{k}$, where the Latin indices take the values *1*, *2*, *3*, may not be satisfied globally, and thus the differentials on surface are, in a sense, arbitrary with respect to space coordinates: they need to be chosen in advance, and thus the surrounding space needs to be described with respect to surface by a reference frame that must be constructed *ad hoc*, not simply *adapted* from among those of the surrounding space. The situation appears at its best if we analyze the conditions of embedding of a surface in general.

Indeed, the classical embedding consists of the choice of a coordinate system satisfying the condition $s^3 = 0$, which should represent, locally at least, the equation of the surface. The embedding equations should then represent actually the most general condition defining a family of surfaces $d\wedge s^3 = 0$, which is satisfied even for s^3 *constant but arbitrary*, *i.e.* for any surface parallel to the one given by $s^3 = 0$. In fact, the most general solution of the equation $d\wedge s^3 = 0$, is *an arbitrary exact differential* 1-form. In view of this, further differential conditions should be provided in order to describe the embedding of a surface in the surrounding space. In the classical theory of surfaces these conditions are usually known as Gauss-Codazzi, or sometimes as Mainardi-Codazzi equations. We will illustrate their necessity by the way of an example involving explicitly the physics from the very beginning of the construction. Obviously, this example mimics, quite appropriately we should say, the very historical order of ways in which the man gained the comprehensive concept of space *via* geometrical knowledge.

Alois Švec insisted upon the fact that an *adaptation* of a geometrical theory of space to a given surface is essential (Švec, 1988), and we can surely say that this is the case insofar as physics is involved, anyway. But, we have to add, that in such a process, essential would be not so much the *adaptation* to surface of an existing reference frame, as much as a kind of *adjustment to the parametrization* of the surface. The most general kind of adjustement according to Švec's idea, is a linear, homogeneous relation between the fundamental differentials of the local geometry of surface, (s^1, s^2) in the notation above, and some known exact differentials of two parameters, u and v say. Therefore, such a relation is given by an exterior differential equation of the type:

$$s^1 \wedge du + s^2 \wedge dv = 0 \tag{3.4.1}$$

Indeed, according to Cartan's algebraical lemma, from equation (3.4.1) we have that the elementary displacements must be expressed in terms of the differentials (du) and (dv) by a relation involving a symmetric matrix:

$$s^1 = \alpha du + \beta dv, \quad s^2 = \beta du + \gamma dv \tag{3.4.2}$$

The original case of Alois Švec (*loc. cit. ante*) is included here for a special choice of the parameters, whereby we have to choose $\beta = 0$. Thus, the fundamental elementary displacements on our surface – which, in this instance, can be properly called *surface of reference – are not* always exact differentials, but are, nevertheless, always linear combinations of such differentials.

This approach of the local geometry of a surface is closer to the spirit of the physics of surfaces, indeed. As we have noticed previously, the statistics does not necessarily involve differential quantities: for instance the curvature parameters of a surface cannot be determined absolutely, but only in our mind, as it were. In reality they are always finite quantities, with respect to a reference state, therefore they can be even fractal, which is, in fact, the general case. If such quantities (α, β, γ) are built in order to satisfy the relation (3.4.2), then the possible curvature properties get into theory only second-hand, as it were. Indeed, from (3.4.2) we obtain, by an exterior differentiation:

$$d\wedge s^1 = \frac{\gamma d\alpha - \beta d\beta}{\alpha\gamma - \beta^2}\wedge s^1 + \frac{\alpha d\beta - \beta d\alpha}{\alpha\gamma - \beta^2}\wedge s^2, \quad d\wedge s^2 = \frac{\gamma d\beta - \beta d\gamma}{\alpha\gamma - \beta^2}\wedge s^1 + \frac{\alpha d\gamma - \beta d\beta}{\alpha\gamma - \beta^2}\wedge s^2 \tag{3.4.3}$$

so that a system defining the curvature properties can further be obtained from this one by some involved calculations, leading to a curvature matrix like that from equation (3.3.16). Let us do this exercise here, in order to decide what is involved in the local geometry of the surface, from a physical point of view. In order to get an interpretation of the equation (3.4.3), we start with the obvious the in-surface displacement should be an exact differential, as in the regular local geometry of the surfaces:

$$d\boldsymbol{x} = s^{\alpha}\boldsymbol{e}_{\alpha}, \quad d\wedge d\boldsymbol{x} = \boldsymbol{0} \tag{3.4.4}$$

with the usual summation over the Greek indices taking the values *1* and *2*. Expanding the left hand side of the second equation, over the expression of elementary displacements given in the first of these equations, leads to

$$d\wedge\left|s\right\rangle + \boldsymbol{\Omega}^{t}\wedge\left|s\right\rangle = \left|0\right\rangle \quad \therefore \quad d\wedge s^{\alpha} + (\boldsymbol{\Omega}^{t})^{\alpha}_{\beta}\wedge s^{\beta} = 0 \tag{3.4.5}$$

where we assumed a Frenet-Serret equation representing the evolution of the reference frame on the surface:

$$\left|d\boldsymbol{e}\right\rangle = \boldsymbol{\Omega}\cdot\left|\boldsymbol{e}\right\rangle; \quad \left|\boldsymbol{e}\right\rangle \triangleq \begin{pmatrix} \boldsymbol{e}_2 \\ \boldsymbol{e}_1 \end{pmatrix} \tag{3.4.6}$$

and the upper index *t* means 'transposed', as before. Therefore the equation (3.4.3) gives us the Frenet-Serret matrix of the evolution of reference frame $|\boldsymbol{e}\rangle$ over the portion of surface located currently by the three coefficients of curvature (α, β, γ). Performing the calculation the matrix $\boldsymbol{\Omega}$ turns out to be the one from equation (3.3.32). This shows that the coframe of the $\mathfrak{sl}(2, \mathrm{R})$ algebra, as given in equation (3.3.15) above, serves in describing a Frenet-Serret equation of motion for the reference frames on the portion of surface considered. There are a few peculiarities of this Frenet-Serret evolution. of the reference frame

First, notice that the matrix $\boldsymbol{\Omega}$ is *not* skew-symmetrical in general, as in the case of the Euclidean theory of surfaces. This shows that the reference frame $|\boldsymbol{e}\rangle$, as described by Frenet-Serret evolution equation, may not be orthonormal; this is the reason that we did not put a caret over the letters symbolizing the base vectors of the reference frame, for that usually indicates unit vectors of an orthonormal frame. In order to have an orthonormal reference frame, the matrix $\boldsymbol{\Omega}$ needs to have a constant determinant, and satisfy to the differential condition $\omega^1 = \omega^3$. These two conditions show that two of the three parameters (α, β, γ) can be expressed linearly in terms of the third one, and therefore the three components of the coframe (3.3.16) simply reduce to two, so these can be expressed in a single variable parameter. However, this is just a very special case. In general, one has to notice the quaternion form of (3.3.16), where the basis quaternions are the 2×2 matrices:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{3.4.7}$$

of which the last three, $\boldsymbol{q}_1$, $\boldsymbol{q}_2$ and $\boldsymbol{q}_3$ say, generating the vector part of the quaternion, satisfy the following commutation relations:

$$[\boldsymbol{q}_1, \boldsymbol{q}_2] = \boldsymbol{q}_1; \quad [\boldsymbol{q}_2, \boldsymbol{q}_3] = \boldsymbol{q}_3; \quad [\boldsymbol{q}_3, \boldsymbol{q}_1] = -2\boldsymbol{q}_2 \tag{3.4.8}$$

These structural relations are characteristic for a $\mathfrak{sl}(2,\mathrm{R})$ algebra: as one can see by direct calculation, they are satisfied by the vectors from equations (3.3.1) and (3.3.2), which we take as the general commutation relations describing this structure. However, no matter if orthonormal or not, the frame $|\boldsymbol{e}\rangle$ still has to satisfy the equation

$$\left|d\wedge d\boldsymbol{e}\right\rangle = \left|\boldsymbol{0}\right\rangle \quad \Leftrightarrow \quad (d\wedge\boldsymbol{\Omega})\left|\boldsymbol{e}\right\rangle + \left|d\boldsymbol{e}\right\rangle\wedge\boldsymbol{\Omega} = \left|\boldsymbol{0}\right\rangle \tag{3.4.9}$$

which means that the matrix of evolution satisfies the Maurer-Cartan differential condition:

$$d\wedge\boldsymbol{\Omega} + \boldsymbol{\Omega}\wedge\boldsymbol{\Omega} = \boldsymbol{0} \tag{3.4.10}$$

We stop here for now, with this presentation of the embedding of a general surface in the 'Švec's manner', as it were, remaining to develop it as we go along with our necessities. Two conclusions are worth noticing, as they will make some essential points of the mathematics to be used by such a theory of surfaces, insofar as it has to serve the physics. First, this theory of surfaces cannot be Euclidean, at least not locally: this fact can be recognized in the Frenet-Serret evolution of the surface reference frame. Secondly, we have to recognize the necessity of introducing the quaternions in the form of 2×2 matrices, the reason of which will become apparent later on, as we go along with our story.

Represented as such, however, *in the manner of Švec* as we said, the geometry of space in the neighborhood of a position from the surface taken as reference, can be a conform Euclidean geometry, completely determined by the a surface and its curvature matrix at the given position. Indeed, the position of a point *in the surrounding space* can be located in a local Euclidean reference frame by the differential coordinates (Rainich, 1925)

$$\begin{aligned} dx &= (1/2)[(1-u^2+v^2)s^1 - 2uvs^2] \\ dy &= (1/2)[-2uvs^1 + (1+u^2-v^2)s^2] \\ dz &= us^1 + vs^2 \end{aligned} \tag{3.4.11}$$

The parameters u and v are *convenient* parameters on the surface. The origin of reference frame is physically defined by the equations $s^1 = s^2 = 0$. Thus, the qualification 'physically' means here that from a mathematical point of view, in this definition there should be some relations between the parameters (u, v) and the parameters (α, β, γ), whose origin can be physical. Further on, (dx) and (dy) from equation (3.4.11) represent homogeneous transforms of the differential forms $s^{1,2}$, *with the help of the surface parameters* themselves.

This suggests a kind of general 'natural philosophy' that can be related to this representation: the surface 'arranges' locally the physics – as it always did in fact, even in the historical order mentioned above – in the sense that the accomodation of space to surface is first given by the condition (3.4.2) on differentials, which need themselves to express the surface physics in a certain way. This very physics is then lent to space by a condition of accomodation of the differentials of the coframe:

$$dx \wedge s^1 + dy \wedge s^2 = 0 \tag{3.4.12}$$

According to Cartan's Lemma, this equation leads to $|dx\rangle = \boldsymbol{M}\cdot|s\rangle$, where $\boldsymbol{M}$ is here a symmetric 2×2 matrix having *entries which depend exclusively on surface*. These entries are concretely expressed by some quadratic forms in the binary parameter (u,v), as shown in the first two equations from (3.4.11). On the other hand, (dz) from the third of those equations clearly represents – when the coefficients (α, β, γ) in equation (3.4.2) are constants, of course – *the height of the point in space* with respect to the tangent plane of the surface, as given by a quadratic form, considered as the second fundamental form of the surface in which case, the matrix $\boldsymbol{\alpha}$ defining that transformation is the curvature matrix of the surface. Obviously this is the case with equation (3.4.12). Indeed, the first fundamental form of the surface is given by the Euclidean metric of the surrounding space, which can be obtained directly from equation (3.4.11). The metric tensor turns out to be:

$$\lambda^2 \begin{pmatrix} \alpha^2+\beta^2 & \beta(\alpha+\gamma) \\ \beta(\alpha+\gamma) & \beta^2+\gamma^2 \end{pmatrix} \equiv (\lambda\boldsymbol{\alpha})^2 \tag{3.4.13}$$

with $2\lambda \equiv 1+u^2+v^2$. Classically speaking, this tensor represents actually *the third fundamental form* of a regular surface, just as (dz) is its *second fundamental form*. The conditions that the differential forms (3.4.11) are exact differentials can be written in the form:

$$d \wedge |s\rangle = \boldsymbol{\Theta} \wedge |s\rangle; \qquad \boldsymbol{\Theta} \equiv \frac{1}{1-\mu^2-\nu^2}\begin{pmatrix} d\mu + \mu d\mu + \nu d\nu & d\nu + (\mu d\nu - \nu d\mu) \\ d\nu - (\mu d\nu - \nu d\mu) & -d\mu + \mu d\mu + \nu d\nu \end{pmatrix} \tag{3.4.14}$$

with $\mu \equiv u^2 - v^2$ and $\nu \equiv 2uv$. When written in the parameters u and v, they lead to the Gauss-Codazzi equations, which enforce a geometric continuity for the physical parameters (α, β, γ) along the surfaace, characterized by the partial differential equations given by Rainich himself (*loc. cit.*):

$$\begin{aligned} (1+u^2+v^2)(\partial_v \alpha - \partial_u \beta) + 2v(\alpha+\gamma) &= 0 \\ (1+u^2+v^2)(\partial_u \gamma - \partial_v \beta) + 2u(\alpha+\gamma) &= 0 \\ u(\partial_v \alpha - \partial_u \beta) - v(\partial_u \gamma - \partial_v \beta) &= 0 \end{aligned} \tag{3.4.15}$$

One can, therefore, rightfully say that these equations accommodate the physical parameters to the given surface.

Chapter 4 The Interpretation Concept and the Gauging Procedure

The problem of interpretation prompts us into thinking of an ensemble of classical material points in equilibrium in any direction at any distance in space. And we can imagine such a *static ensemble*, by thinking of it in terms of *central forces* of Newtonian type: it can be an ensemble of material points endowed with gravitational mass and charges, both electric and magnetic. These physical features of a classical material point are, indeed, connected with innate Newtonian forces, as our experience shows, and thereby we may imagine ensembles of such material points, each one of them stationary with respect any another one located in any direction, at any distance with respect to it, just like in the case of Newton's prescription for his star system. However, these cannot be but *fictitious* ensembles, inasmuch as, from the very same point of view of our experience, any *real* ensemble of material particles endowed with masses and charges *cannot* be in static equilibrium (see §3.1 above). As we have shown, the kind of nonequilibrium described through Newtonian forces generated by mass and charges is a feature depending on the *space scale* where we consider that nonequilibrium. For instance, at the transfinite scale of space, the gravitation is thought to dominate and, therefore, for the ensembles of identical particles the forces at this scale are dominantly attractive. On the other hand, at an infrafinite or finite scales the electric charges dominate and, for any known ensembles of identical particles the forces should be dominantly repulsive. The bottom line is that the problem of interpretation cannot be solved here but only by involving at some moment of its solution some fictitious ensembles of particles, forasmuch as *we are allowed* to think of an ensemble of *identical particles in equilibrium* under Newtonian forces, but such an ensemble is not a fact of human experience, but of human imagination. However, in view of the fact that we deal here only in Newtonian forces, such an ensemble can be defined by a mathematical condition independent of space scale. Every particle of such an ensemble is stationary, for it is in equilibrium with any other identical particle, located at any distance in any direction in a certain space. Only, we are under obligation not to extend this imagination to an incidental reality, because our possible experience does not contain such a case. In other words, there is no physical reality coping with our imagination in this sense. We shall come back to this issue later on, as we go along with this work.

We will call the classical material point making the element of such a fictitious ensemble a *Hertz material particle*, in view of its original inventor, who also indicated the somewhat precise – being defined in mathematical terms – scale character of an incidental geometry describing this kind of ensemble and its elements. This is a case apparently never taken in consideration by the natural philosophy of any persuasion – forgotten at its very birth, if we may say so – and which was aroused, the first and only time, by Heinrich Hertz in his *Principles of Mechanics* (Hertz, 2003). Probably, even Hertz himself did not know what to do with the concept – the course of his beautiful work follows only the mathematical line of presenting the principles of mechanics, 'in a new form', as he declares himself, right from the subtitle – and so it happened that only the wave and quantum mechanics unveiled the true gnoseological capabilities of this concept, all converging mostly into the definition of interpretation due to Charles Galton Darwin. The interpretation turned out to be a fundamental concept of physics in the new views on mechanics, and around it, explicitly or implicitly, the whole *modern* physics has been gradually built (Mazilu, Agop, Mercheș, 2019). The best illustrative example is the discussion around the cosmological problem that was started by Einstein in 1917 (Mazilu, Agop, Mercheș, 2020). This example revealed the important position of the Einsteinian point of view in the natural philosophy, and the clear difference between this natural philosophy and the old classical Newtonian point of view. It also revealed that the two points of view – Newtonian and Einsteinian – in the natural philosophy can be 'reconciled', if we may say so, into a general, apparently more realistic, natural philosophy, whereby the wave mechanics plays an essential part. As here we merely have to pinpoint just that essential part played by mechanics, it seems therefore best to start from the very Hertz's concept.

We reproduce and discuss, for now, only the necessary original definitions and commentaries [(Hertz, 2003), pp. 45–46)], keeping in store our understanding, in order to be revealed gradually and, of course, specifically, as we go along with our work. Quoting, therefore:

> **Definition 1**. A *material particle is a characteristic* by which we associate without ambiguity a given point in space at a given time with a given point in space at any other time.
>
> Every material particle is *invariable and indestructible*. The points in space which are *denoted* at two different times by *the same* material particle, *coincide when the times coincide*. Rightly understood the definition implies this.
>
> **Definition 2**. The number of material particles *in any space*, compared with the number of material particles *in some chosen space at a fixed time*, is called *mass* contained in the first space.
>
> We may and shall consider the number of material particles in the space chosen for comparison to be infinitely great. The *mass of the separate material particles* will therefore, by the definition, be *infinitely small*. The mass in any given space may therefore have any rational or irrational value.
>
> **Definition 3.** A *finite* or *infinitely small mass*, conceived as being *contained in an infinitely small space*, is called a *material point*.
>
> A material point therefore consists of any number of material particles *connected with each other*. This number is always to be infinitely great: this we attain by supposing the material particles to be of a *higher order of infinitesimals* than those material points which are regarded as being of infinitely small mass. The masses of material points, and especially the masses of infinitely small material points, may therefore bear to one another any rational or irrational ratio. (*our emphasis, n/a*)

The trend of progressing of his *Mechanics* does not seem to indicate that Hertz followed a *program* as outlined in this list of definitions, at least not from the points of view later revealed in physics. After all, it seems just normal: the reasons of such a presentation of mechanics, as listed by Hertz in the *Preface* to his treatise, show that he followed mainly the soft spots in the contemporary comprehension of *concept of force* at that time. To be fair, that way of understanding the concept of force still persists, in fact, even today. This is perhaps the reason that the treatise does not play today, as it never did in fact, the foundational part it deserves in our physical knowledge. However, an early analysis by Henri Poincaré suggests that Hertz's work has to be taken by far more seriously even as it is, for even as such it touches fundamental issues of the human knowledge. Quoting:

> I insisted on this discussion longer than Hertz himself; I meant to show though that Hertz didn't simply look for quarrel with Galilei and Newton; we must agree to the conclusion that in the framework of the classical system *it is impossible to give a satisfactory idea for force and mass*. [(Poincaré, 1897); *our translation, original emphasis*]

The exquisite analysis of the great scholar, from which we excerpted this fragment – and the analysis of anyone else ever interested in the *Mechanics* of Hertz, for that matter – does not appear to take due notice of the concepts involved in the definitions listed above. Fact is that the definitions in that excerpt from Hertz, which is apparently referring only to mass, touches actually, even though partially and perhaps only implicitly, both of the two fundamental ideas mentioned by Poincaré here, and with them the objective reasons of the subsequent general relativity and wave mechanics. In this respect, two things are worth noticing right away, bearing directly on our subject-matter of this work.

First, the important feature of the Hertz's definition of a material particle: it has to be 'invariable and indestructible'. This should be very important for us, for, in view of our discussion thus far, it reflects an implicit consideration of the space scale. Namely, in his material particles one can easily recognize the very classical material points as, in fact, we mentioned above: *positions endowed with physical properties*. Again, in view of our Newtonian definition of the three physical properties connected to the innate forces of our world, such material points cannot be but fictitious. On the other hand, a *material particle* of Hertz endows, according to Newtonian view of a material point, a position in space not only with an identity, a task for which the definition was mainly intended, but with an indestructible material 'anchor', as it were, when that position is in the matter. Indeed, it is not too hard to see that, besides being an 'indicator' for a point in space, such a material particle is, incidentally, the most convenient *point of application of a force describing a physical field* according to the third principle of classical dynamics. For, naturally, the Hertz's particle can also logically support the third principle of dynamics, inasmuch as it is conceived as dimensionless and, more than that, "invariable and indestructible", as Hertz himself hasten to comment. This fact is, of course, crucial in building an interpretation in physics.

Now, there is a critical difference between a *position in space* and a *position in matter*: in the first kind of position, a particle can acquire an observable *motion*, in the last it does not. However, let us recall that, in modeling the reality around us we have only the possibility to work with material points in the acceptance of Hertz. Indeed, we are scientifically aware that a star, for instance, or even a galaxy for that matter, is actually an extended body, and only from a distance we perceive it as a point. Therefore we have to accept that a material point is itself a complicated structure, and this fact is duly noted in Hertz's formalism: *the material points are made of material particles*! This very definition liberates our spirit from the obligation of removing the forces from the development of a physical theory, as the general relativity claimed sometimes to have been doing, and as, in fact, Hertz himself aimed to do. Indeed, a material particle can support an acting force on it, and within an equilibrium ensemble of material particles this acting force is always present with its defining reacting force. In a word, in the structure of a material point, a material particle is 'as free as it gets' in a given environment, according to the precepts of classical mechanics. As a matter of fact, this was entirely Newton's initial philosophy in building the concept of forces!

There is a subtle point here that unfortunately has not been exploited along the time, because of the prevailing concept of vector attached to force. In order to reveal it, let us notice that Hertz's definitions *implicitly* show that there is a real difference between *motion* and *displacement*. This difference implicitly enters the definition of a *material point*: forces may act on a material point only through its constituent material particles. These material particles, however, can only be *displaced* by forces. It follows that the motion – a thing we observe, therefore a thing of experience – may *not* be a direct consequence of the force as in the Newtonian axiomatics. As a matter of fact strange situations may even occur, where the point of application of a force acting on a material point is outside *anyone* of the material particles from the constitution of that material point. This seems to be, for instance, the situation in that condition of definition of the wave surface, serving for the definition of de Broglie's concept of physical ray, which we deemed as 'strange' in §2.1 above: 'as one *approaches at constant time* a light particle following its trajectory', the amplitude of the wave in the physical ray, must go inversely with the distance. This statement is referring to approaching *a particle* from the ensemble serving for interpretation, but the 'process' is to be taken in the sense of Eddington, as only a mind process. As long as we maintain the geometrical image of vector for a force, like Hertz did, we may not have too much of a choice in overcoming such a difficulty but to define further *physical* features that are "concealed" [(Hertz, 2003), pp. 223–225]. We believe that the real lesson to be learned here is that, generally, we have to speak of a *material point* in the sense of Hertz when describing a motion we happen to *observe*, and of *material particles* in the sense of Hertz when in need of properly describing the action of forces that might go along with this motion. However, when it comes to describing the force as an *effect of motion*, we need to pay close attention, because some statistics may come into play, as dictated by the space scale where we contemplate the things. After all, a material point is, first and foremost, and *ensemble of material particles*! And when it comes to ensembles, the statistics is the most appropriate method to use in the process of comprehension.

Nowhere was this philosophy as clear as in the historical moment of birth of electrodyamics. Here, for the first time in its existence, our spirit had to face the problem created by a situation where there is not but a faint 'guarantee that $\boldsymbol{dx}$ is traversed by a particle', as Arthur Eddington would say. The conception upheld by Eddington (see §2.1 above) is, in fact, a very old one in the natural philosophy – it occurred once a principle of describing the motion appeared as necessary to our knowledge. However, on the occasion of birth of electrodynamics in the 19$^{\text{th}}$ century, more specifically, with the occasion of Ampère's works on the electrodynamic forces, a particular aspect of it has been revealed. As we see it, this aspect has been rightfully expressed on that occasion only because with Ampère, we can talk of a 'rebirth', as it were, of the Newtonian forces, in a new form, forced upon our spirit by the existence of a new phenomenology related to the motion of charges. The point of view was voiced by François-Alexandre Reynard, and on it we shall return with details later in this chapter. For now, we only need to mention that the new Newtonian forces in question were those invented by André-Marie Ampère, and they carry a special meaning. Quoting, from the relevant work of Reynard, we have the following view, enticed by the Ampère's electrodynamical forces:

> The *actions of currents cannot be assimilated to the ones that take place between the molecules of matter*, for they *are not innate to the matter of the fluid currents*, *or to that of the conductors followed by these currents*. Here, it is the motion, *a thing immaterial, a being of reason*, which is the agent of action, and yet it is a material body, a conducting wire, that receives this action. One *cannot have equal and directly opposing action and reaction* between what produces the effect and what receives this effect: *these are things of a different nature*.
>
> There is, in one point, a movement that causes another movement in another point. This fact *requires an intermediary*. This intermediary can only be a common medium, and the transmission of movement through this medium does not seem to be possible but *by a normal pressure on the body which is set in motion*. This is, indeed, proved by experience, so that starting from a contrary idea, Ampère was, nevertheless, compelled to establish his formula in such a way that *the resulting action given by the calculation should always be a normal force*. Since one is forcibly brought to it, why not adopt *the principle of normal action* right away, which would simplify the procedure?
>
> In Ampère's theory, *one cannot get an idea of how the movement of electricity in conducting wires can produce attraction or repulsion between these wires*. In the hypothesis of a transmission of movement through a medium, though, one can very well conceive the possibility of the production of such actions through this transmission. [(Reynard, 1870); *our translation and emphasis*].

In other words, our spirit was forced to recognize that there are things *unnoticeable* producing motion, which is an indisputable *observable*. However, there are cases, like the motion of charges through the wires, where this motion itself is a 'being of reason', inasmuch as it produces the motion of wire, and only this one is accessible to our senses. It feels like our duty, then, to notice today that such ideas were not any different ever since the invention of the principles of Fermat (around 1640) and Maupertuis (around 1740), referring to the description of the motion. It was then, as it is nowadays, only a matter of imagination to assert that the light – an observable phenomenon, for which the *motion can only be assumed* in the form of propagation – should be somehow equivalent to material particles, for which the *motion itself is observed*, but no assumption could be found appropriate, for no object of assumption could ever be thought as equivalent to propagation. The spirit struggled to complete this equivalence up to the beginning of the 20th century, when Louis de Broglie found the expression – and an adequate accompanying mathematical description – that we all think today is the right one: the material point is actually a 'wave phenomenon'.

The excerpt above from Reynard, is referring to one critical moment from the history of our spirit, when it was taken by surprise, so to speak, by the fact that the motion became *undeniably* an object of assumption, exactly as Eddington asserted fifty years later. One can say that this was a 'de Broglie moment' of our knowledge – even though *avant la lettre*, as it were – and we recognize here as our duty having to make the best of it, by analyzing it from the point of view suggested by the name of André-Marie Ampère. This great personality of the science of physics was, every now and then, characterized as the *second Newton* of science, an not without good reasons. With the concept of central Newtonian forces that Ampère so aptly generalized, he opened the door for the idea, only implicit in the very Newton's researches based on the concept of inertia, that the *forces are a general* *expression of the memory of a universe*. Let us therefore follow, in the present chapter of our work, some details of this historical moment of knowledge, and thereby show what we think it actually tells us. The general idea of choice of these details resides in the main theme of our work: *a wire can be assimilated with a de Broglie tube serving for the definition of the physical ray*. The chief one among the details is a closer perception of that 'order of infinitesimals' stipulated by Hertz in his definitions.

4.1 Newton's Forces: the Observable Memory of a Universe

A good story should start here with a specific description of the concept of force as it was instituted in our knowledge by the magnificent Isaac Newton. The force was, of course, a fact of social experience, but Newton 'invented' a related concept, in order to describe *in actuality* facts thought to have been happening in the past of the universe, and *imprinted in its structure*. The notorious such structure that led to the construction of the force concept, made by Newton into the condition of 'imprint', was the Kepler motion, thought as an *everlasting* material structure, and presented by Newton as the mark of an accident in a possible evolution of the universe, therefore as an *expression of the memory* of such an event. That possible evolution of the universe was a supposed evolution, of course, leading to an event imaginable by by sheer analogy with nowadays facts of experience on Earth: the *free fall of the bodies* toward what, again, *is thought* to be the unique center of the Earth. Quoting from a first letter to Bishop Richard Bentley:

> …it seems to me that *if* the matter of our sun and planets, and all the matter of the universe, were evenly scattered throughout all the heavens, and every *particle had an innate gravity* towards all the rest, and the whole space throughout which this matter was scattered was but *finite*; the matter on the outside of this space would, by its gravity, tend towards all the matter on the inside, and, by consequence, *fall down into the middle of the whole space* and there compose one great spherical mass. But *if the matter was evenly disposed throughout an infinite space*, it *could never convene into one mass*; but some of it would convene into one mass, and some into another, so as to make *an infinite number of great masses*, scattered at great distances from one to another, throughout all that infinite space. And thus might the sun and fixed stars be formed, supposing the matter were of a lucid nature. But how the matter should divide itself into two sorts, and that part of it which is fit to compose a shining body should fall down into one mass and make a sun, and the rest which is fit to compose an opaque body should coalesce, not into one great body, like the shining matter, but into many little ones; or if the sun at first were an opaque body like the planets, or the planets lucid bodies like the sun, whilst he alone should be changed into a shining body, whilst all they continue opaque, or all they be changed into opaque ones, whilst he remains unchanged; I do not think explicable by mere natural causes, but am forced to ascribe it to the *counsel and contrivance of a voluntary Agent*. [(Bentley, 1838), pp. 203 – 204, *our Italics*]

The structure of matter in this primary instance is simple: it is made out of particles. Now, *if* these particles had innate gravity, then they would fall toward each other, insofar as the gravity manifests itself attractively according to our experience. As one can see right away, the Newton's condition for the multiplicity of matter formations in the universe is *the infinity of space of its existence*. Only under this proviso can we consider escaping from the condition of finiteness of the classical view of the universe. One can say that, according to such a natural philosophy, the world is only perceived by its components at a transfinite scale of space. Now, according to this view, the science discovered an apparently eternal matter structure involving a motion, that is another motion than the free fall: the Kepler motion. And, leaving aside the facts of faith, Newton made out of this, *just through a process of thinking by analogy*, the first one of the physical structures used today as *depositories of memory*. Quoting from a second letter to Bishop Richard Bentley:

> To the last part of your letter, I answer, first, that if the earth (without the moon) *were placed any where with its centre in the orbis magnus*, and *stood still there without any gravitation or projection*, and *there at once were infused into it both a gravitating energy towards the sun*, and *a transverse impulse of a just quantity moving it directly in a tangent to the orbis magnus*; the compounds of this attraction and projection would, according to my notion, cause a circular revolution of the earth about the sun. But the transverse impulse must be a just quantity; for if it is too big or too little, it will cause the earth to move in some other line. Secondly, I do not know any power in nature which would cause this transverse motion without the divine arm. Blondel tells us somewhere in his book of Bombs, that Plato affirms, that the motion of the planets is such, as if they had all of them been created by God in some region very remote from our system, and left fall thence towards the sun, and *so soon as they arrived at their several orbs, their motion of falling turned aside into a transverse one*. And this is true, *supposing the gravitating power of the sun was double at the moment of time in which they all arrive at their several orbs*; but then the divine power is here required in a double respect, namely, to turn the descending motions of the falling planets into a side motion, and, at the same time, to double the attractive power of the sun. So, then, gravity may put the planets into motion, but, without the divine power, it could never put them into such a circulating motion as they have about the sun; *and therefore for this, as well as other reasons, I am compelled to ascribe the frame of this system to an intelligent Agent* [(Bentley, 1838), pp. 209 – 210, *our Italics*]

There is, therefore, *in the mind of Newton*, an event *imprinted* in the 'transverse' circular motion, that can be described in the following fashion: the initial motion of the planets is radial towards a center represented by the Sun; this radial motion turned into transverse motion at the moment when 'they arrived at their respective orbs'. This description is only true provided some incidentals occur, in the form of a gravitating power of Sun, which has to be double at the moment when, in their free fall, the planets reached their 'orbs'. All these are rational conjectures, produced by imagination according to the rules of logic; no doubt about that. The only reality connected to them is their present motion, that stands witness to the imaginary past event thus described, thereby representing the material form of its memory. This material form can then be taken as a depository of such a memory in the very modern sense of such a device. The rest of Newton elaborated opinion is a matter of faith, as we said, for there is no other possible explanation.

The things then evolved as the great *Principia* shows: based upon quotidian experience, where we have an innate form of memory in the daily manifestations – to wit, *the inertia* – Newton created a system that helped understand, in human terms, how this whole universe may work. One of the most important achievements of this system was the possibility of explanation of forces in the form of a concept introduced by Newton himself. We reproduce this explanation in the analytic form, presented long ago by James Whitbread Lee Glaisher. In fact starting from a certain moment of the 19th century, that explanation was explored on most of its faces, if not all of them. However, Glaisher's work we are considering now (Glaisher, 1878) does nothing more than showing that all of the results in characterizing the forces conceptually introduced by Newton, can be recovered from the very fundamental Newton's own propositions, by casting them into an *analytical* form. And the basic proposition Glaisher has chosen to put into such analytic form was the Corollary III of the Proposition VII from Book I of Newton's *Principia*. Quoting, therefore:

> The force by which the body P *in any orbit* revolves about *the center of force* S, is to the force by which the same body *may revolve in the same orbit*, and *the same periodic time*, about *any other center of force* R, as the solid SP·RP^2, contained under the distance of the body from the first center of forces, and the square of its distance from the second center of force R, to the cube of the right line SG, drawn from the first center of force S parallel to the distance RP of the body from the second center of force R, meeting the tangent PG of the orbit in G. For the force in this orbit at any point P is the same as in the circle of the same curvature. [(Newton, 1974), p. 51; *our Italics*]

As we see it, this is a definition of the force based on a precise *idea of measurement* of a genuinely classical sort, used in the operation of any apparatus: *the comparison*. Two forces are simply compared with each other in a ratio that can be effectively constructed for a certain reality. The real and, in fact, physically necessary ingredients here, are *the orbit* and *the periodic time*. Of the two points, *S* and *R*, helping in defining the force by its direction and distance, one is our choice – the point toward which we want to calculate the acting force – while the other one can be arbitrary at random, provided we know the force acting towards it. Apparently, as Newton presents the idea, both of these points should be chosen inside the orbit. However, with the analytical development of the theory, even this condition became obsolete, allowing later on, for instance, the Newtonian definition of the force correlated with the light phenomenon, a situation that can be taken, again, as 'strange', in the exact sense we used for the de Broglie's condition from the §2.1 above (see, further, §2.2 for details).

That last mention in the excerpt above, referring to "the circle of the same curvature", seems to send our thinking to a practical tool using the 'measurement' of two forces acting along different directions, by one another: the *slingshot*. And what we mean here, is not the modern materialization of such a device, since the emergence of rubber into the life of humanity, but the classical thing used by little David to kill the giant Goliath. The strings of this device allow putting to work two forces acting clearly in different directions: the force of *inertia* and that of *gravity*. Moreover, the device is the only one of its kind incorporating even the idea of *projection*, as employed by Newton in the excerpt above. Newton uses the idea of a sling only once in his *Principia* [(Newton, 1974), pp. 2 – 3] in order to illustrate the definition of centripetal force, as being "that force, *whatever it is*, by which the planets are continually drawn aside from the rectilinear motions". The idea of projection, which is the essential *practical* reason of existence of the device, does not seem to reappear in his works. This is why we feel worth insisting a little longer on this issue, from a modern point of view on 'projection' inspired by Kurt Gödel's work on general relativity (Gödel, 1949, 1952).

The difference between inertia and gravitation is most clearly illustrated by sling shooting, indeed, if the stone to be projected is first "whirled", to use Newton's own word, in a horizontal plane, above the head, using a classically often illustrated posture. In this case the force of inertia acts 'horizontal-wise' as a centrifugal force, while the force of gravitation acts 'vertical-wise', as it permanently does. Therefore, in this ideal case, the device – which obviously works based on the equilibrium of forces – allows, in fact, a 'measurement' proper, of the force of gravitation by the force of inertia – or vice versa, of course – in any instantaneous vertical plane. In our opinion, this is a clear illustration of a manifest difference between the *compass of inertia* and the *compass of gravity*, and this is the very fact stressed by the works of Gödel, cited above. In the case of sling shooting, though, contrary to the main modern illustration of the concepts, it is the compass of gravitation that points always in the same direction, while the compass of inertia changes continuously the direction. And, we might add as well, this is the way Ernst Mach has understood what later came to be known as the "Mach's principle". Starting from this sling shooting image, and considering only the concept of force, without any further specification of its nature – *i.e*. accepting, with Newton, the force as 'whatever it is' – we can dispense with the strings, and imagine a planet in motion, as in the case of a Kepler problem, the way Newton himself did. Thus, the attraction *per se* can be considered as *the action of a force* which is 'measured' against another force, which may or may not be a force of inertia. Newton's own case in the above excerpt is referring to *two attraction forces*. However, historically speaking at least, the things evolved in such a way that the difference between inertia and gravitation has been brought to bear again on our knowledge, from the general relativistic point of view. In fact, this was the case for the whole foundation of the general relativity, as its application to cosmology – an application for which, we might say, it was in fact initially intended, if we take into consideration the Einstein's principle of equivalence – clearly shows. It is for this reason that we shall follow, by and large, this history a litte closer, in order to properly introduce Gödel's ideas referring to the *concept of compass*.

Still, before getting into the details of this subject, we may be allowed to show the importance of the idea of a 'compass', used extensively, even as a concept we might say, by the general relativity in its capacity of a cosmological theory. Realization that the manner of working of the age-old sling involves *two compasses*, one of inertia and the other of gravitation, could not be achieved otherwise. Nor could we be able to comprehend the more involved idea that, in a Kepler motion, the manner of working of these two compasses is just about the same as in the sling shooting, with two little different details, which make, nonetheless, a big difference: first, there is no *projection* whatsoever in the modern idea of compass, a fact which seem to be corrected by the modern Gödel's general relatvistic theory, thus suggesting a fresh reconsidering of the old Newtonian theory of gravitation; secondly the compass of gravitation aims to *a fixed point in space* – the Newtonian center of force in a modern mathematical rendition of the problem – *not in a fixed direction* as, apparently, it does in the case of sling shooting, while the compass of inertia changes its orientation exactly like in the case of the old sling. One can see here the difference between that 'coordinate space' description of things, and the 'ordinary space description' noticed, as we have shown, by Charles Galton Darwin in connection with Schrödinger equation.

Now, speaking of *inertia* and *gravitation* concurrently, a physicist is naturally led to think about *Mach's principle*, as mentioned above, according to which the inertia would be determined by the action of the distant masses only, the masses out of the reach of our experimental possibilities, ideally those masses located at infinity with respect to the body being described by inertia. This principle was introduced to the theoretical physics of 20th century by Albert Einstein, and the idea stays at the foundations of general relativity. One can safely say that *there is not* a formulation of Mach's principle due to Ernst Mach himself. [See, for this, the collection of dedicated works (Barbour & Pfister, 1995), especially the contribution of John D Norton starting that collection, which we choose to cite separately, due to its importance for the history of this issue (Norton, 1995)]. Nevertheless, browsing the celebrated *Science of Mechanics*, we have been able to find a suggestive expression – which seems to us as being not only the closest to a concept of principle, but also indicative for the very concept of a compass of inertia – in the following excerpt explaining as clearly as possible Mach's own position with respect to main Newtonian course of cosmological ideas:

> The expression "absolute motion of translation" Streintz (in *Physikalische Grundlangen der Mechanik*, *Leipzig 1883, a/n*) correctly pronounces as devoid of meaning and consequently declares certain analytical deductions, to which he refers, superfluous. On the other hand, with respect to *rotation* (*original emphasis, a/n*), Streintz accepts Newton's position, that *absolute rotation can be distinguished from relative rotation*. In this point of view, therefore, one can select *every body not affected with absolute rotation as a body of reference* for the expression of the law of inertia.
>
> I cannot share this view. For me *only the relative motions exist*, and I can see, in this regard, *no distinction between rotation and translation*. When *a body moves relatively to the fixed stars, centrifugal forces are produced*; when it moves relatively to some *different bo*dy, and not relatively to the fixed stars, *no centrifugal forces are produced*. I have no objection to calling the first rotation "absolute" rotation, if it be remembered that nothing is meant by such a designation except *relative rotation with respect to the fixed stars* (*original emphasis, a/n*). Can we fix Newton's bucket, *rotate the fixed stars*, and *then* (*original emphasis, a/n*) prove the absence of centrifugal forces? [(Mach, 1919), Appendix XX, pp. 542 – 543; *our emphasis everywhere*, *except as indicated*]

First, one can notice that Mach was against the idea of any absolutes as it were, mostly when it comes to describing the motion. Newton's bucket was intended to prove the inertia as an effect of rotation with respect to absolute space [(Newton, 1974), Volume I, pp.10 – 11]. Apparently, for Mach it was hard to understand the existence of such a thing that can act "but cannot be acted upon" [(Einstein, 2004), p. 58] like the absolute space. So, according to Einstein, "he was led to make the attempt to eliminate the space as an active cause in the system of mechanics". Thus, when it comes to action at a distance, Mach contended that *only matter can act upon matter*, and thus the inertia should not be due to the absolute space, but to distant matter that could not be represented otherwise, but as the matter contained in the distant stars. Thus, in a way, Mach reinstated, so to speak, the fixed stars in their own rights, according to the very Newtonian idea of action at a distance. But then, the proof or disproof of the existence of centrifugal forces in the case of Newton's bucket experiment, would be pending not on the rotation of the bucket, but on the rotation of the distant stars. Which, of course, is not to be accomplished within the capabilities of mankind. However, the idea of such a rotation pervades the modern Gödel's theory.

In fact, at a certain moment in history, it was realized, and observationally even proved, that the distant stars are not fixed as Newton believed, but seem to move *only along the visual direction*, more to the point, to depart from us (Hubble, 1929); this is why they appear as fixed for the purely optical observation devices. In the language of general relativity, this would mean that *the compass of inertia points exactly along the direction of the compass of gravitation*, as in the case of free fall, and it is here the point where Einstein's equivalence principle cuts in, explaining that both these actions are somehow settled with the assistance of the *compass of light*, as forced upon us by the electrodynamical nature of light. However, the compass of inertia came under suspicion along with the unfolding of the ideas of general relativity, as being differently oriented from the compass of gravitation, exactly as it was to Newton initially, on the occasion of constructing the concept of force. The first such suspicion was therefore bound to be raised sometimes, and its expression in the sound modern terms of the very Einstein's general relativity was given by Kurt Gödel, as we said, in the works just cited above.

Fact is, that the compass of inertia was always thought as being realizable like a regular magnetic compass, pointing out in a fixed direction – as the compasses usually do – exactly in the manner described by Ernst Mach in the excerpt above: a piece of elongated material suspended like a magnetic needle, in the interior of a material sphere for instance [see (Harrison, 2000), p. 243]. This way, the inertia acts to maintain the direction of the needle, but in the spirit of the first law of classical dynamics – as long as no other forces act it maintains its state, *i.e.* the orientation – only with a little 'twist' of *imagination*, naturally, brought up by Gödel's idea. Namely, the compass points in the very same direction *during the motion within a 'substratum' of the universe*, which would represent the matter at a cosmic scale according to Einstein's ideas, and would have to be recognized in a specific rotation of the sphere of device. Then, in order to validate Mach's idea of the compass of inertia, two essential ingredients would be necessary, at least from a purely theoretical point of view. First, comes the need to prove that the fundamental property of this compass – *viz*. that of remaining constantly oriented while the sphere rotates in any possible way – *is independent of the space extension of the sphere*. This is exactly the point that the Gödel's idea has brought up and, we believe, it was also the reason of that often-cited high appreciation of Einstein for Gödel's achievement. However, according to our knowledge, this point was never an object of independent study like, for instance, its celebrated counterpart – *the thermal radiation* – which has been independently studied, experimentally and theoretically, from this point of view, in order to finally realize that it has a *spectral distribution* independent of the dimensions of the 'hohlraum' [the Wien's displacement law; see (Mazilu, 2010)]. The second, and by far more important ingredient necessary here, is a way to describe the action of the rotating matter of the enclosing material sphere upon the very compass: it is this interaction that has to be invariant with respect to the change in the dimensions of the material sphere containing the compass of inertia. The classical theory of forces has, apparently, no possibility of describing it. However it is describable in a general relativistic setting, as the *Lense-Thirring effect*; of course, without considering, though, the change in dimensions of the hollow sphere [for a critical account, updates and translations of the original articles from 1918 of Hans Thirring and Joseph Lense, see (Mashhoon, Hehl, & Theiss, 1984)]. We shall have to come back to these two isssues, in order to show how they have been transacted in physics.

Now, after this historical digression and, in fact, an implicit sketching of some of our future tasks in this work, let us get back to our present point: Glaisher's argument on the excerpt above from Newton, and its connection with the idea of compass. Using some geometry of a triangle in order to handle the concepts in that excerpt, Glaisher reduces the ratio of those two forces, as it was defined by Newton himself, to the expression

$$\frac{FORCE \;\; to \;\; R}{FORCE \;\; to \;\; S} = \left(\frac{\overline{SN}}{\overline{RM}} \right)^3 \frac{\overline{RP}}{\overline{SP}} \tag{4.1.1}$$

This expression involves, on one hand, the perpendiculars *SN* and *RM* of the two centers of force onto the tangent in *P* to the orbit, and, on the other hand, the distances *RP* and *SP* of the moving point to the two centers of force. The expression is particularly prone to an analytical form and, further on, even to a differential calculus, which is, in fact by far more important from a classical dynamical point of view. Let us reproduce the calculations here, inasmuch as they are instrumental in describing what we believe can be taken as the original *Newtonian compass of inertia*: the Kepler motion. Choosing *S* as the origin of a reference frame – which, by the way, means a reference frame fit to the gravitational force towards S – and referring the generic coordinates, (ξ,η) say, in the plane of motion, to such a frame, the equation of orbit in a Kepler motion, can be written as

$$C(\xi,\eta) \overset{def}{=} a_{11}\xi^2 + 2a_{12}\xi\eta + a_{22}\eta^2 + 2a_{13}\xi + 2a_{23}\eta + a_{33} = 0 \tag{4.1.2}$$

where C here is taken as meaning a '*C*onic'. The tangent to this conic is one essential ingredient in Newton's procedure. In the current moving point P, of coordinates (x, y) say, the equation of such a tangent direction to its orbit around the center of force S is:

$$(a_{11}x + a_{12}y + a_{13})\xi + (a_{12}x + a_{22}y + a_{23})\eta + a_{13}x + a_{23}y + a_{33} = 0 \tag{4.1.3}$$

Thus one can calculate the distances from the two centers of force to this tangent of the orbit in question, by a well known analytical procedure. One gets the following expressions dependent on the coordinates of the point of application of force:

$$\overline{SN} = \frac{1}{\sqrt{\boldsymbol{z}\cdot\boldsymbol{z}}}, \quad \overline{RM} = \frac{1+\boldsymbol{z}\cdot\boldsymbol{\varepsilon}}{\sqrt{\boldsymbol{z}\cdot\boldsymbol{z}}} \tag{4.1.4}$$

where $\boldsymbol{z}$ is the vector of components

$$\frac{a_{11}x + a_{12}y + a_{13}}{a_{13}x + a_{23}y + a_{33}}, \quad \frac{a_{12}x + a_{22}y + a_{23}}{a_{13}x + a_{23}y + a_{33}} \tag{4.1.5}$$

and $\boldsymbol{\varepsilon} \equiv \overrightarrow{SR}$. The equation (4.1.1) can then be written in the form

$$FORCE \quad to \quad S = (1+\boldsymbol{z}\cdot\boldsymbol{\varepsilon})^3 \cdot \frac{r}{\sqrt{\boldsymbol{r}^2 + \varepsilon^2 - 2\boldsymbol{r}\cdot\boldsymbol{\varepsilon}}} \cdot FORCE \quad to \quad R \tag{4.1.6}$$

Therefore, *if we know* the force toward R, *we can calculate* the force toward S, and for such an occasion Newton analyzed a number of particular cases in order to be used appropriately in his work. One of these cases shows that, if R is in the center of the conic, then the force between P and R is proportional to the distance between them [(Newton, 1974), p. 54; Corollary I of Proposition X, Problem V]. Using this condition, and choosing R as the center of the conic section representing the trajectory of motion, the expression (4.1.6) simplifies to

$$FORCE \quad to \quad S = \mu \cdot r \cdot (1+\boldsymbol{z}\cdot\boldsymbol{\varepsilon})^3 \tag{4.1.7}$$

Here μ is a constant coming from the law of force acting towards the center of conic, provided that force allows, according to Newton's Corollary III of Proposition VII excerpted above, for this conic as orbit too: such a force *is known* to be proportional to the distance, as we said. Using (4.1.5) in order to calculate the dot product from the parenthesis of equation (4.1.7) gives

$$FORCE \quad to \quad S = \mu \cdot a_{33}^3 \cdot \frac{r}{(a_{13}x + a_{23}y + a_{33})^3} \tag{4.1.8}$$

which is the main of Glaisher's results. It was established even earlier by William Rowan Hamilton, *via* synthetic geometrical arguments though, which is why it is also known under his name (Hamilton, 1847). Based on the equation (4.1.3) we can formulate this theorem in words: the force toward the center S acting on P is proportional to the distance of P to S, and inversely proportional to the cube of the distance from P to the straight line conjugated to S with respect to the orbit (the polar of this center of force with respect to orbit). Using the equation of the orbit (4.1.2), the expression (4.1.8) can still be written as

$$FORCE \quad to \quad S = \mu \cdot a_{33}^3 \cdot \frac{r}{(a_{11}x^2 + 2a_{12}xy + a_{22}y^2 - a_{33})^{3/2}} \tag{4.1.9}$$

Consequently, (4.1.8) and (4.1.9) are the only expressions of force, allowed by the geometry of the Kepler problem. Or so it seems, for there is more to it along the lines of physics.

The principles of dynamics were, apparently, not involved here, so we cannot make a comparison between inertia and gravitation, as the concept of compass of inertia rightfully requires. However, to be fair, we should say that the dynamics *per se* was certainly involved, *but only to an infrafinite time scale*. Indeed, Newton used the idea of impulse of a force in order to physically guarantee a comprehension of the idea that a central force acting continuously, obeys *the second of the Kepler laws* [(Newton, 1974), p. 40]. So the legitimate question occurs, as to what would happen if we would apply explicitly the classical equations of motion to find the trajectory, for only then we could talk of the *inertia as a force*. In this respect, it is an historical fact that when it occurred for the first time (Bertrand, 1877), the modern problem of finding the *possible expressions* of the central forces acting on planets in the observed motions, was not conceived otherwise, but only in connection with the second principle of dynamics. Even Glaisher's geometrical argument above was triggered by the same historical incident whose inception point was the problem posed in 1877 by Joseph Bertrand. The Glaisher's solution above, is nevertheless unique among all the other solutions presented to this problem – which, to be fair, are all genuinely unique, each one with its own merits – by the fact that it uses explicitly *only* geometry and the original definition of Newton for forces.

Significantly, though, Joseph Bertrand was interested in eliminating the second and the third laws of Kepler from the logic related to the procedure of obtaining the expression of central force that determines the motion of celestial bodies (Bertrand, 1877), and we think we can even trace the reason for that elimination. In the second of the work we have in observance here (Bertrand, 1877b), he mentioned the name of Yvon Villarceau who "in his beautiful research on *binary stars*" would have already found the expression of the central force which, pulling toward a given point in a plane, "can determine an elliptical pathway". We went ahead further and located this, indeed beautiful, work of Villarceau in the collection (Villarceau, 1850), p.71, and largely commented on its meaning within Newtonian theory of forces (Mazilu & Porumbreanu, 2018). Here, and for now, we are only interested in the reason that Joseph Bertrand has been compelled to choose only *the first law of Kepler* in order to select the analytic expression of the magnitude of central force. Even though not mentioned by Bertrand explicitly, the reference to Vilarceau seems to elucidate here: it stays in those *arbitrary points in plane* from the Corollary III of Newton excerpted above by us, toward which the two forces, measuring each other by their action on a planet, are supposed to point during their actions.

Fact is that the binary stars do not obey the Kepler laws without 'tweaking' a little bit their original formulation, if we may say so. The binary stars are systems of two neighboring stars, out of which one – the stationary star – seems to be stationary, while the other – the mobile star – has observationally an orbit around it. The pathway of this motion of the moving star of a binary system is always an ellipse, but this is *the only* real resemblance with the original Kepler motion. For, the center of force is, indeed, in the stationary star of the system, but that star *is never located in the focal point of the ellipse* as in the planetary system described by Kepler. That stationary star can, however, be recognized as the center of force in the very same manner Newton himself described the procedure: the position vector of the companion *with respect to it*, sweeps equal areas in equal times. This is the second law of the Keplerian system, indeed, but *not* with respect to the focus of the orbit, as in the case of a genuine Kepler system.

It is worth imagining, for the sake of a better comprehension of the case, that one may suspect that in defining the centripetal forces as in the Corollary III reproduced above, Newton might have been aware of the existence of such systems, which, by the way, started being signaled in the sky just about that epoch [around 1650; see (Aitken, 1918)]. People, especially those inclined to a conspiration theory, for instance, may also theorize that Newton had some information on these systems, judging by the formulation of that Corollary III we are talking about here. However, knowing the possibilities of Newton's imagination, this does not seem likely. More to the point, if he was able to *invent* forces based on the generalization of the idea of a sling shooting, that very feat leaves no doubt that, knowing that there are *real* binary systems whose center of force can be anywhere inside the ellipse, Newton would have been able to improve on the very Hooke's idea of physical light ray, in order to give a concept of modern physical light ray that was, by and large, obvious to physics only in the times of Louis de Broglie. But, let us get back to our present subject matter, making some comments and drawing some conclusions on this issue – and some other important collaterals in fact – as we go along.

As it appears from the facts, Joseph Bertrand realized that only the first law of Kepler is of essence for the problem of forces *per se* and, by his reference to Yvon Villarceau, we feel entitled to assume that this awareness is due to his knowledge about the binary stars. This seems to be especially indicated by Bertrand's continual dedication to the Kepler problem, on which he even formulated a fundamental theorem – of a cosmological extraction, we should say – on forces (Bertrand, 1873): the only central forces capable of generating closed orbits according to the classical laws of dynamics are the force of magnitude *proportional to distance* and the force of magnitude *inversely proportional to the square of distance*. The first one of these two laws of force was given by Newton himself, *directly* we might say, as signaled before [(Newton, 1974), p. 54; Corollary I of the Proposition X, Problem V]. The second one can only be given only *indirectly*: one has to assume the first law of force, in order to give it by the procedure of measurement indicated in the Corollary III, as shown above. This way, however, it becomes noticeable that, with respect to the Kepler motion, the first kind of law of force – the one proportional to distance, *i.e*. the *elastic* force – is not 'real' in the classical sense. Indeed, there is nothing material in the center of orbit, at least in the case it is an ellipse, to be suspected as creating the force, for the material body assumed to creating the gravitational field is in the focus of the orbit. This nonexistence of something material in the center of force serving in the Newtonian process of measurement, even became a characteristic of the physics of the light phenomenon, especially after Augustin Fresnel, and the procedure has been pushed to its extreme by the energetical approach of the problem of motion, culminating with the uprising of quantum mechanics (Heisenberg, 1925).

Coming back to our main subject-matter here, we can say that, probably as a result of such a continual preoccupation, in 1877 Joseph Bertrand set out to prove an important theorem connected to the fundamentals of Newtonian theory of forces, that would make a compass of inertia, indeed, out of the Kepler motion. Quoting from original:

> If Kepler would not have deduced from observations but a single one of his laws: *the planets describe ellipses whose focus is occupied by the Sun*, one could conclude, from this result alone *taken as general principle* (*our Italics here*, *n/a*), that the force governing them is directed toward the Sun and is inversely proportional with the square of distance [(Bertrand, 1877a); *our translation*, *original Italics, except as mentioned*].

And he even proves it, indeed. One can notice here the identity of the 'general principle' adopted, with the first of the Kepler laws in its initial formulation. Probably realizing the necessity of a more specific generalization as indicated by the existence of binary stars, Bertrand then proposes, in the final of the article just quoted, such a generalization:

> *Knowing that the planets describe conic sections, and assuming nothing else ever, find the expression of the components of the force that drives them, expressed as functions of the coordinates of their application point.* [(Bertrand, 1877a); *our translation*, *original Italics*].

There is a wealth of responses to this real challenge, giving solutions from different points of view, and with different manners of attack of the problem. As we already mentioned, Glaisher's own work was enticed by this challenge. A significant one of these works, was an article by Paul Appell, which we take as instrumental in defining the inertial properties of the physical surfaces [see (Mazilu, Agop, & Mercheş, 2019), Chapter 7, equation (7.22) ff]. Mention should also be made of the important contributions of Gaston Darboux from France (Darboux, 1877, 1884) and Ugo Dainelli from Italy (Dainelli, 1880), as being of essence in describing the difference between the inertia and gravitation for the case of Kepler motion, which thus *can be taken, indeed, as a natural compass of inertia* [see (Mazilu, Agop, & Mercheş, 2020), Chapter 5, equation (5.10) ff]. For now, however, we are only interested in showing that the well-known classical dynamical problem which describes the Kepler motion within the framework of Newtonian mechanics, assumes a *modern gauging*, whereby the mass is a 'convenient parameter' indeed – to use the Poincaré's observations in the work from which we excerpted previously (Poincaré, 1897) – and can be taken as a statistical gauging parameter in a genuinely modern sense, involving a cosmological invariance of the Newtonian forces. In hindsight, though, this very possibility of gauging seems to sanction our own idea that that the Kepler motion is actually an imprint of memory in the structure of the universe we inhabit. We, therefore, make our duty in revealing how the memory works, for such an understanding helps us in building comprehension for any universes.

The considerations that follow, and which are intended to support these last conclusions in some of their necessary details, are associated with the name of Ivan Vsevolodovich Meshchersky, the founder of the modern theory of the variable mass body motion [see (Mazilu & Porumbreanu, 2018), for further details]. Let us follow the reason for the case considered by Meshchersky himself [(Mestschersky, 1902); see also (Мещерский, 1952), pp. 199 – 213], but strictly dealing with the Keplerian motion. In order to do this, we shall write the equations of motion in the current coordinates (x, y) as used above in this section for the generic moving point in the plane of its motion, taken with respect to the center of force:

$$\ddot{x}+\kappa\frac{x}{r^3}=0,\quad \ddot{y}+\kappa\frac{y}{r^3}=0,\quad r^2\equiv x^2+y^2 \tag{4.1.10}$$

Now, Meshchersky notices that if we gauge the coordinates and the time *differently*, in the form:

$$\xi=x\cdot f(t),\quad \eta=y\cdot f(t),\quad d\tau=dt\cdot\varphi(t) \tag{4.1.11}$$

then under the condition connecting the two gauging functions:

$$\varphi(t)=k\cdot f^2(t) \tag{4.1.12}$$

where k is a constant, the system (4.1.10) becomes:

$$k^2\xi''+\frac{\kappa}{f}\frac{\xi}{R^3}+\frac{2\dot{f}^2-f\cdot\ddot{f}}{f^6}\xi=0,\quad k^2\eta''+\frac{\kappa}{f}\frac{\eta}{R^3}+\frac{2\dot{f}^2-f\cdot\ddot{f}}{f^6}\eta=0 \tag{4.1.13}$$

where $R^2=\xi^2+\eta^2$ is the square of the gauged position vector. The last terms of these equations are the components of an elastic force, but directed toward the center of the force going inversely as the square of distance. This fact is obvious, especially if we take the coefficient depending on the time derivatives of f as a constant, which happens only if:

$$f(t)=(\alpha t^2+2\beta t+\gamma)^{-1/2} \tag{4.1.14}$$

Thus, with an appropriate choice of the constants in this expression, the system (4.1.13) becomes:

$$\xi''+\frac{\kappa}{f}\frac{\xi}{R^3}+(\alpha\gamma-\beta^2)\xi=0,\quad \eta''+\frac{\kappa}{f}\frac{\eta}{R^3}+(\alpha\gamma-\beta^2)\eta=0 \tag{4.1.15}$$

so that, we can conclude that by the special gauging:

$$\boldsymbol{r}\to\frac{\boldsymbol{r}}{\sqrt{\alpha t^2+2\beta t+\gamma}},\quad d\tau\to\frac{dt}{\alpha t^2+2\beta t+\gamma} \tag{4.1.16}$$

the equations of motion (4.1.15) become

$$\xi''+\kappa_0\frac{\xi}{R^3}+(\alpha\gamma-\beta^2)\xi=0,\quad \eta''+\kappa_0\frac{\eta}{R^3}+(\alpha\gamma-\beta^2)\eta=0 \tag{4.1.17}$$

provided we choose

$$\kappa\cdot\sqrt{\alpha t^2+2\beta t+\gamma}=\kappa_0 \tag{4.1.18}$$

In our case, this last condition is referring to the product between the gravitational constant and the mass of the body creating the gravitational field. The observation can, therefore, be made that if the gravitational constant is, indeed, a genuine constant, then the gravitational mass of the material point creating the field, is actually a parameter serving for gauging the coordinates according to equations (4.1.16). In the framework of classical mechanics – which is actually all Meshchersky followed – this means a variation of the mass generating the gravitational field with time. This conclusion is bound to change the ideas in the two-body problem, for instance.

However, for the moment Meshchersky concentrates in proving that equation (4.1.18) is *a unique choice* in the most general conditions of invariance of the equations of motion under Newtonian forces of magnitude inversely proportional with the square of distance. In order to prove this, he uses an important result of Paul Appell that sheds a special light upon the transformation (4.1.5) giving the auxiliary vector $\boldsymbol{z}$ used in calculating the forces (Appell, 1889). Namely, Meshchersky takes notice that a general, nonhomogeneous gauging of the coordinates of moving point in the Kepler motion, *i.e.* a gauging of the form:

$$\xi=f_1(x,y;t),\quad \eta=f_2(x,y;t),\quad d\tau=dt\cdot\varphi(x,y;t) \tag{4.1.19}$$

can be realized as a *projective Appell transformation*, which we write in the form:

$$\xi=\frac{a_{11}x+a_{12}y+a_{13}}{a_{13}x+a_{23}y+a_{33}},\quad \eta=\frac{a_{12}x+a_{22}y+a_{23}}{a_{13}x+a_{23}y+a_{33}},\quad \varphi=\frac{1}{(a_{13}x+a_{23}y+a_{33})^2} \tag{4.1.20}$$

with the time entering the stage *via* an implicit dependence of the transformation through coordinates and the coefficients. Indeed, using (4.1.20), the equations of motion (4.1.10) become

$$\xi''=\frac{1}{\varphi}\left[(a_{11}x+a_{12}y)\frac{\kappa}{r^3}+\frac{\partial}{\partial t}\left(\frac{1}{\varphi}\frac{\partial f_1}{\partial t}\right)\right],\quad \eta''=\frac{1}{\varphi}\left[(a_{12}x+a_{22}y)\frac{\kappa}{r^3}+\frac{\partial}{\partial t}\left(\frac{1}{\varphi}\frac{\partial f_2}{\partial t}\right)\right] \tag{4.1.21}$$

At this point Meshchersky *requires the radial motion to be conformal*, a class of transformations that includes the transformation (4.1.11) as a special case. This leads to a limitation of the transformation (4.1.20) for the space variables, which he writes in the gauged coordinates as:

$$x = l\frac{\pm A\xi - B\eta}{c_1\xi + c_2\eta + c_3}, \quad y = l\frac{\pm B\xi + A\eta}{c_1\xi + c_2\eta + c_3} \tag{4.1.22}$$

Here l is only a function of time, while all the other coefficients are constants. This transformation reduces – up to a *linear homogeneous transformation*, inconsequential at this moment of our discussion but quite relevant later on, so that we have to mention it here – to (4.1.11) for the case $c_1 = c_2 = 0$, so that, whatever we have to say for the general transformation (4.1.22) remains valid for its purely linear counterpart thus obtained from it. And what we have to say is of essence: the equations of motion (4.1.21) become

$$\begin{aligned} \xi'' &= \mp\frac{(A^2+B^2)^{3/2}\xi}{(c_1\xi + c_2\eta + c_3)^3}\left[(c_1\xi + c_2\eta + c_3)^3\frac{\kappa l}{R^3} + (l^3\ddot{l})(A^2+B^2)^{3/2}\right] \\ \eta'' &= \mp\frac{(A^2+B^2)^{3/2}\eta}{(c_1\xi + c_2\eta + c_3)^3}\left[(c_1\xi + c_2\eta + c_3)^3\frac{\kappa l}{R^3} + (l^3\ddot{l})(A^2+B^2)^{3/2}\right] \end{aligned} \tag{4.1.23}$$

The time dependence can occur here only through the products (κl) and ($l^3\ddot{l}$), which, for conformity with the classical theory, are to be taken as constants, so that:

$$l^3\ddot{l} = const, \quad \kappa l = const \tag{4.1.24}$$

The first of these can be integrated right away, so that we have the final result:

$$l(t) = \sqrt{\alpha t^2 + 2\beta t + \gamma}, \quad \kappa \propto (\alpha t^2 + 2\beta t + \gamma)^{-1/2} \tag{4.1.25}$$

Thus, the last proportionality here shows that the equation (4.1.18) is, indeed, the unique solution of this problem of gauging. The property is, of course, maintained for the case $c_1 = c_2 = 0$, as mentioned, but let us notice that this case describes an ensemble of positions generated by the Appell-Glaisher transformations (4.1.5), having the same gauge function $l(t)$, as given by the first of the equations (4.1.25). The corresponding ensemble of Hertz material particles serving for a physical interpretation here, must have distinguished properties to be revealed in due time in our present venture. To be more specific, recall Hertz's stipulation from his Definition 2, reproduced in the excerpt above, namely:

> The number of material particles *in any space*, compared with the number of material particles *in some chosen space at a fixed time*, is called *mass* contained in the first space.

The Merchersky results then show that the "fixed time" from this definition should be connected to the time of our observation – variable t of the theory – only at the infrafinite scale of time, *i.e.* by a differential equation:

$$\frac{dt}{d\tau} = \alpha t^2 + 2\beta t + \gamma \tag{4.1.26}$$

They also show that, according to Hertz's definition, the mass may be expressed, at least in the special case of invariable gravitational constant, by an equation of the form

$$\frac{m(t)}{m_0} = \frac{1}{\sqrt{\alpha t^2 + 2\beta t + \gamma}} \tag{4.1.27}$$

Our contention is that discovering what t and τ really mean, should offer us the key to physical characterization of that "fixed time" and that "space" containing "the number of material particles" represented by $m(t)$. What we have for sure at this moment, is that the space in question has to be a *coordinate space*, if it is to use the definition of Charles Galton Darwin, and that the "number of material particles in some chosen space" has to be m_0. For the rest, let us follow some historical moments of our physical knowledge in order to get a indication in understanding.

4.2 The Berry-Klein Gauging Philosophy

The Berry-Klein dynamical theory of invariance of the force fields under expansion of the coordinate space (Berry & Klein, 1984) appears to us as the only theory *deliberately* 'sanctioning' the concept of scale transition as a criterion of doing physics. There is no such theory in physics, not even for the celebrated case of the spectrum of blackbody radiation. That would be the only place where the sanction appears almost explicitly as necessary, and yet the demonstration of invariance to expansion is only fortuitous as it were, not playing the fundamental part we think it should play. This notion, as we are capable to see it with the aid of Berry-Klein theory of behavior of the force fields under expansion, was indeed used before even by Newton, on the occasion of invention of the concept of forces. However, on that occasion, it was used only implicitly, therefore still 'colaterally', if we may say so. Indeed, it is by using a continuous sequence of *collision events*, with percussions acting toward a unique point in space, that Newton was able to sanction – to use the same word as before, meaningful in context – from a natural philosophical point of view, the existence of forces responsible for the Kepler motion.

Ideally, that is to say, for the case of the classical material points, the collision events can be considered as *dynamical* events involving forces that act within infinitesimal time intervals – instantaneously, in modern terms – *i.e.* at the *infrafinite scale* in time, within the modern terminology of Nicholas Georgescu-Roegen, closely related to the subject-matter of scale and the theory of continua associated to it (Georgescu-Roegen, 1971). The collision forces are accidental forces according to human experience, with accidentality 'measured', if we may say so, by the time extent of their action: ideally, they are *local* and *instantaneous*. In view of this, using Georgescu-Roegen's expression, we can say that Newton used forces acting at the infrafinite scale in time and space, in order to define field forces – forces acting everywhere and permanently – *i.e.* forces acting at the *transfinite scale* in both space and time. Historically speaking, Newton's would then be therefore the first case of a specific *gedanken* experiment involving the use of a scale transition, at least from the physical point of view. The transition between scales is indeed implicit, as we said, made *via* an *assumed identity* of the physical concepts invented by Newton with the accidental forces of our daily experience. However, it is by no means the only case of such a transition in history: we can find, in fact, a few more, if we follow the idea of scale transition. Let us, therefore, proceed along this path of unfolding the ideas.

Once again, in order to do this, we have to assume the relevance of Berry-Klein expansion theory in two important points of its intervention in the *Hamiltonian formalism*, both of them cosmological. First of all, it can be considered as a modern approach of classical *Newton's original cosmology* (see especially the Book III of *Principia*) an approach that ought to be applied, at any rate, to the modern *Newtonian cosmology* (McCrea & Milne, 1934), in order to make it truly 'modern'. As it happened, though, in this modern instance it was, again, only implicitly applied: the cosmology had not, by then – an still does not have even today, we should say – a selection criterion involving the expansion. The hallmark of a modern cosmology at large, appears to be, indeed, the expansion of the universe: this seems to be even a fact of observation of the universe around us, which can serve as a test for every law of physics. The truly modern example of conformity with such a test is provided by the Wien's displacement law: physically demonstrated for thermal radiation, conceived as an adiabatic thermodynamical system, it appears as an exact – that is, independent of the expansion rate of the enclosure containing radiation – therefore universal law, serving as a criterion of selection for any of the laws of radiation. The universality in question can be proved by the *scale invariance*, using a purely geometrical point of view, as expressed by Eddington in the excerpt above. This way one can prove that such a law, true at any *finite* laboratory scale, is valid at the *infrafinite* microscopic scale, and also at *transfinite* cosmological scale (Mazilu, 2010), inasmuch as the pure geometry of a coordinate space can be 'filled' with a physics that copes with the Wien's displacement law, in order to make a sound natural philosophy out of it, *at any scale*. And this is, indeed, the very case, up to a point: the cosmological scale manifestation of Wien's displacement law, has been definitely demonstrated by the data on the cosmic background radiation, whose 3K spectrum satisfies the Planck's law of radiation, which is a direct consequence of Wien's displacement law (Fixsen, Cheng, Gales, Mather, Shafer, & Wright, 1996). Therefore Planck's law, sanctioned experimentally at the finite space scale of the terrestrial laboratories, should be also valid at the transfinite space scale of the universe we inhabit.

Now, coming back to Newton's own cosmology, its foundation is provided by the forces he invented, an ingredient, if we may say so, which is just as important for that cosmology as the law of radiation is for an expanding cosmology. No one has ever considered, though, that in creating a Newtonian cosmology along the Einsteinian line (McCrea & Milne, 1934), the *field of forces* has to 'pass the expansion test', as it were. This is the second important point of the Berry-Klein theory: it can be taken as ascertaining that the Newtonian force field passes, indeed, the expansion test, so that from the modern cosmological point of view provided by *the fact* of expansion of the universe, Newton's own cosmology is indeed 'modern', and can be aptly considered as a Newtonian cosmology, as in fact it always was, but only implicitly. This observation has remarkable consequences: the chief among them is that *we cannot dispense with forces in physics*, we just have to understand them properly. Historically, the first step in this process of understanding of forces was the creation of the very modern theoretical concept of field, independently of the dynamical Kepler problem. One can say that with the concept of field thus created, the physics came out of the 'coordinate space' into the 'ordinary space', if it is to use again the Darwin's expressions. Indeed, the field equations of Laplace and Poisson, just like Schrödinger's equation for the free particle, are equations in the ordinary space. Let us, therefore, get to 'ordinary space' *via* a Berry-Klein gauging philosophy.

An issue pops up right away, when trying to approach this problem: what is the expression of this scale invariance at the infrafinite space scale, *i.e.* that space scale which includes the microscopic world? After all, that one is the world out of which the idea of quantum came forth, and 'leapt' into Max Planck's mind, if we may say so. The answer to this question is one of the outstanding undertones of the Berry-Klein definition of scale invariance of the force fields. Just for completeness, let us reiterate this definition in order to reveal what we deem as its essentials. Quoting:

> A system is specified by its Hamiltonian function H: let it be that of a particle of mass m moving in a *non-conservative potential field* V in a space of N dimensions. The linear scale factor $l = l(t)$ governs an *isotropic expansion*, *or contraction*, of this field, so that

$$H(\boldsymbol{r},\boldsymbol{p},l(t))=\frac{\boldsymbol{p}^2}{2m}+\alpha(l(t))\cdot V(\frac{\boldsymbol{r}}{l(t)}) \tag{4.2.1}$$

> where the multiplier $\alpha(l)$ scales *the strength of the expanding field*. Such a Hamiltonian includes the important case of an *expanding cavity*, for which, at all times, the potential term would vanish inside the cavity and be infinite outside it. [(Berry & Klein, 1984); *our Italics*]

It is in the spirit of this quotation that we shall discuss the previous results, as well as those which follow, and construct the mathematical tools serving for some incidental new results. Only, our discusssion is limited to the three dimensions ($N = 3$) of ordinary space. Notice the notation of the gauge length l. We can disclose right away that it is not chosen just by chance: it should be, indeed, a solution of the very same gauging differential equation as the previous one, *viz*. the equation (4.1.24). First, let us transcribe, in our notations, the Berry-Klein basic dynamical equations, resulting from the above Hamiltonian. [(Berry & Klein, 1984); §2]. The equations of motion used by our authors are:

$$m\ddot{\boldsymbol{r}}=-\alpha(l)\cdot\nabla_{\boldsymbol{r}}V(\frac{\boldsymbol{r}}{l}) \tag{4.2.2}$$

where a dot over a symbol means derivative on the measured time, as usual. This is a classical dynamical equation, in a time-dependent force field given by the space gradient of the potential, however not in a conservative way. An important point to notice for what shall follow in this work, is that the time dependence is implicit though, *via* the gauging factor $l(t)$. Going over to a new time, function of the old one, and a new position given by the scaled vector in equation (4.2.2):

$$t\to\tau(t);\quad \boldsymbol{r}\to\boldsymbol{R}=\frac{\boldsymbol{r}}{l} \tag{4.2.3}$$

we have to perform the replacements:

$$\ddot{\boldsymbol{r}}=\ddot{l}\boldsymbol{R}+(2\dot{l}\dot{\tau}+l\ddot{\tau})\boldsymbol{R}'+l\dot{\tau}^2\boldsymbol{R}'';\quad \nabla_{\boldsymbol{r}}=l^{-1}\nabla_{\boldsymbol{R}} \tag{4.2.4}$$

resulting in the following differential equation:

$$(l^2\dot{\tau}^2)\boldsymbol{R}''+(l^2\dot{\tau})^{\cdot}\boldsymbol{R}'+(l\cdot\ddot{l})\boldsymbol{R}+\frac{\alpha}{m}\nabla_{\boldsymbol{R}}V(\boldsymbol{R})=0 \tag{4.2.5}$$

Here the accent means derivative with respect to the newly introduced time $\tau(t)$. In order to find a physical interpretation of this equation, Berry and Klein take notice that under the following constraints:

$$\alpha l^2=1;\quad \dot{\tau}l^2=const;\quad ml^3\ddot{l}=const(\equiv k) \tag{4.2.6}$$

it becomes easily interpretable, as it gets a well known form:

$$m\boldsymbol{R}''+\nabla_{\boldsymbol{R}}\left(V(\boldsymbol{R})+\frac{1}{2}k\boldsymbol{R}^2\right)=\boldsymbol{0} \tag{4.2.7}$$

So, in the *gauged positions* and with the corresponding *gauged time*, we have a world where the *forces are conservative*, even though not quite as 'pure' as the original forces of the classical dynamics from which we have started. In the original words of our authors:

> This result allows the following interpretation. Let *measuring rods expand* with the system to measure $\boldsymbol{R}$ instead of $\boldsymbol{r}$, and let clocks be *recalibrated* to read τ instead of t. In this *frame of reference*, the particle appears to move in a *conservative field* consisting of $V(\boldsymbol{R})$ plus *a central force* $-k\boldsymbol{R}$ (which is *inertial* in origin, reflecting the fact that *observers in the expanding frame*

would *accelerate*). The central force attract particles to the centre of expansion if $k > 0$, and repels them if $k < 0$. [(Berry & Klein, 1984), *our emphasis*]

At this point we leave the Berry and Klein specific analysis, directing the reader interested in its details to the original text, for we have a few questions, a possible answer of which is guaranteed to take us way beyond this interpretation of our autors. These questions are obviously related to the words emphasized in the excerpt right above. To wit: how the measuring rods are expanding? Are the 'rods', as physically understood, unique physical objects to serve in 'measuring'? How are the clocks 'recalibrated', and, as a matter of fact, what can be made into a clock actually? What is the nature of those 'observers' that would accelerate in the 'expanding frame', and how is this very frame actually defined? How shall we describe all of these issues and, most of all, how are we to describe the 'acceleration field' of those observers?

In this very section we only touch the fundamentals. The other special issues connected with this theory will be spread all along the present work, and will be solved to our best, as we go ourselves lengthways with the story. To start with, we have the fundamental question: *what is the length of a measuring rod*? The answer to this question will enable us to see what a measuring rod can be from a physical point of view. And that answer was provided by Vittorio de Alfaro, Sergio Fubini, and Giuseppe Furlan in 1976: the length in question is *a wave mechanical field describable by a realization* of the SL(2,R) group algebraical structure (de Alfaro, Fubini, & Furlan, 1976). Such a field is connected to the partial differential equation (1.1.6). This is, in fact, quite a vague characterization, to say the least, but let us take our longer journey starting with it, in order to show how one can get to its essentials, and thus find what is the interpretation of the field in question.

Everything starts from the equation listed in (4.2.6): the third of those equations can be written in the form of equation (4.1.24) above:

$$l^3\ddot{l} = \kappa^2; \quad \kappa^2 \equiv k/m \tag{4.2.8}$$

which has the general solution of the form (4.1.25)

$$l^2 = at^2 + 2bt + c \tag{4.2.9}$$

with a, b and c some constants, as duly noted by Berry and Klein in their original work. Then, with a convenient choice of the constant involved in it, the second equation (4.2.6) can be written as

$$d\tau = \frac{dt}{l^2} \quad \therefore \quad \frac{dt}{d\tau} = at^2 + 2bt + c \tag{4.2.10}$$

so that we find again the equation (4.1.26), at least formally speaking. A first observation, occasioned though by the current context of this work: from the point of view of the interpretation concept, the Berry-Klein theory is the interpretation of the dynamics in an enclosed space, *viz.* a coordinate space in the connotation of Darwin. From this point of view the theory above is akin to the interpretation of the blackbody radiation by Albert Einstein, that led in its time to the concept of quantum (Einstein, 1905b). There was a continuum then, that *he interpreted as a molecular gas ensemble*. Recall, indeed, his conclusion that…

… radiation of *low density* behaves as though it consisted of a *number of independent energy quanta* [(Einstein, 1905b), *our Italics*]

We have to notice, though, that at those historical moments the interpretative ensemble was, as a rule, thermodynamically decided, so it had to be thought of as a physical object: an ideal gas in evolution by successive equilibrium states, so that the evolution had to be *adiabatic*, in concordance with the character of mechanical theoretical explanation of the *adiabatic invariants*. Here, on the other hand, the interpretation procedure is not quite so handy. One thing still has to be noticed, in agreement with what seems to be one of the points of Sir Michael Berry's general philosophy, and this is the fact that *the problem of coordinate space should be free from the adiabatic condition*. It is a fundamental physical problem, which, in fact, only incidentally popped up in thermodynamics, where the coordinate space was ever-present by the physical necessity of containers.

It is at this juncture, that we have to notice that the problem of interpretation takes a significant turn, marked, as it were, by an outstanding solution of an equally oustanding *gauging procedure*. Namely, notice that the equation (4.2.9) offers the radial equation of motion for an ensemble of *classical free material points*, which is the undisputable conterpart of the classical ideal gas, which stays at the foundation of thermodynamics. Indeed, we can produce the Berry-Klein gauge length from equation (4.2.9) by considering the vectorial equation of motion of a free particle in an Euclidean reference frame. Thus, we have:

$$\boldsymbol{l} = \boldsymbol{v}t + \boldsymbol{l}_0 \quad \therefore \quad l^2 = \boldsymbol{v}^2 t^2 + 2(\boldsymbol{v}\cdot\boldsymbol{l}_0)t + \boldsymbol{l}_0^2$$

proving the fact: the Berry-Klein gauge length can be considered the free path of a classical material point in an ensemble of such points. Therefore, the gauge length in a Berry-Klein gauging can be connected, indeed, with an ensemble of free particles, with the freedom kinematically defined by the classical equations of motion of those particles. The gauging length *is the radial coordinate of that motion*, which is in fact the analogous of a mean free path in the case of ideal gas. This statement may seem occasionally confusing but, in fact, it sets our thinking on the right track, at least historically speaking.

First, it jumps one's eyes that each and every one of the free particles of the interpretative ensemble should have, according to what seems to be just a matter of course, its 'own' gauge. Obviously, we are not quite so sure that, as such, we have a correct gauge at all. Indeed, a correct gauge has to involve, if not a firm physical magnitude like the mass or the charge, at least a statistic representing an estimate over a certain ensemble, like the mean free path mentioned above. However, the idea of 'individual', at least in the classical sense of the word, apparently does not involve either one of these situations. But only *apparently*, indeed: as we have seen in §3.1, the charge and mass can be classically assigned at random to a Hertz material free particle, so even from this point of view we should not have a problem with the gauging procedure. On the other hand, historically we have the case of de Broglie's assignment making a 'wave phenomenon' out of the classical material point, as de Broglie himself expressed it. Before any such considerations on the gauging procedure should be further elaborated, let us, however, show how they can be connected with the dynamics.

That task seems to be simple, in view of the solitonic theory produced by us in the §3.1 [see equations (3.1.17) ff] and destined to describe the space distribution of the physical magnitudes describing the material particles: the transformation (4.2.3) looks like a *Mariwalla transformation*, and such a transformation leaves the classical dynamics, based on Newtonian central forces going inversely with the square of distance, invariant (Mariwalla, 1982). Indeed, if we write the Berry-Klein gauging equations as:

$$d\tau = \frac{dt}{l^2}; \quad \boldsymbol{R} = \frac{\boldsymbol{r}}{l} \tag{4.2.11}$$

it becomes just a matter of identification of the gauge length l with a linear form in the coordinates of the moving particles, in order to make a Mariwalla transformation out of it. The two share the same general philosophical feature: the Berry-Klein measuring length that serves in 'fixing the scale' – to use a modern notion here – does the job in the *infrafinite range in time*, and in a *finite range in space*, just like the Mariwalla's gauging factor. Further on, this gauging factor can be viewed as the distance from a plane on space, which is the essential variable of a solitonic theory as we presented in §3.1 [see, *e.g*. the equation (3.1.27)].

The transformation (4.2.11), just like Mariwalla transformation, never involves *the same* space and time scales. This would mean, for instance, that the calibration of clocks depends on this gauge factor. In other words, a Berry-Klein gauging length can serve to throw a bridge between the microscopic and daily worlds, just as the Planck's constant once did, and this is quite an advantage, in view of our previous observations. If in the case of space we may be able to think of a way of fixing the scale, for instance with a rod if nothing else, in the case of clocks we certainly have to elaborate on a construction of some kind, for the things are not altogether clear. The classical expression, though, of the connection between length measuring and time, namely the *concept of velocity*, provides an important clue: the measuring rod length *can also be a distance*. This means that even a 'piece of vacuum' can serve, just as well, as a 'measuring rod', and this is quite in agreement with the Katz's natural philosophy of the charges (again, see §3.1, above); an essential preliminary idea, in need of a further rational elaboration, of course.

The positive fact to start with here, is that the Mariwalla's transformation only goes with the Newtonian forces, *i.e*. central forces of magnitude inversely proportional to the square of the distance between observers. And this is the key of everything that follows. To wit, this would mean that the equation (4.2.7) is invariant to the transformation (4.2.11) only if under the gradient operation we have a Newtonian potential, otherwise the equation changes, perhaps uncontrollably. This means that the first of the equations (4.2.6) cannot take place but just for central forces of magnitude inversely proportional to the square of distance, whose dynamics *only*, is, therefore, invariant with respect to the Berry-Klein gauging procedure. This condition selects then only the Newtonian forces going inversely with the squared distance between bodies. But this is not all of it: as we said, the algebraical structure of the original Mariwalla transformation shows that, in case we take the Berry-Klein gauging procedure in the classical way, the gauge length is not independent of the coordinates used in describing the position in the ordinary space of the bodies involved in the classical Kepler problem. Therefore, the Berry-Klein gauging procedure, like the very classical Kepler problem thus connected to it, is simply the key to constructing an application from the coordinate space to ordinary space, and vice versa. Let us elaborate on these statements in order to show explicitly what we mean by it.

In fact, Mariwalla has shown that Newton's equations for planetary motion, which describe dynamically the Kepler problem, are invariant with respect to the simultaneous transformations of coordinates and time given by the formulas

$$\boldsymbol{R} = \frac{\boldsymbol{r}}{1+\boldsymbol{k}\cdot\boldsymbol{r}}, \quad d\tau = \frac{dt}{(1+\boldsymbol{k}\cdot\boldsymbol{r})^2} \tag{4.2.12}$$

where $\boldsymbol{k}$ is, on this occasion, an arbitrary constant vector characterizing the transformation through its components. As one can see, this is, indeed, a *mixed* transformation in terms of scales of space and time, and creates the issue we emphasized here quite a few times before: it is referring, on one hand, to the *finite space scale*, with the space represented by the components of the position vectors in an Euclidean reference frame and, on the other hand, to the *infrafinite time scale* (marked by the use of differentials) for any meaningful time in that reference frame. Of course, it is quite obvious that the definition of the finite scale is actually a matter of pure convention, once it is related to a choice of the Euclidean reference frame, even if we do not ask if such a frame exists, to say the least.

Before going any further, we have to produce a reasonable proof for equation (4.2.12). It involves only straight calculations, to be done in the spirit of previous Berry-Klein calculations, as follows: if, in general, we choose a scale function, l say – in order to be in line with the Berry-Klein notation used in equation (4.2.11), and with the notation from previous section – then we have:

$$\frac{d\boldsymbol{R}}{d\tau} = l\cdot\dot{\boldsymbol{r}} - \dot{l}\cdot\boldsymbol{r} \quad \therefore \quad \frac{d^2\boldsymbol{R}}{d\tau^2} = l^2(l\cdot\ddot{\boldsymbol{r}} - \ddot{l}\cdot\boldsymbol{r})$$

Newton's equations of motion in the coordinates $\boldsymbol{R}$ and time τ then look like

$$\frac{d^2\boldsymbol{R}}{d\tau^2} + (r^3\ddot{l} + \kappa l)\frac{\boldsymbol{R}}{R^3} = 0 \tag{4.2.13}$$

so that, in cases where the dynamical problem involves *the very same Newtonian type of force* at different scales, we must have, l_0 being a constant length:

$$r^3\ddot{l} + \kappa l = \kappa l_0 \tag{4.2.14}$$

and the equations of motion deriving from the second principle of classical dynamics are indeed form-invariant. This means that a human being living, for instance, in a world where it establishes the event (x, y, z, t), uses the same equations to describe Kepler motion as a human being living in a world where it establishes the event ($\boldsymbol{R}$,τ). These two worlds are 'similar by a scale factor', as it were, provided the similarity goes at a finite scale in space and an infrafinite scale in time, in either one of the two worlds. Now it is quite obvious that if in equation (4.2.14) we choose

$$\ddot{l} = \boldsymbol{k}\cdot\ddot{\boldsymbol{r}} \tag{4.2.15}$$

where $\boldsymbol{k}$ is an arbitrary constant vector, then we have

$$l = l_0 + \boldsymbol{k}\cdot\boldsymbol{r} \tag{4.2.16}$$

no matter of the constant κ. Therefore the similarity factor between the two worlds can be simply determined by the *projection of acceleration along any constant vector*. There are thus multiple Newtonian worlds in a given reference frame, distinguished from one another by scale factors determined by accelerations! However, once again, we need to take proper notice that this kind of scale factors – specifically *linear in the position in a reference frame* – are referring to the finite space scale and infrafinite time scale of those worlds. As long as the classical dynamical equations of motion are involved in the description of such a world, this condition is inescapable.

This is, in fact, a kind of invariance discriminating in favor of Newtonian forces for, in general, we have

$$\frac{d^2\boldsymbol{R}}{d\tau^2} + \kappa\frac{\boldsymbol{R}}{R^3} = l^3\left(\ddot{\boldsymbol{r}} + \kappa\frac{\boldsymbol{r}}{r^3}\right)$$

and any force *other than the Newtonian one* is liable to break the invariance of the equation of motion. By the same token, any equation of motion, *other than that involving acceleration as the second order derivative* is also liable to break the invariance. Mariwalla, for instance, even noticed that the geometrical properties of Kepler motion (area law proper, as well as the preservation of the plane of motion) are consequences of this type of invariance, which thus gives us all the invariants we need for the central motion under the *inverse square force*. Another property comes with the observation that, in case *some other kinds of Newtonian forces* act simultaneously between the bodies – a situation we are to consider essential in the definition of the process of interpretation, as shown previously in the §3.1 – so that instead of (4.2.14) we may have

$$r^3\ddot{l} + \kappa l = \kappa' l_0$$

with κ′ ≠ κ. Then the scale function from equation (4.2.16) changes to

$$l = (\kappa'/\kappa)l_0 + \boldsymbol{k}\cdot\boldsymbol{r} \qquad (4.2.17)$$

Thus, the Kepler problems for gravitational mass and charge, for instance, as we had them before, are not simply equivalent: *they are indeed formally identical, but for different scale factors* satisfying the description of Mariwalla's type of invariance in space. The forces though, involved in this kind of invariance, should be of the same family: central forces, of Newtonian character, *i.e.* with *their magnitude going inversely with the square of distance*. We are entitled then, to use these forces in constructing a Hertz material point, according to a 'cosmological principle', as we already indicated in the introduction to this very chapter.

Whatever has been said, up to this point in the present section, regarding the idea of interpretation, is based only on the equation (4.2.9), and concerns the interpretation in terms of ensembles of free particles in the classical sense. One might say that such a gauge length is suitable in constructing the second of the equations (4.2.11). However, we are not to forget that the gauge length also enters the equation (4.2.10), therefore the first of the equations (4.2.11), and consequently, while essential in a finite space scale, it may be regarded as essential in the infrafinite time scale too. What is, then, the meaning of the equation (4.2.10)? It does not seem to be proper interpreting it as a length or even as a distance. Or, if we do so, that length or distance should have a statistical further meaning. So, can it be taken as a statistical condition? In order to answer to this question, let us discuss in more detail the equation (4.2.8), which describes the time evolution of the gauge length in Berry-Klein theory. That equation is *form-invariant* with respect to the transformation of time and gauge length given by equations

$$T = \frac{\alpha t + \beta}{\gamma t + \delta}, \quad L = \frac{l}{\gamma t + \delta} \qquad (4.2.18)$$

i.e. it is invariant in the following sense:

$$\ddot{l} = \frac{\kappa^2}{l^3}, \quad L'' = \frac{K^2}{L^3}; \quad K = \frac{\kappa}{\Delta} \qquad (4.2.19)$$

Here $\Delta \equiv \alpha\delta - \beta\gamma$ is here the determinant of the time transformation, and the accent mark means derivative on T. This equation of motion is, indeed, not quite invariant with respect to the transformation (4.2.18), but only form-invariant as we said, therefore quasi-invariant. It is invariant only with respect to time transformations of unit determinant, but the quasi-invariance (4.2.19) may prove occasionally salutary, in view of our observation that, at some point of the theory, it may become necessary to take the gauge length as a linear form in the coordinates, according to Mariwalla theory. Indeed, then one can argue that the coefficients of the transformations are to be determined from those physical features related to Hertz material particles generating the innate Newtonian central forces responsible for the equilibrium of the interpretative ensembles of such particles. As one can figure out, this is a real advantage in a physical theory, so that we will avail of it, by maintaining the quasi-invariance (4.2.19) of the Berry-Klein gauging equation of motion.

The two equations (4.2.18) give a closed-form SL(2,R)-type of realization of an action represented at the infinitesimal level by three differential vectors, as follows:

$$\boldsymbol{X}_1 = \frac{\partial}{\partial t}, \quad \boldsymbol{X}_2 = t\frac{\partial}{\partial t} + \frac{1}{2}l\frac{\partial}{\partial l}, \quad \boldsymbol{X}_3 = t^2\frac{\partial}{\partial t} + tl\frac{\partial}{\partial l} \tag{4.2.20}$$

These operators satisfy the algebraical structural relations (3.4.10), characteristic for a $\mathfrak{sl}(2,\mathrm{R})$ algebra, which are also satisfied by the vectors from equations (3.3.1) and (3.3.2), and which we take as the standard commutation relations for this algebraic structure. Then the equation (4.2.9) can be viewed as a result of invariance with respect to the general action of the realization (4.2.20) of this algebra. Indeed, the most general vector of this space is a linear combination of the base vectors (4.2.20). By definition, the invariant functions with respect to such an action, is a solution of the partial differential equation:

$$(c\boldsymbol{X}_1 + 2b\boldsymbol{X}_2 + a\boldsymbol{X}_3)f(t,l) = 0 \quad \therefore \quad (at^2 + 2bt + c)\frac{\partial f}{\partial t} + (at+b)l\frac{\partial f}{\partial l} = 0 \tag{4.2.21}$$

which turns out to be a continuous function of the ratio

$$\frac{l^2}{at^2 + 2bt + c} \tag{4.2.22}$$

Therefore, the solution (4.2.9) of the Berry-Klein gauging equation of motion is actually an *invariant function* under the action (4.2.20) of the $\mathfrak{sl}(2,\mathrm{R})$ algebra. Revealing a few more properties of this kind of action can be in order on this occasion.

We have seen above, in the §3.2, how productive is the *Stoka theorem* in checking the mathematical form of different joint invariants of group actions in the construction of the physical functions (Stoka, 1968). The Gaussian statistics and, consequently, the Cauchy statistics connected to it, are such obvious examples. In geometry, the metrics can be thus produced, with statistical meaning, and we shall see in fact more of these examples as we go along with this story. Here, and for now, we are interested to apply the Stoka theorem to two actions (4.2.20), one in the variables (l,t), the other in the variables (L,T). This comes down to solving the system of partial differential equations:

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial T} = 0, \quad t\frac{\partial f}{\partial t} + T\frac{\partial f}{\partial T} + \frac{1}{2}\left(l\frac{\partial f}{\partial l} + L\frac{\partial f}{\partial L}\right) = 0, \quad t^2\frac{\partial f}{\partial t} + T^2\frac{\partial f}{\partial T} + tl\frac{\partial f}{\partial l} + TL\frac{\partial f}{\partial L} = 0 \tag{4.2.23}$$

producing the general solution as a continuous function of the following algebraic expression:

$$\frac{L \cdot l}{T - t} \tag{4.2.24}$$

If the gauge lengths and the times differ only infinitesimally, then the differential $d\tau$ from (4.2.13) can be considered a joint invariant, for, then, the expression

$$\frac{dt}{l(l+dl)} \tag{4.2.25}$$

is such an invariant, exactly. This is the invariance-connected aspect of the first Berry-Klein equation in the transformation (4.2.11). Another aspect is the one connected to equation (4.2.10), and can be achieved when touching the issue of simultaneity, as follows.

Assume that we need to interpret the *condition of simultaneity T = constant*, from the first formula of transformation (4.2.18), which leaves the Berry-Klein equation of evolution for the gauging length form-invariant. This will provide the set of moments of time t, all corresponding to one and the same value of T. Assuming that these time moments correspond to some space events, these events are then indeed simultaneous, with the simultaneity defined by the first equation from (4.2.18). The *generic* condition of such simultaneity, *i.e.* the one condition independent of the value of T, can be given by the differential equation $dT = 0$, and amounts to a connection between the differential of time t and the differentials of the parameters α, β, γ, δ. This, of course, assumes that dT is calculated by the usual rules of differentiation on both time t and the four parameters. The result is

$$d\frac{\alpha t+\beta}{\gamma t+\delta}=0 \quad \therefore \quad dt=\omega^1 t^2+\omega^2 t+\omega^3 \tag{4.2.26}$$

where the coefficients of the quadratic polynomial are the following differential forms:

$$\omega^1=\frac{\alpha d\gamma-\gamma d\alpha}{\alpha\delta-\beta\gamma}, \quad \omega^2=\frac{\alpha d\delta-\delta d\alpha+\beta d\gamma-\gamma d\beta}{\alpha\delta-\beta\gamma}, \quad \omega^3=\frac{\beta d\delta-\delta d\beta}{\alpha\delta-\beta\gamma} \tag{4.2.27}$$

These differential 1-forms are a coframe for the $\mathfrak{sl}(2,\mathrm{R})$ algebra, for they satisfy the structural relations (3.3.22). As a matter of fact, for symmetric matrices they reduce to the coframe (3.3.15) as expected.

Therefore, the interpretation of *simultaneous events* corresponding to the same moment of time T, is provided by a set of positions described as a $\mathfrak{sl}(2,\mathrm{R})$ metric space in coordinates (ω^1, ω^2, ω^3), corresponding to moments of time given by differential equation (4.2.26). These moments amount to a time perceived in observations, put in order in the form of sequences with the help of calibrated motions – of clocks, for instance, to take the best known, and almost exclusively used case. As the equation (4.2.26) is an equation in pure differentials, the association position-moment of time in the structure of a such event still remains, to some extent, arbitrary. However, the property of the $\mathfrak{sl}(2,\mathrm{R})$ of being a Lie algebra, saves the day, as it were. Indeed, in that case the space of positions of the events is a metric space, with the metric basically provided by the discriminant of the quadratic polynomial from equation (4.2.26). Along the geodesics of this space the differential forms (4.2.27) are exact differentials proportional with the elementary arclength, ($d\tau$) say:

$$\omega^1=a^1 d\tau, \quad \omega^2=2a^2 d\tau, \quad \omega^3=a^3 d\tau \tag{4.2.28}$$

where (a^1, a^2, a^3) are constant rates with respect to the arclength of geodesics. One can say that these three equations are conservation laws, which is, in fact, the way such a theory is usually presented, at least in some special cases [(Schutz, 1982); see §§3.12 and 5.8]. Be it as it may, the net result is that the differential equation from (4.2.26) becomes an ordinary differential equation of the Riccati type:

$$\frac{dt}{d\tau}=a^1 t^2+2a^2 t+a^3 \tag{4.2.29}$$

and therefore its solutions provide a connection between t and τ at the differential level – consequently at an infrafinite time scale – which is of the form provided by the first equation of Berry-Klein transformation from (4.2.11). And thus, the Berry-Klein scaling theory hands us 'on a plate', as it were, the statistical interpretation needed for the Hertz's definition of mass according to Meshchersky's equation (4.1.27). Only in passing, for now, we notice that this condition requires a certain connection between the Hertz's mass and the gauge length.

Notice, further, that the above equation (4.2.29) is essentially of the form (4.1.26). Let us assume, for the moment, that they are *identical* and, based on that, explain what 'essentially' means. According to the way it was obtained, equation (4.2.29) describes an ensemble: this is the ensemble of all 2×2 gauging matrices of the form (3.3.32), with entries given by the parameters (a^1, a^2, a^3). These matrices represent, each one of them, material particles, evolving in the 'time' τ along the geodesics of the $\mathfrak{sl}(2,\mathrm{R})$ Riemannian space. For every moment of time t, we have an ensemble of such positions on different geodesics of the Riemann space. Then t must be connected to a statistics, and this is, indeed, the case: it is the *mean* of an *exponential family of distributions* with *quadratic variance function* (Morris, 1982). The *variance* function is in this case ($dt/d\tau$) and is provided by the right hand side of the equation (4.2.29), which justifies the name of this family of distributions. Therefore, if one takes the equation (4.2.26) along the geodesics of the $\mathfrak{sl}(2,R)$ metric space, we can get the parametric equation (4.2.29): this equation requires some conservation laws to be enforced on the geometry. More about this subject, with special emphasis on the celebrated Planck's distribution describing the first quantization ever, can be found in (Mazilu, 2010).

4.3 Ermakov-Pinney Equation: the First Quantization in Disguise

Incidentally, someone can ask: if the Planck's name is involved in this 'statistical geometry' as it were, where is the quantization here!? To the extent it is connected to the harmonic oscillator, it is incidental too, and that even to a high degree: one can say that the harmonic oscillator is just as fictitious for the Planck's theory, as the elastic force was for Newton, on the occasion of his definition of the forces invented to explain the world order. Indeed, let us take notice that the Berry-Klein theory can also describe a classical free particle: it is a harmonic oscillator, but *in the gauged coordinates and the gauged time*. Indeed, the equation of motion of a free particle, can be obtained from equation (4.2.7) making the potential V constant, specifically zero. Assuming that the gauged time of such a problem can be the natural measured time of classical physics, the Berry-Klein equation (4.2.7) becomes:

$$\boldsymbol{x}'' + \omega^2 \boldsymbol{x} = \boldsymbol{0} \tag{4.3.1}$$

proving what we just said. Insofar as such a free particle serves in building an interpretation, a corresponding Berry-Klein gauging is realized by its radial coordinate equation, as we have shown above. One, therefore, needs the radial equation of motion of this Berry-Klein free particle whose general motion is described by equation (4.3.1). That equation can be obtained from (2.3.21) and (2.3.22) so that, preserving the notations from §2.3, the radial motion corresponding to (4.3.1) is described by a differential equation of the form:

$$r'' + \omega^2 r = \frac{R^2}{r^3}, \quad r \triangleq \left\| \boldsymbol{x} \right\| \tag{4.3.2}$$

where we used a customary notation of the norm of a vector. This shows that the radial coordinate of a *Berry-Klein free particle* can be the gauge length of the Berry-Klein corresponding gauging procedure, only in the case $\omega = 0$ in equation (4.3.1). What is the meaning of this situation?!

In order to reveal this meaning, let us start by noticing the fact that equation (4.2.8) can be produced by making stationary the action corresponding the following Lagrangian (de Alfaro, Fubini, & Furlan, 1976), to be denoted here by the symbol Λ (we do not use L as typical, in order to avoid some possible confusion with a previous notation for the gauging length):

$$\Lambda(l,\dot{l}) = \frac{1}{2}\left(\dot{l}^2 - \frac{\kappa^2}{l^2} \right), \quad i.e. \quad A(t_1,t_2) = \int_{t_1}^{t_2} \Lambda(l,\dot{l})\, dt \tag{4.3.3}$$

where the time t of the classical dynamics is used. One important fact that recommends the gauge length l as a distance, is the solution (4.2.9) of the equation of motion (4.2.8) resulting from the variation of this action: that solution represents the 'radial motion', as it were, of a generic *classical* free particle in an arbitrary Euclidean reference frame, as we have shown in §4.2. It may also represent the radial motion, in such an arbitrary reference frame, of a position from the surface of a wave propagating with constant speed in a homogeneous medium. In this last case, we have here a suggestion of interpretation in classical terms, *i.e.* a kind of interpretation like that of Louis de Broglie, by an ensemble of free classical material points.

The *physical* action from equation (4.3.3) can be simply viewed as the *time average* of the difference of the two terms involved in the definition of Lagrangian. The action from equation (4.3.3) represents, indeed, just such an average, with an important specification though, regarding the manner in which the average is taken: it is the time average of the Lagrangian over a time sequence of *equally probable* moments of time, arranged in a given order by the free particles used in the interpretation. These times can be, in fact, decided with arbitrary clocks but – and this is of a overwhelming importance for what we have to say here – the 'equal probability' in question is statistically conceived in terms of a *uniform distribution* of the time moments, having a constant probability density and described by the elementary measure *dt*. Reformulating, therefore, this conclusion in specific theoretical statistical terms, it sounds like: the equation (4.2.8) can be mathematically produced by the physical condition of stationarity of the *time average* of the difference involved in the definition of Lagrangian, *over a sequence of equal probability moments, characterized by a uniform distribution*, *i.e.* a probability distribution of the time moments, having constant density.

Once adopting this *physical meaning* for the physical action – the necessity of which shall be clear soon – we can cite a result of Morton Lutzky, that settles the position of the equation (4.3.2) – regardless of the time involved in it – usually called the *Ermakov-Pinney equation* in physics [for this see (Ermakov, 2008); see also (Pinney, 1950)], which is the name we also adopt here. As we see it, this is one of the most important equations of mathematical physics, and for this statement we have firm reasons soon to become completely obvious. The result of Lutzky we just mentioned, can be summarized with reference to the following observation (Lutzky, 1978b). Modifying the Lagrangian from equation (4.3.3), to the following algebraical form, describing the radial motion of a 'combined' free particle:

$$\Lambda(r,\dot{r}) = \frac{1}{2}\left(\dot{r}^2 - \frac{R^2}{r^2} - \omega^2 r^2\right) \tag{4.3.4}$$

produces the Ermakov-Pinney equation of motion (4.3.2), by the stationarity of the corresponding action constructed as time average on the usual uniform time distribution. The 'combined' free particle, means here a free particle of the classical dynamics, but 'gauged' Berry-Klein style. That equation of motion was always interpreted – in physics, of course – *dynamically*: it describes a harmonic oscillator *perturbed* with a force going inversely with the third power of elongation. The Berry-Klein gauging theory changes this optics, switching the emphasis *on the kinematics*, as it were. While the dynamical case is just as important, mostly from a geometrical point of view as we shall see, we will abide here, occasionally, by the kinematics too, in order to describe the idea of interpretation. So, let us see where we stand with this gauging theory.

Mathematically speaking, the Ermakov-Pinney equation has a few distinctive properties on which we need to insist at some length. Assume indeed, that we have to consider *two* posssibilities of Berry-Klein gauging for a particle free according to equation (4.3.1), for the same time t: one with the gauge length p, the other for the gauge length q. Both of them satisfy a corresponding Ermakov-Pinney equation for the same ω, meaning the same elastic forces, if it is to use a here a dynamical view, which turns out to be particularly suggestive:

$$\ddot{p}+\omega^2 p=\frac{R_1^2}{p^3};\quad \ddot{q}+\omega^2 q=\frac{R_2^2}{q^3} \tag{4.3.5}$$

What is the relevance of these equations for what we have to say here, based on dynamics!? First of all, our notations p and q are intended to suggest a *phase plane* of the two symbols: one of them can be taken, for instance, as a generalized coordinate, the other can be taken as generalized momentum. Now, in case $\omega = 0$, for either p or q, its Lagrangian reduces to the de Alfaro-Fubini-Furlan Lagrangian (4.3.3), and the corresponding equation of 'motion' for this gauge length reduces to that of *a proper Berry-Klein gauge length* (4.2.8). Therefore the (p,q) phase plane includes the radial motion of a free particle, and so this phase plane is bound to represent – and describe also – those scale transitions correlated with the properties that led to the very classical quantization of Max Planck. This is, indeed, the case, and we shall describe it here in some detail, but in connection with what we find as significant, in order to reveal its meaning in full.

First of all, by extending an observation of Colin Rogers and Usha Ramgulam, we can give the following theorem (Rogers & Ramgulam, 1989): the ratio of solutions of two Ermakov-Pinney equations, like the ones from (4.3.5), is also, formally speaking, a solution of an Ermakov-Pinney equation. The qualification 'formal is referring to the following: the 'time' of this new equation derived from the two initial ones, is dictated by the equation that gauges the time in the corresponding Berry-Klein transformation (4.2.11). In order to properly understand this result we have to prove it in detail. To start with, by successive direct differentiations on t, and choosing the transform of time as dictated by the gauge length p, we get

$$\frac{d}{d\tau}\left(\frac{q}{p}\right)=p\dot{q}-q\dot{p},\quad \frac{d^2}{d\tau^2}\left(\frac{q}{p}\right)=p^2(p\ddot{q}-q\ddot{p});\quad d\tau=\frac{dt}{p^2}$$

Now, by further using here the equations from (4.3.5), specifically the second of those equations, we get indeed an Ermakov-Pinney equation for the ratio of q and p, which *does not involve* the pulsation ω anymore, and therefore the elastic forces and inertia properties that come with it:

$$\left(\frac{q}{p}\right)''+R_1^2\left(\frac{q}{p}\right)=R_2^2\left(\frac{p}{q}\right)^3 \tag{4.3.6}$$

Here an accent means differentiation with respect to gauged time τ. Had we have used the gauge length q for the time transform instead of p, we would have gotten a similar equation but for the reciprocal ratio of the two gauge lengths, with the constants R switching places. By direct integration of this equation or, if one likes, by constructing the Hamiltonian of the corresponding Lagrangian (4.3.4), we get another one of Lutzky's important results, amounting to the constant integral of motion

$$I(p,p';q,q')\equiv R_1^2\left(\frac{q}{p}\right)^2+R_2^2\left(\frac{p}{q}\right)^2+\left(\frac{q}{p}\right)'^2=const \tag{4.3.7}$$

This result presents the gauge lengths in their interdependence: given two gauge lengths, solutions of the Ermakov-Pinney equation *for the same time variable*, either one of them can be used to 'fix the gauge', as it were, by a Berry-Klein transformation of the form:

$$l=\frac{q}{p},\quad d\tau=\frac{dt}{p^2}\qquad or \qquad l=\frac{p}{q},\quad d\tau=\frac{dt}{q^2} \tag{4.3.8}$$

This allows passing to the time t, in which case we have:

$$I(p,\dot{p};q,\dot{q}) \equiv R_1^2\left(\frac{q}{p}\right)^2 + R_2^2\left(\frac{p}{q}\right)^2 + (p\dot{q}-q\dot{p})^2 = const \tag{4.3.9}$$

Either for $R_1 = 0$, or for $R_2 = 0$, this is the original *Lewis invariant*, which is *a direct generalization of the classical Planck's constant* (Lewis, 1968). This can be made intuitively obvious as a simple fact: for both $R_1 = 0$, and $R_2 = 0$, the invariant I is the square of the elementary area rate of the phase plane, which can therefore be connected with the Planck's constant. But the things are by far more intricate, and we need to insist further on this important subject, inasmuch as the Planck's constant is considered today one of cornerstones of the modern physics.

The physics here, has a geometrical explanation, just like any other physical, well-explained fact. However, in the very beginning, when the quantum mechanics was created, this geometrical explanation was not so obvious: it became clear only after that work of Harold Ralph Lewis from 1968, we just cited above. The physical fact was the energy conservation for the harmonic oscillator, which was brought front and center by the Planck's procedure of statistical quantization in the case of light. This fact triggered the necessity of a detailed study of the harmonic oscillator, with the following results: for the linear *undamped* harmonic oscillator, whose equation of motion is

$$m\ddot{q} + kq = 0 \tag{4.3.10}$$

there is an obvious constant of motion obtained by the direct integration:

$$H(p,q) = \frac{1}{2}\left(\frac{p^2}{m} + m\omega_0^2 q^2\right);\quad p \triangleq m\dot{q},\quad \omega_0^2 \equiv \frac{k}{m} \tag{4.3.11}$$

This is the Hamiltonian, and *can be identified with the energy*, for it extends the concept of energy for the classical free particle, which is only the kinetic part $p^2/2m$ of this quadratic form. Notice now that equation $H(p, q) = const$, warranted by the equation of motion (4.3.10) as the conservation of energy, represents a conic in the phase plane. So the Hamiltonian can be used in generating the motion of the harmonic oscillator, just as the classical potential is used in generating the force, only a bit more sophisticated mathematically – according to the definition of the momentum from equation (4.3.11) – by the Hamilton equations:

$$\dot{p} = -\frac{\partial H}{\partial q},\quad \dot{q} = \frac{\partial H}{\partial p} \tag{4.3.12}$$

The first one of these equations is the equation of motion proper, as given in (4.3.10), while the second one is the definition of the momentum from equation (4.3.11). If E is the value of energy, then the area covered by points internal to the ellipse $H(p, q) = E$, is given by the integral [(Cramer, 1962), equation (11.12.3)]:

$$I_E \equiv \iint_{H(p,q)\le E} dp\, dq = \pi\frac{E}{\omega_0} = \pi\hbar \tag{4.3.13}$$

where the last equality comes here with the idea of Planck quantization procedure for the electromagnetic field. Now, a clear difference between th invariant (4.3.9) and the Planck's constant starts to unravel. First (4.3.9) is a *differential invariant* representing, physically speaking, the square of an energy, while (4.3.13) is an *integral invariant* physically representing an action. The difference between these mathematical notions needs to be clearly delineated, for it is instrumental for physics [see (Hannay, 1985) and (Berry, 1985)], and we shall do it in this work via continuous group theory. In our opinion, this is the real root of the often-mentioned difference between energy and Hamiltonian, that became critical once the wave and quantum mechanics made their appearance within our spirit (Kennedy & Kerner, 1965).

Obviously, the previous theory cannot be more viable than in cases where the electromagnetic field is a reality in mechanics, and such a case occurred – just fortuitously we should say, if it is to consider it from a purely historical point of view – on the occasion of Bohr's quantization of the planetary atom, at the beginning of the 20th century (Bohr, 1913). A rationale might be found, therefore, for the quantization for the equation (4.3.13) as a *time invariant*, just like some conservation laws, but because the time in physics was universal at the historical moment of Bohr's quantization – more to the point, the time was *unique*, for it was taken as in the classical mechanics – the quantity (4.3.13) could not be appropriated as just an ordinary time invariant: it was an *adiabatic invariant* [see (Ehrenfest, 1917), and the previous literature cited in this detailed summary of Paul Ehrenfest]. It is based on this concept, that quantum mechanics came to light (Heisenberg, 1925). This meant then – as it means even now, actually – that the energy varies so slowly in the time of evolution given by equation (4.3.10), that its variation is not quantitatively significant enough over a period of motion described by that equation, in order to be noticed as a change of that quantity. In the equation of motion the 'slowness' is described by the time rate of a parameter hidden somewhere in the expression of the energy, in such a way that its variation over a period of motion is insignificant (Landau & Lifschitz, 1966). In other words, the energy may not be really conserved, however one can say that, the motion proceeds only *as if* the energy is conserved, and this is a practical thing that needed to be raised to a theoretical understanding.

The issues got complicated, but the advantage of a hindsight allows us nowadays to frame this complication as being connected to the necessity of recognizing here *a time scale transition*: *the energy is conserved in the time of motion, while the action is conserved at an entirely another time scale*. As we will shortly show here, this essential difference of behavior, took a precise meaning historically, which we try to relegate to the fact that the old adiabatic parameter is the a time, and its rate simply reflects a transition between the two time scales. Maintaining, however that old view connected to the idea of adiabatic changes, the contention was that the new time has to be connected to the evolution of the universe, and this is the main point where the space scale transition gets into play. The first description of this transition apparently belongs to Lorentz, and is based on a special isotropic transformation of a coordinate space [(Lorentz, 1901); see especially §4 of that work]. The general mathematical principles of an extension of classical natural philosophy, in order to make it capable of including the new facts of theoretical physics brought about by the 19th century at its close, were described by Paul Ehrenfest as early as 1905. Two physical theories are recognized by Ehrenfest in his work, as counting as final productions of the previous century, within the new natural philosophy that started unravelling at the turn of the 20th century: these are the *Planck's theory of thermal radiation* on one hand, and the *Lorentz's theory of the electric structure of matter*, mentioned by us before, on the other hand. In recognizing that the light turns out to be the only phenomenon that naturally transits the scales of space and time in the universe, these theories bring to the fore the brightest of the realizations of the 19th century: the electromagnetic nature of light. This aspect changed the means of theoretical physics, to an even greater extent than the contemporary special theory of relativity, by facilitating a fresh approach of the concept of a *coordinate space*. Insofar as this concept needs to be made precise in view of the necessary transition to the ordinary space, we may very well start that description with the problem of quantization here. Quoting, therefore:

According to Planck's model – linear oscillators with pure radiation damping within a completely reflecting enclosure – a radiation state Z_I exists. In the expression of Planck's theory, the following statements may apply to it:

1. There is a natural radiation (disorder hypothesis)
2. This is a stationary state, where Σ_I and Σ_{II} do not vary in time
3. The total energy is E_I
4. The radiation density is Δ_I
5. The spectral energy distribution is $s_I = \varphi(\lambda)$

The stationary radiation state Z_I consists of a series of rapidly changing fields and oscillations of all resonators. These processes are controlled

a) *by Maxwell's equations*

b) by the boundary conditions at the reflecting enclosure

c) *by the equations of oscillation of resonators*

All three groups of equations are linear-homogeneous in the electric force ($\mathfrak{E}$), the magnetic force ($\mathfrak{H}$) and in the vibration vector (f) of the individual resonators and their spatial and temporal derivatives.

If thus a series of processes satisfy these equations, then one can immediately derive a ∞^1-fold series of processes, which again satisfy the equations: Scale up $\mathfrak{E}$, $\mathfrak{H}$ and f in the ratio of $1{:}m^2$, where m^2 is a constant quantity in time and space, but can be arbitrary (we will never choose m^2 very large in the following, so that we do not have to prove that all estimates of Planck's theory are also valid for the new series of operations). [(Ehrenfest, 1905); *our rendition and emphasis*]

This is the the moment that brings to fore the *concept of resonator*, introduced by Planck in order to account explicitly for the interaction of matter with the cavity radiation according to the Kirchhoff's laws governing the equilibrium thermodynamics of this physical system. Thereby, the very concept of oscillator came under a closer scrutiny, first and foremost by completing the dynamics of the harmonic oscillator with a term representing a kind of transition. The results of this can be logically framed within the following philosophy, which brings us back to the problem of Morton Lutzky.

The equation (4.3.10) has some sound natural philosophy ingrained in it, in the spirit of the Berry-Klein gauging procedure: the far universe acts locally to arrange the inertial mass, while the close universe acts locally to arrange the elastic strength necessary to the dynamics embodied in the description of the harmonic oscillator. If the oscillator does not work forever – practically, it is always damped – then the equation of motion changes. It becomes, in the simplest of the cases (the so-called *linear damping*):

$$m\ddot{q} + 2\beta\dot{q} + kq = 0 \tag{4.3.14}$$

where β is the *damping coefficient*, describing here a 'fading' of the motion due to a resistance proportional to the instantaneous speed; the factor *2* was chosen just for convenience. This equation too, admits a constant of motion (Denman, 1968), but this time we can hardly identify it with the energy, for it does not reduce itself to that quantity in obvious instances like the previous Hamiltonian. As the equation (4.3.14) stands, the result of integration is:

$$m^2\left((p/m+\lambda q)^2+\omega^2q^2\right)\cdot\exp\left(-2(\lambda/\omega)\tan^{-1}\frac{p/m+\lambda q}{\omega q}\right)=const \tag{4.3.15}$$

where the following notations are used, over the ones introduced previously:

$$\lambda \triangleq \frac{\beta}{m}, \quad \omega^2 \triangleq \omega_0^2 - \lambda^2$$

Now, there is a problem: while the equation (4.3.10) *can be obtained* from the principle of action connected to a physical Lagrangian

$$\Lambda(\dot{q},q) = \frac{1}{2}\left(m\dot{q}^2 - kq^2\right), \quad i.e. \quad A(t_1,t_2) = \int_{t_1}^{t_2} \Lambda(\dot{q},q)\,dt \tag{4.3.16}$$

from which then we can get the Hamiltonian as energy *via* a usual recipe, one cannot say the same for the expression (4.3.15). Indeed, that expression can hardly count as energy. However, Denman's calculations can be associated with more general group methods, for which the Lagrangian is only a particular approach (Sarlet, 1978). Here we choose to follow these group methods, however only selectively, just following the pure connection with the physical reason already noticed before, that applies here too, with outstanding results. Namely, as Harry Bateman once noticed (Bateman, 1931), the equation of motion (4.3.14) can be obtained from a variational principle, exactly like the one formulated based on equation (4.3.16), only referring to the Lagrangian

$$\Lambda(\dot{q},q;t) = \frac{1}{2}e^{2\beta t}\left(m\dot{q}^2 - kq^2\right) \tag{4.3.17}$$

This Lagrangian differs from the one leading to equation (4.3.10) just by an exponential factor. However, the corresponding action, still can be explained physically by making use of the statistical properties. Indeed, the physical action (4.3.16) can be seen as the time average of the difference between the kinetic energy and the potential energy. Now, if we construct the action integral with this Lagrangian from equation (4.3.17), it sounds:

$$A(t_1,t_2) = \int_{t_1}^{t_2} e^{2\beta t}\Lambda(\dot{q},q)\,dt \tag{4.3.18}$$

This physical action can still be physically interpreted as the time average of the difference between the very same kinetic and potential energies of a free particle gauged by Berry-Klein procedure, over a time sequence beginning with t_1 and ending with t_2. The only real difference of the action (4.3.18) – which leads to equation of motion (4.3.14) – from the classical case (4.3.16) – which leads to equation (4.3.10) – stays in the fact that the *distribution of the time moments is not uniform* anymore, but is governed by an elementary measure of the time range having the form

$$e^{2\beta t}dt \tag{4.3.19}$$

i.e. it is a distribution of an *exponential type* – the type almost exclusively used in physics for statistical purposes (Lavenda, 1992). And thus, even though the relationship between energy and motion may be lost, as it were, the statistical element still stands and, moreover, this statistics turns out to be of a *Planck nature*, that led to the idea of quantum, even though applied now to time (Mazilu, 2010).

And now, for the true point of these conclusions, which turns out to be, in fact, a *kinematical* reading of the Ermakov-Pinney equation, of the kind used in Berry-Klein gauging procedure. This procedure reveals the fact that the harmonic oscillator can be taken as a free particle the radial motion of which defines the Berry-Klein gauging length [see equation (4.3.1) and (4.3.2)]. The discussion to follow spins around this idea, in order to show a connection between kinematics and dynamics. Indeed, as we have seen above, there equal room, so to speak, for kinematics as well as for dynamics in this gauging theory and, therefore, there does not seem to exist a place where their domains are delimited with respect to each other. And yet, if one adopts the physical interpretation of the Lagrangian by a time statistics, as discussed above, there seem to be a place neatly described by the Berry-

Klein gauging theory, where the dynamics goes into kinematics and vice versa, allowing us to distinguish that time we call classical, which is the time of dynamics.

Fact is that the physical action (4.2.16) for the undamped harmonic oscillator is not invariant with respect to groupal action (4.2.20). As we have seen above, even the genuine $\mathfrak{sl}(2,R)$ action, only results in a *relative* invariance, whereby the Lagrangian needs a gauging in order to be properly used. What would be, in these conditions, the place of harmonic oscillator, so clearly delineated in a Hamiltonian dynamics by the Berry-Klein theory? The answer can be found, in our opinion, in incidental result given by H. J. Wagner in the form of following theorem (Wagner, 1991): there is a Lagrangian of the form

$$L(q,\dot{q},t) = \frac{1}{2}\frac{1}{p^2}\left((p\dot{q}-q\dot{p})^2 - \kappa_1^2\frac{q^2}{p^2}\right) \tag{4.3.20}$$

producing the Euler-Lagrange equation

$$\ddot{q} - \frac{q}{p}\left(\ddot{p} - \frac{\kappa_1^2}{p^3}\right) = 0 \tag{4.3.21}$$

Thus, the theorem can be reformulated in a simpler and more suggestive way: q is a harmonic oscillator if, and only if, p is an Ermakov-Pinney particle. If p is a Berry-Klein gauge length proper, then q is a *classical* free particle. Of course, another Lagrangian can be constructed right away, with the roles of q and p switched, but the conclusion is the same. In words, it sounds: kinematically speaking, the reading of Ermakov-Pinney equation is not that suggested in (4.3.2), whereby the inverse cubic term appears as a perturbation of the harmonic oscillator from the left hand side, but rather that suggested by (4.2.8), whereby the right hand side provides a linear fluctuation of an acceleration field, with respect to *a background provided by some inverse cubic accelerations*.

In order to assess the situation, it is best to provide a demonstration for the Wagner's theorem. An observation based on the theorem of Morton Lutzky from equation (4.3.9), provides the key to proof: applying his theorem above for the particular case $R_2 = 0$, we get the Lewis invariant:

$$I(p,\dot{p};q,\dot{q}) \equiv \left(\frac{q}{p}\right)^2 + p^4\left(\frac{d}{dt}(\frac{p}{q})\right)^2 = const \tag{4.3.22}$$

where we chose the constant R_1 as unity, which is, in fact, Wagner's choice. Now, switching to a new time, τ say, defined by $d\tau = dt/p^2$, the Lewis invariant becomes: $(q/p)^2+[(q/p)']^2$ where the prime means derivative on τ. This is the energy of a harmonic oscillator working *in the new time* according to an equation of motion produced by the Lagrangian $[(q/p)']^2 - (q/p)^2$, as well known. Switching back to the time t in this new Lagrangian, gives the Wagner's Lagrangian (4.3.20) that produces the equation of motion (4.3.21). The action corresponding to this equation of motion is, physically speaking, the time average of the difference between the area rate in the phase plane of the coordinates (p,q) and their ratio adjusted by an appropriate constant for dimensional reasons. The average is to be calculated with respect to a time measure given by $d\tau = dt/p^2$.

However, the true significance of the Wagner's theorem concerns a construction enticed by the idea of application function introduced by Louis de Broglie (see §2.4 above): it is destined to describe the concept of charge in connection with that of field in a geometry of charge built entirely based on the natural philosophy of Katz as described by us previously, in the §3.1. Fact is that, because it stands upon the Ermakov-Pinney equation, the Berry-Klein gauging procedure does not necessarily need a gauge length *per se*: it is quite sufficient a charge, for instance, in order to accomplish it! Then by Wagner's theorem, the electric charge describes, according to

Katz's natural philosophy, a Berry-Klein gauged free particle if, and only if, it is has a magnetic charge, and *vice versa*: it can describe a magnetic free particle according to Katz's natural philosophy, only if it has an electric charge. Let us, therefore, get into some details of the theory of charges.

4.4 Ermakov-Pinney Equation: the Time of Charges

The Berry-Klein gauging procedure opens a new view on the world, due to the opportunity of connection of different possibilities of defining the freedom of the particles making the elements of the interpretative ensembles. In the previous section, for instance, we have shown what the Berry-Klein freedom means, as a result of the Wagner's theorem. This needs *two* Berry-Klein gauging lengths, and involves some important properties of the Ermakov-Pinney differential equation, whose importance becomes thereby overwhelming in matters physical. But this importance is by no means limited to describing a generalization of Planck's quantization procedure, the way we just presented it above starting from the idea of adiabatic invariants. It goes deeper, to more fundamental things, if we may be allowed such an expression: the Ermakov-Pinney equation describes the 'phase plane of charge', as it were, which is an essential issue in the Katz's natural philosophy (see §3.1). This natural philosophy comes with a mathematical problem whose solution leads directly to the Ermakov-Pinney equation [for the basics see (Katz, 1965), §V, and refer especially to Fig. 1 of that section].

The fact that a Hertz material particle possesses two charges in an undecidable manner, over any class of imagined experiments in which it may be involved, can be mathematically sustained by the idea of a charge phase, that can be introduced naturally by an equation like (3.1.13). There are two levels of undecidability, inasmuch as e can be an electric charge, just as well as a magnetic charge. So, if we are to be thorough with our theory, according to this natural philosophy there are two split angles: one for the case of the electric charge, say θ_E, and one for the magnetic charge, say θ_M. These will provide the following charges among the Hertz particles of an interpretative ensemble:

$$q_{EE} = e_E \cos\theta_E, \quad q_{EM} = e_E \sin\theta_E; \quad q_{ME} = e_M \cos\theta_M, \quad q_{MM} = e_M \sin\theta_M \tag{4.4.1}$$

where e_E is the charge in the electric case, while e_M is the charge in the case magnetic. Now it is a matter of a simple observation that q_{EE} and q_{EM}, for instance, may be taken as the two fundamental solutions of the second-order differential equation

$$q'' + q = 0 \tag{4.4.2}$$

where the accent means derivation with respect to θ_E. Obviously, the same goes for the magnetic case. Consequently, e_E as defined by the sum of squares as in equation (3.1.13) is, accordingly, a solution of the Ermakov-Pinney equation

$$e'' + e = \frac{1}{e^3} \tag{4.4.3}$$

The very same analytical characterization is valid for the magnetic side of the charge, but with respect to the independent variable θ_M: q_{ME} and q_{MM} are two fundamental solutions of the second-order differential equation (4.4.2), while e_M is a solution of the Ermakov-Pinney equation (4.4.3). Therefore we have a situation as described above, whereby the ratio of e_E and e_M is a solution of an Ermakov-Pinney equation, because each one of them is a solution of an Ermakov-Pinney equation [see equations (4.3.5) and (4.3.6) above]. Obviously, by applying the

Wagner's theorem to the pair (q, p), where q satisfies the equation (4.4.2) and $p \equiv e$, where e satisfies the equation (4.4.3), we get the result formulated at the end of the precedent paragraph: a free Berry-Klein particle is conditioned by the dynamics of its magnetic charge while a free magnetic particle is conditioned by the dynamics of its electric charge. But there is more to it, along the line of discerning between the two cases.

Even though perhaps mathematically irrelevant, we need to ask what differentiates between the electric and magnetic charges. The fact that the charges satisfy equations (4.4.2) and (4.4.3) is just generic, so to speak: the difference between the two cases can be, mathematically speaking, a matter of convention at most. Physically this is not sufficient, and here is the point where an idea of a *time sequence* can rightfully enter the stage. The equation (4.4.2), for instance, has just incidentally the aspect inviting the idea of a dynamics, due to the the particular geometry of the charge, enticed, as it were, by the Katz's natural philosophy. Fact is that the equations (4.4.2) and (4.4.3) are referring to a *class of possible experiments*, whereby the charge of particle can have a value or another, as described by the phase θ and the amplitude e. This is a purely mathematical fact: the physics can enter here only with further considerations, and these are possible along two lines of thought.

The first line of such considerations is statistical, and comes with the Feynman's idea of interpretation (see Chapter 3 of the present work). Paraphrasing the words of great physicist…

> … a particle is either here or there, but wherever it is, it is the very same particle, as identified by the gravitational mass, the electric charge and the magnetic charge, in the fictitious Newtonian world…

Thus, the construction of an ensemble of particles serving for interpretation can be viewed, from physical point of view, as a set of measurements of identical particles located *instantaneously* in a coordinate space. Once again, the 'coordinate space' has here the connotation assigned by Darwin which, physically, would mean an enclosure of a given, but unspecified space extension. In order to go over from this coordinate space to an *ordinary space* proper, one has to prove that the ensemble thus constructed is independent of the space extension. It is not too hard to see that such a construction cannot be a physical structure: the word *instantaneous* is obviously restrictive when it comes to a physical characterization. However, we have shown before that the mathematics allows us to think of the space of location as of a metric space – specifically a $\mathfrak{sl}(2,\mathrm{R})$ Lie algebra – entirely based on statistical precepts, in spite of the fact that this statistics does not seem to have any bearing on a ordinary space for the moment being (see Chapter 3, *passim*). Leaving this connection open for a later mathematical characterization, we just have to notice for now the existence of a significant moment of physics that, in our opinion, properly perceived, is destined to elucidate the necessity and the overwhelming importance of the idea of scale transition. This moment involves the transition from the classical term 'instantaneous', to the relativistic term 'simultaneous', known to be related closely to the electromagnetic phenomena.

Fact is that the word 'instantaneous' involves a *different* scale of time, as the physics of last century leaves us to realize, but only with the due assistance of the concept of motion: it is that time scale of simultaneous events that, when connected through a characteristic signal according to the precepts of relativity, *appear as instantaneous*. This means that the signal of *synchronization* giving us the idea of simultaneous, is propagating with a speed beyond any physical possibilities at the finite time scale of the world we live in: in short, *instantaneously*. In a world of charge, like that of Feynman above, we can guess that such a signal is only electromagnetic. Consequently, for once, we are to see in this the objective reason of occurrence of the special relativity in connection with electrodynamics (Einstein, 1905a), and thereby learn the appropriate lesson from this occurrence. And that appropriate lesson should be starting from the fact that Einstein introduced the concept of motion where, classically speaking, there is no room for motion, just in order to maintain the physics in place. Apparently, that physics would not have any *raison d'être* without the motion. This fact obstructed, for the physics that followed, the regular appearance of the concept of scale transition, which, according to the observations right above, would need something like a *static ensemble*, connected to an interpretation process. So much the more, nevertheless, it should be gnoseologically significant that the idea of scale transition emerged as being fundamentally connected with the relativistic ideas (Nottale, 1992, 2011). A first step in applying this lesson should be the definition of simultaneity used by us in §4.2, for instance, in order to define the host space of Hertz's material particles [see equation (4.2.26) ff]. That step would imply, first and foremost, a rational connection between the phase of charge and the time of events, so the question pops up: *is that feasible*?

In a word, the answer is *yes*, provided a *dynamical analogy* is enforced, in order to introduce the time. More to the point, assume that the charge is applied to a field – for instance, in the manner envisioned by Louis de Broglie, and presented by us in §2.4 – to the effect that it can be perceived as a signal, continuous in a convenient time, but still involving the phase of charge:

$$q(t) = A(t)e^{i\theta(t)} \tag{4.4.4}$$

If the amplitude of such a signal would be a constant, then no doubt, this signal would be a solution of the equation (4.4.2), no matter of t; but if only a time dependent amplitude is necessary, the things get a little more complicated. Assuming that $q(t)$ gets physical significance as a periodic signal of the harmonic oscillator type in time, we will have the following identifications:

$$\ddot{q}(t) + \omega_0^2 q(t) = 0 \quad \leftrightarrow \quad \begin{aligned} &\frac{\ddot{A}}{A} + \omega_0^2 = \dot{\theta}^2 \\ &\frac{\dot{A}}{A} + \frac{\ddot{\theta}}{2\dot{\theta}} = 0 \end{aligned} \tag{4.4.5}$$

Thus we have right away a connection between the amplitude and the phase of such a representation:

$$A(t)\sqrt{\dot{\theta}(t)} = const \tag{4.4.6}$$

and thereby an equation for the amplitude alone of the representation:

$$\ddot{A} + \omega_0^2 A = \frac{R_0^2}{A^3} \tag{4.4.7}$$

where R_0 is the square of the constant from equation (4.4.6). Thus, the amplitude of this representation has to be a solution of the Ermakov-Pinney equation with respect to time.

Before going any further, let us treat the case of a damped harmonic oscillator, described by equation (4.3.14), just for completeness, if nothing else. In this case, for the representation (4.4.4), we shall have instead of (4.4.5) the conditions

$$\ddot{q}(t) + 2\lambda\dot{q}(t) + \omega_0^2 q(t) = 0 \quad \leftrightarrow \quad \begin{aligned} &\frac{\ddot{A}}{A} + 2\lambda\frac{\dot{A}}{A} + \omega_0^2 = \dot{\theta}^2 \\ &\frac{\dot{A}}{A} + \frac{\ddot{\theta}}{2\dot{\theta}} + \lambda = 0 \end{aligned} \tag{4.4.8}$$

leading to the following equation for the phase as a function of time:

$$\dot{\theta}^2 = \omega_0^2 - \lambda^2 - \frac{1}{2}\{\theta, t\} \tag{4.4.9}$$

Here $\{\cdot,\cdot\}$ means *Schwarzian derivative* of the first symbol in curly brackets with respect to the second. However, the notation is not unique in the mathematical literature, even though the definition of the symbol is the same:

$$\{\theta, t\} \equiv S(\theta)(t) \triangleq \frac{d}{dt}\left(\frac{\ddot{\theta}}{\dot{\theta}}\right) - \frac{1}{2}\left(\frac{\ddot{\theta}}{\dot{\theta}}\right)^2 \tag{4.4.10}$$

The notation S(·), for instance, is sometimes used for geometrical purposes, for instance in the case of Lorentz surfaces of constant curvature (Duval & Ovsienko, 2000): the Schwarzian derivative is a second order differential which has the meaning of a curvature (Flanders, 1970). These things aside, coming back to our line here, the connection between amplitude and phase for the case of a damped harmonic oscillator, is not as simple as before, for it involves the time explicitly: instead of (4.4.6), we have

$$A(t)e^{\lambda t}\sqrt{\dot{\theta}(t)} = const \tag{4.4.11}$$

However, this fact does not change the previous conclusions on $A(t)$, but just rephrase them in a way, because the equation (4.4.9) is actually an Ermakov-Pinney equation 'in disguise', as it were. Indeed, we can write

$$-\frac{1}{2}\{\theta, t\} = \frac{\ddot{\xi}}{\xi}, \quad \xi \equiv \left(\sqrt{\dot{\theta}(t)}\right)^{-1} \tag{4.4.12}$$

so that (4.4.9) becomes an Ermakov-Pinney equation for ξ:

$$\ddot{\xi} + (\omega_0^2 - \lambda^2)\xi = \frac{1}{\xi^3} \tag{4.4.13}$$

Using then equation (4.4.11) we can find the amplitude of this representation in the form

$$A(t) = const \cdot \xi(t) \cdot e^{-\lambda t} \tag{4.4.14}$$

where $\xi(t)$ is a solution of (4.4.13).

Let us discuss, up to a point, the physics involved here: essential in the previous mathematics is the *first time derivative of a phase*, which can be taken as an *instantaneous frequ*ency, like in optics, for instance (Mandel, 1974). The basic equation on which we choose to discuss here in the statistical spirit of time-frequency analysis (Cohen, 1995), is the equation (4.4.9): it is the only equation that can offer the instantaneous frequency in physical terms of a damped harmonic oscillator, and we take it as meaning a special approach to statistics of a time signal. Notice that the instantaneous frequency is a well-defined mechanical frequency

$$\dot{\theta}^2 \equiv \omega^2 = \omega_0^2 - \lambda^2 \tag{4.4.15}$$

only in the special cases where the phase is a linear rational function of time:

$$\{\theta, t\} = 0 \quad \therefore \quad \theta(t) = \frac{\alpha t + \beta}{\gamma t + \delta} \tag{4.4.16}$$

Here we have used the result (4.4.12) in order to integrate the Schwarzian differential equation. Therefore if the phase is linear rational in time one can say that the recorded time signal can be represented by a harmonic oscillator. In view of (4.4.15) the first equation (4.4.8) for the amplitude of representative oscillator becomes

$$\ddot{A}(t) + 2\lambda\dot{A}(t) + \lambda^2 A(t) = 0 \quad \therefore \quad A(t) = (at + b)e^{-\lambda t} \tag{4.4.17}$$

with a and b two constants. This seems to be just about the only case admitting the definition of the parameters from a physical point of view, *i.e.* as the parameters of an oscillator. Indeed, using the second equation for amplitude in (4.4.8), where according to (4.4.16) we have to put

$$\frac{\ddot{\theta}(t)}{2\dot{\theta}(t)} = -\frac{\gamma}{\gamma t+\delta} \quad \therefore \quad A(t) = (\gamma t+\delta)e^{-\lambda t}$$

so that, once the equation for phase is (4.4.16), the constants a and b cannot be quite arbitrary. In view of this, the whole situation can be interpreted in the spirit of the classical time measurements, whereby the frequency is defined as the reciprocal of a time period. The signal (4.4.4) should be of the form

$$q(t) \propto (\gamma t+\delta)\exp(-\lambda t + i\frac{\alpha t+\beta}{\gamma t+\delta}) \tag{4.4.18}$$

Thus, one can say that the transformation (4.2.18) encompasses the idea of phase as physically defined, and the idea of a gauge quantity as well, with an important observation: *that quantity does not necessarily have to be a length*. Every physical process satisfying a second order differential equation can be taken as providing such a gauging quantity. The classical case is, again, that of light: the addition of difraction to the phenomenology of light revealed the importance of periodic functions of phase (Fresnel, 1827), and these functions satisfy – by default, as it were – a second order differential equation. One can say that the classical dynamics entered here only incidentally, through such an equation, in order to explain the light as a motion phenomenon. In general, however, such an explanation has to rely upon equation (4.4.13).

The necessity of interpretation, reveals another side of this issue, though. In this framework, that is the Berry-Klein gauging theory, the harmonic oscillator can be taken as a free particle. The equation (4.4.13) then allows us to associate an instantaneous frequency to that particle, based on physical reasons, provided

$$\lambda^2 + \frac{1}{2}\{\theta,t\} = 0 \tag{4.4.19}$$

In terms of harmonic oscillator, this represents a state of *no transfer* between the far and close environments of the Berry-Klein free particle. In view of (4.4.12) this is a second order differential equation

$$\ddot{\xi}(t) - \lambda^2\xi(t) = 0 \tag{4.4.20}$$

having solutions of the form

$$\xi(t) = A\cosh(\lambda t+\phi) \quad \therefore \quad \dot{\theta}(t) \equiv \omega_0 = \frac{1}{A^2\cosh^2(\lambda t+\phi)} \tag{4.4.21}$$

where A and ϕ are integration constants. This leads to a phase of the form

$$\theta(t) = \theta_0 + \frac{1}{A^2\lambda}\tanh(\lambda t+\phi) \tag{4.4.22}$$

The equation (4.4.20) has relevance in physics as the equation of an *inverted harmonic oscillator* (Barton, 1986), a physical structure relevant in closed spaces occupied with matter, for instance in tunneling problems (Baskoutas & Jannussis, 1992). This whole theory only means a definition of the frequency through a time sequence, as one does usually in the practice of measurements of this quantity. Only, in this special case, the practice is simply relying upon equation (4.4.21), so that equation (4.4.22) gives a succession of phases corresponding to the time sequence in question.

However, the equation (4.4.20), and therefore its solution (4.4.22), are just a particular case of a more general situation, to be described as follows: if (4.4.9) should be regarded as physical, then we need to have

$$\lambda^2 + \frac{1}{2}\{\theta,t\} = \mu^2 \quad \therefore \quad \ddot{\xi} + (\mu^2-\lambda^2)\xi = 0$$

where μ is a new damping coefficient entering the 'updated' definition of frequency of the damped harmonic oscillator serving for the physical definition of parameters. Therefore the frequency can be defined by a time sequence, according to equation

$$\xi(t) = A\cos(\omega t + \phi) \quad \therefore \quad \dot{\theta}(t) \equiv \sqrt{\omega_0^2 - \mu^2} = \frac{1}{A^2 \cos^2(\omega t + \phi)} \tag{4.4.23}$$

if $\mu^2 > \lambda^2$, with the phase connected to the time sequence by equation

$$\theta(t) = \theta_0 + \frac{1}{A^2\omega}\tan(\omega t + \phi); \quad \omega^2 \equiv \mu^2 - \lambda^2 \tag{4.4.24}$$

The case given by equation (4.4.22) can occur if $\lambda^2 > \mu^2$.

For future purposes, we need to present the previous theory regarding the connection between the linear second order differential equation and the Ermakov-Pinney equation from still another point of view. Namely, there is an interplay between three differential equations here. Take the second-order differential equation

$$\ddot{u} + a(t)\dot{u} + b(t)u = 0 \tag{4.4.25}$$

and choose a basis of two independent solutions $u(t)$ and $v(t)$. With these, construct a quadratic form with constant coefficients, which defines two new variables as follows:

$$q \triangleq \alpha u^2 + 2\beta uv + \gamma v^2 \triangleq r^2 \tag{4.4.26}$$

Then $q(t)$ is a solution of the third order differential equation [see (Bellman, 1997), p. 179, Exercise 3]:

$$\dddot{q} + 6a\ddot{q} + 2(\dot{a} + 4a^2 + 2b)\dot{q} + 2(\dot{b} + 4ab)q = 0 \tag{4.4.27}$$

while $r(t)$ is a solution of the second order differential equation [see (Eliezer & Gray, 1976), §7]:

$$\ddot{r} + a(t)\dot{r} + b(t)r = (\alpha\gamma - \beta^2)\frac{(v\dot{u} - u\dot{v})^2}{r^3} \tag{4.4.28}$$

Between these last two equations, there is therefore a connection based on the association of q with r that results from equation (4.4.26). Keeping these results in reserve for now, let us go over to an entirely different side of electrodynamics, the one involving the classical Newtonian forces.

Chapter 5 A Newtonian World of Charge: the Central Forces in Coordinate Spaces

The Reynard's observation reproduced by us in the introduction to previous chapter, namely that…

> The actions of currents *cannot be assimilated to the ones that take place between the molecules of matter*, for they are not innate to the matter of the fluid currents, or to that of the conductors followed by these currents.

puts the Lorentz's theory of electric matter [(Lorentz, 1892), see §3.2 above] on a hot spot, as it were. Indeed, Lorentz contended that, in principle, whatever cannot be conceived for a *classical fluid*, in general, cannot be applied to electric matter either. And this in spite of the fact that a classical fluid is, as a general rule, conceived as a continuum, while Lorentz considered the material charged particles as moving inside matter. The physical property of having charge involves Newtonian forces – more to the point Coulombian forces – so one cannot speak (anyway, at least not always!) of a classical fluid in the case of a gas of electrically charged particles. Such an occasion asks for a special analysis, in order to discern the conditions that need to be satisfied by the Newtonian forces in order to be capable of suporting an interpretation. Until the time comes for accomplishing such an analysis within the present work, we have to notice that the observation of Reynard was enticed by the very concept of force, as this was adapted in order to serve the purposes of upcoming electrodynamics. The analysis of this historical moment with conclusions on the possible generalization of the Newtonian forces, constitutes the substance of the present chapter of our work.

5.1 Ampère's Forces: the Necessity of Scale Transition

With André-Marie Ampère, the concept of Newtonian forces – and with it, naturally, the concept of action at a distance – came to a crossroad. In order to offer a better understanding of the issue at hand, we find necessary to present this moment of knowledge not so much by quotations from Ampère himself, according to our custom so far, but mainly by a discussion of the concept of *central forces*. The properties of such forces are key points in the arguments that, during the 19th century, led to the ideas that lie today at the foundations of the field theory. And, in our opinion, we should have to agree with Poincaré who once expressed how much the central forces meant for the development of physics:

> The hypothesis of central forces *contained all the principles*; it entailed them as necessary consequences; it entailed both the *conservation of energy* and *that of masses*, and the *equality of action and reaction*, and the *law of least action*, which appeared, it is true, not *as experimental truths, but as theorems*; and whose enunciation had, at the same time, that I don't know what of a more precise and less general than their present form. [(Poincaré, 1905), p. 196; *our rendition and emphasis*]

The moment Ampère of human knowledge is the moment where the very central forces came under scrutiny, and we chose to detail it for the good reason that some essential ones among 'all the principles', took then a turn which, eliminating "that I don't know what" of the central forces, as Poincaré would say, brought them in "their present form", which is not the most appropriate in building, say, the physics of brain, for instance. In short, this kind of physics needs now, as it always did, quite an alternative turn, to be found only in that very moment of our knowledge, because it was subsequently obliterated by the development of main-stream physics. We further choose to be assisted in such an atypical endeavor by an exquisite work of André Assis, which uses a suggestive notation for the action at distance, that we would like to standardize, as it were, for reasons that will become clear as we go along our work (Assis, 1994). One can also refer to a work of Olivier Darrigol for the same kind of presentation of the very same idea, but in a broader physical context (Darrigol, 2002). Another exquisite, but more recent production we recommend to the readers, mostly to those who follow the historical, critical and physical details of the the early electrodynamics' ideas, is a work referring exclusively to Ampère's electrodynamical contributions, that also contains English translations of a few ones from among the essential productions of the great scholar (Assis & Chaib, 2015).

Now, the first in the order of things to be done here, is to clarify what are those key points which enticed the discussions that helped create the electrodynamics along the lines initiated by Ampère, *i.e*. with no consideration of the Faraday's induction phenomenon whatsoever. To start with, the Newtonian forces of physics were then –

and we have to recognize that they still are now, to a great extent – central forces. No wonder then, that Ampère would try to find a way to use these forces, along the way initiated by his illustrious predecessor, this time, though, in electrodynamics, where the essential premise of their existence is quite problematic, to say the least. That way can be perceived from the fact that Ampère was fully aware that there is a clear difference between the genuine Newtonian forces describing the action of gravitation or that of static electricity, and the forces acting between two *currents* (Ampère, 1823). In order to see that difference, let us present the issue from a definite perspective on the central forces.

When talking of forces connected to the action at a distance, the physics of the 19th century following the Poisson's work (Poisson, 1812), took the habit of understanding the force as a vector having the following three characteristics:

(1) *Centrality*, which is the feature that gives the name of Newtonian forces, *i.e.* the property of forces to act only along the line joining the two particles involved as partners in the action at distance.

(2) *Magnitude depending only on the distance* between the two particles involved in the action at distance.

(3) *Conservativity*, *i.e.* the property of the vector force to be derived from a potential function by the differential operation of gradient.

These features of forces have in the background either the idea of classical material point, or that, more appropriate when it comes to the concept of interpretation *per se*, of Hertz's material particle (see the introduction to Chapter 4 above). Indeed the central forces can be *ideally* assumed to exist when the particles involved in the action at a distance *have no space extension* whatsoever, *i.e.* they are positions endowed with physical features generating that action: after all, this kind of manifestation was the one allowing their very discovery. Indeed, *practically*, the fundamental condition of classical material point of having no spatial extension, can never be satisfied, and this explains why the action at distance was first described in the form of forces between matter formations capable of individual existence, but so far away from each other, that they can be considered space points in a geometrical sense. Hertz's natural philosophy is closer to reality: for it, these matter formations can only be either material particles or material points, *i.e.* some kinds of 'higher order infinitesimals', as Hertz expressed it. Therefore, we take the above attributes of forces as referring to material particles, with the proviso that the resulting description of matter cannot be more than an *interpretation* of it, in the sense defined for the necessities of wave mechanics (Darwin, 1927). More to the point, such a description *cannot be a physical model*.

Now, the attributes above of the classical concept of force of 19th century, have been so much used, and so indiscriminately, that they became somehow 'embedded' in the background, as it were, and, when used, not even mentioned as conditions, as if they would belong to the force concept by default. The misconception was even aroused in time that they should be somehow equivalent. The source of this misapprehension seems to be the fact, notably stressed by Hermann Helmholtz, that the forces having the first two properties from the list above also possess the last one, which is the most important property of forces in theoretical physics: it is the basis of energetics (Helmholtz, 1869). Fact is that it just happens that these three characteristics were occasionally taken as equivalent among themselves in a very precise manner: two of them imply the third one. This can easily lead one to the idea that, theoretically speaking, one of those properties is redundant. Only a clever observation may be able to point out a crack in their circle, and this observation has been made, indeed, by J. C. Burns in a short but, at least for us, inspirational notice (Burns, 1966). At a closer scrutiny, if the forces are conservative and have a magnitude depending only on the distance between locations *they are not necessarily central*, for instance.

More important, though, from the perspective of the electrodynamics rising in those times of 19th century, is the property that allowed Ampère's equation of forces between *elementary currents*: if the forces are conservative and central, *their magnitude does not necessarily depend only on the distance* between the two places of the action at distance. This can be seen right away, assuming a typical central conservative force with the magnitude depending on coordinates separately, *i.e.* a force that can be written as a vector in an arbitrary Cartesian reference frame, where we write it as: $\boldsymbol{f}(\boldsymbol{r}) \equiv f(x, y, z)\cdot\boldsymbol{r}$. This field has to satisfy the *Helmholtz conditions*: a scalar one amounting to $\nabla\cdot\boldsymbol{f}(\boldsymbol{r}) = 0$ and a vectorial one, in the form $\nabla\times\boldsymbol{f}(\boldsymbol{r}) = \boldsymbol{0}$; these are automatically satisfied by the original Newtonian forces. Then, from the second of those conditions we have, in detail:

$$\nabla f(x,y,z)\times\boldsymbol{r} = \boldsymbol{0} \quad \therefore \quad \nabla f(x,y,z) \propto \boldsymbol{r} \tag{5.1.1}$$

if the force is not to be a constant in the chosen reference frame. The first Helmholtz condition becomes:

$$\boldsymbol{r}\cdot\nabla f(x,y,z) + 3f(x,y,z) = 0 \tag{5.1.2}$$

and shows that the magnitude of force must be a homogeneous function of degree *–3* in the coordinates. It is only in the particular cases where this function is r^{-3}, that we get the Newtonian forces going inversely with the distance squared. Otherwise, the magnitude $f(x,y,z)$ can very well be the inverse of third degree homogeneous polynomial in the three coordinates, or the a *–3/2* power of a homogeneous quadratic form, as in fact happened in the original Newton's case [(Glaisher, 1878), see §4.1 above] or, in fact, any other form leading to a homogeneous function having the degree *–3*. Combining (5.1.1) and (5.1.2) we find that the most general force satisfying both Helmholtz conditions, is a vector of the form

$$\boldsymbol{f}(\boldsymbol{r}) = -(\frac{r^2}{3})h_{-5}(x,y,z)\boldsymbol{r} \tag{5.1.3}$$

where h_{-5} has to be a homogeneous function of degree *–5*, as indicated by its lower index. This expression of the vector force can be rearranged to appear as proportional to a Newtonian force:

$$\boldsymbol{f}(\boldsymbol{r}) = h_0(x,y,z)(\frac{\boldsymbol{r}}{r^3}), \quad h_0(x,y,z) \equiv -(\frac{r^5}{3})h_{-5}(x,y,z) \tag{5.1.4}$$

with the coefficient h_0 – a function *homogeneous of degree zero in coordinates*. In other words, the most general force field satisfying the two Helmholtz conditions – taken as the most general and fundamental properties of Newtonian field of forces and extended as fundamental properties of any force – must be proportional to a genuine Newtonian force field, with the proportionality described in a system of Cartesian coordinates, either by a function of the ratios of coordinates, or by a constant, as in the genuine Newton's case. Then, simply put, but in quite general terms from a geometrical point of view, a central force can have a magnitude depending on the coordinates of the positions involved in the action at distance *via* some trigonometrical functions, multiplying the classical Newtonian force. One can therefore say that equation (5.1.4) generalizes the classical Newton case, which is what André-Marie Ampère apparently upheld, both by his stated task and factually, when building his electrodynamic force.

This generalization, however, is not quite complete, for it is referring only to a *space geometry*, with no involvement whatsoever of the physics. In the classical case of genuinely Newtonian forces, one would have the masses or the charges, or even both, located at the two positions involved in the interaction at distance: the forces are *bilinear* in those physical quantities. In keeping with the idea of continuity and space extension of matter, we may think of some mass elements, or charge elements – Hertzian 'higher order infinitesimals' – located at the two positions involved in the action at distance, as entering in bilinear expressions, *i.e.* by their product, and so may appear the equation of force in the case of currents. A problem remains, though, concerning the *measure* of the current elements: while in the case of genuinely Newtonian forces one can think of the differentials of mass or charge, in the case of currents issues of relative directions occur. The common view at the time of Ampère was that a current element should be represented by the time rate of variation of the charge – known as the intensity of the current – multiplied by the element of wire, thought to be a *line* in space: $I\boldsymbol{dl}$.

And thus, the force between two elements of wire, $\boldsymbol{dl}$ and $\boldsymbol{dl'}$ carrying currents of intensities I and I', as provided by André-Marie Ampère in his pioneering work on electrodynamics (Ampère, 1823), can be written, in a form using modern notations, as [see (Assis, 1994), Chapter 4, Equations (4.14 – 15); see also (Darrigol, 2002), Appendix A, equation (A.3)]:

$$d^2\boldsymbol{f}(\boldsymbol{r}) = -I \cdot I' \cdot \left[\boldsymbol{dl} \cdot \boldsymbol{dl'} - \frac{3}{2}(\hat{\boldsymbol{r}} \cdot \boldsymbol{dl})(\hat{\boldsymbol{r}} \cdot \boldsymbol{dl'}) \right] \cdot \frac{\boldsymbol{r}}{r^3} \tag{5.1.5}$$

Here $\hat{\boldsymbol{r}}$ is the unit vector of $\boldsymbol{r}$; this notation represents a change with respect to that used by us before in this work (see §2.3 above). As a little digression: the reason of this change is that the vector variable $\boldsymbol{r}$ is now starting to show up in the transition between the 'ordinary space' – as used previously, in §2.3 – and the 'coordinate space', serving physical purposes, and its orientation is a vector by itself, not a part of a reference frame. Expression (5.1.5) of the force is a *bilinear symmetric form* in the two current elements involved in the action at distance, so that the equation (5.1.5) can, indeed, be understood, according to our discussion above, as a natural generalization of the classical central force, for the specific case of the current elements. It is in this circumstance that the vector $\hat{\boldsymbol{r}}$ becomes a field to be geometrically considered by itself, but this raises the problem of reading the formula of forces thus provided. For once, we shall take the equation (5.1.5) as this has been taken starting with Ampère himself, and is suggested by Darrigol's notation: the key of reading is the general vector relation $\boldsymbol{r}^2 \equiv (\boldsymbol{l} - \boldsymbol{l'})^2$, showing what should be meant by the relative position of the two elements of current. Namely, $\boldsymbol{l}$ and $\boldsymbol{l'}$ are the *position vectors* of the two locations in the chosen reference frame, of the two current elements $I\boldsymbol{dl}$ and $I'\boldsymbol{dl'}$ involved in the action at distance. Then, according to Edmund Whittaker, the expression from equation (5.1.5) can be taken as the right expression incorporating the observation that the action of a closed circuit on an elementary current is perpendicular to this current [(Whittaker, 1910), pp. 89 ff]. Be this reading as it may, a hint of the depth of Ampère's inovation in using the central forces is apparent and, from our point of view, needs to be uttered explicitly right away. It concerns the answer to the natural question: why would anyone need the force between two current *elements*, while the specific experience is always referring to *finite* currents or *finite parts* of currents?

The answer is, in our opinion, obvious: *there is nothing*, *in this specific experience*, *analogous to the Kepler motion*, in order to sustain the idea of a central force for the case of finite currents, as this motion once did for Newton. So, if one wanted to continue Newton's tradition on forces, as Ampère declares in the very beginning of his work, one would need to extract from that experience situations equivalent to that envisaged by Newton himself. This operation cannot be accomplished but only conceptually, for it obviously involves matter formations without space extension – ideally differentials, if it is to give them a measure, that is the 'highest order infinitesimals' of Heinrich Hertz – in which case one can indeed operate with central forces, in the way suggested above. The formula given in equation (5.1.5) is *abstracted* from a whole set of experiment with finite currents, divided into four classes, destined to cover all typical situations of the relative positions of the two *finite* conductors involved in the action at distance [see (Jamin, 1866), 70th Lecture, ff; see also (Maxwell, 1873), Volume II, Part IV, Chapter II, (Whittaker, 1910) and (Assis & Chaib, 2015)]. Then we can conclude that, after such a careful classification and conclusions concerning the finite currents, Ampère made a *transition of scale* from *finite* to *infrafinite* currents – the highest order of infinitesimals in a Hertzian natural philosophy – in order to be able to use the theory of Newtonian forces as it was designed by its illustrious creator. Only, then, Ampère was forced to calculate the forces at a finite scale by a procedure of integration, so as to be further able to use the experience. This explains the differential notation – we first met it to Assis and then to Darrigol (*loc. cit.*), for no one of authors, at least not those in our study, seems to be concerned with its use – from the left hand side of equation (5.1.5): the *whole* force needs a *double integration* along the finite paths followed by each wire in turn. Not only this, but noticing such a detail, some other ones start showing up: an element of *real* current, for instance, may have a direction entirely independent of the path followed by its designated position. For, in reality, the wires are not lines, but bodies, having space extension, and in such a space – a proper coordinate space – the current can go in many different directions, not just one. Therefore, as the great Gauss would say, this is precisely the place to see the remarkable difference between the 'geometria situs' and the 'geometria magnitudinis', and to adapt the mathematics as a consequence of this observation (Gauss, 1833).

Now, every formula of calculation of the magnitude of force between two current elements, ever presented in those times of the birth of modern electrodynamics – no matter if static, like the original one of Ampère, or kinematic, like the one presented later, more specifically, in 1846, by Wilhelm Weber [see, for instance, (Assis, 1994), Chapter 3, equation (3.34)], – respects this trend of generalization of the Newtonian forces. To wit, any such a formula gives a Newtonian force of magnitude inversely proportional to the square of distance, up to a factor, which is a homogeneous function of degree zero, usually trigonometrical, in space coordinates and time. However, such a formula – like any human production for that matter – is by no means free of contradictions. This time the contradiction concerns the very concept of force defined according to Poisson's equation, as Hermann Grassmann has shown for the Ampère prototype. Quoting, therefore from Grassmann:

> Even the intricate form of this formula should arouse a suspicion against it. This suspicion is amplified when trying to use it. For example, if we consider the simplest case, where both current elements are parallel, *i.e.* $\varepsilon = 0$, $\alpha = \beta$, then the Ampère's expression results in:
>
> $$\frac{ab}{r^2}(2-3\cos^2\alpha) \tag{5.1.6}$$
>
> from which it follows that if $cos^2\alpha$ equals 2/3 or, which is the same, if $cos2\alpha$ equals 1/3, *i.e.* if the *center of the attracted element lies on a cone surface whose apex is in the attracting element*, and whose angle cosine is 1/3, *no action takes place*, for, an *inside repulsion takes place with a simultaneous attraction outside of the cone*. This result is actually very unlikely, in order not to raise suspicion on the assumption from which it emerges, no matter how much this assumption may be apparently supported by analogy with *all other forces*. Furthermore, the application of this analogy to our subject *is not quite so well founded*. In the case of all other forces *we have,*

> *originally, point-like elements*, *i.e.* elements with no privilleged directions, acting on each other, and in this case the inevitability of a mutual effect along their connecting line can even be seen *a priori*; but *what can justify us to transfer this analogy to a quite strange subject*, where *the elements are endowed with privilleged directions*? Also the formula itself, *which is in no way similar to the formula for gravitational attraction*, makes it clear enough that the analogy does not work this way [(Grassmann, 1845); *our rendering and emphasis*; see also (Tricker, 1965), p. 203, and (Assis & Chaib, 2015), pp. 219–220].

That 'intricate form' in the beginning of this excerpt, is given by us in equation (5.1.5), showing that Grassmann's a and b are the magnitudes of the current elements. Carrying on the comparison further, shows what the other Grassmann's notations are: ε is the angle between the two elementary currents, while α and β are the angles between those currents and the line joining their middles. Notice the important fact that the angle ε can only be defined by a *parallel transport* of one of the two current elements at the location of the other, along the straight line joining their positions, oriented as a Cartesian vector. This fact is instrumental: one can say that it compels us to an Euclidean description of the matter, therefore to a Cartan-type description of it, whereby the *torsion of space* is an essential geometrical and physical ingredient [see (Mazilu, 2020b), Chapter 4)].

For once, the observation regarding similarity from the last phrase of the excerpt above should be taken *cum grano salis*, as it were, to say the least. The dynamics here is only implicit: the elements involved in interactions are continua that can be interpreted as ensembles of material points in the sense of Hertz, existing *simultaneously* in the form of geometrical objects, and this is the whole point of Grassmann. All of the theories on the interaction of elementary currents, starting with that of Ampère, involve static forces and, as we have shown above, the force laws should involve generalizations of the Newtonian forces, whereby the products of masses or charges are replaced by *multiplying factors*, *functions of the ratios of coordinates*. One can hardly find any other kinds of law of force in the whole history of 19$^{\text{th}}$ century physics, even when the kinematics started being considered (Whittaker, 1910). Thus, we reckon that a multiplying factor can, in fact, count as a particular form of similarity, so there is, after all, *at least one way* in which such a formula "is similar to the formula for gravitational attraction".

The penultimate phrase of the excerpt above, however, contains a reasonable question, which we translate into: what can justify us in transferring the existing analogy – which, by itself, is referring only to the *finite* scale of experience, and is thus controllable by our experience – to an *infrafinite* scale, where such a control is, obviously, lost? This is a *necessary* arbitrariness for the problem at hand: in order to *calculate theoretically* the forces between finite currents while he did not have anything but only a scientific understanding of central forces, Ampère was compelled to consider situations as close as possible to material points involved in the classical interactions. So he *assumed* infinitesimal circuit elements, which can be as close as one can imagine to the condition of material points, but are, nevertheless, innately oriented objects, and this can raise important issues, as in fact happened. An idea of what such an issue can be, and how deep can it run into our knowledge, is given by a notable polemics between Joseph Bertrand and Hermann Helmholtz, regarding the statics of forces involved in the interaction of finite currents [for a good account on this important historical issue – with no solution, because there isn't any! – one can consult (Poincaré, 1901), §262, pp. 275 – 277; the great scholar even hints to the fact that a solution of this problem will need the change of the whole classical system of mechanics (Poincaré, 1897), p. 238].

Going further with the observations, the way we already started, *i.e*. in the inverse order of the issues raised in the above excerpt from Grassmann, we finally find an essential one, whose right solution is, in our opinion, the key to the solutions of them all: the formula of Ampère accepts an *ensemble of positions* in space where the force does not act like "all other forces". In other words, Ampère's formula does not yield the magnitude of force according to its concept derived from the Newtonian idea of centrality, whereby the force can be *either* attractive in the whole space, like gravitation, *or* repulsive in the whole space, like the electric charges of the same sign, but nowhere in that space *concomitantly* attractive *and* repulsive. Indeed, as Grassmann noticed, the formula (5.1.6) has a *null manifold* represented by the ensemble of points of a special conic surface with the apex in the attracting current element: if the attracted current element is parallel with the attracting one and is located anywhere on such a conic surface then the interaction force is zero. According to the classical precepts of statics, this would mean that in specific points along the direction of the two current elements there are two equal central forces acting oppositely, so that their sum is zero. Furthermore, in this case the force is not force in the whole space, according to its field concept: inside the cone it is repulsive, while outside it is attractive according to Ampère's formula. As we already have noticed before, this situation may not be entirely realistic when considering the basic condition of reality of experiments. Indeed, such a consideration shows that the wires supporting currents only have no *length*: being differentials in the definition of current elements, according to Newtonian philosophy [(Newton, 1974), Book II, Section II, Lemma II] they are only *moments of genitae*, *i.e*. the 'affirmative, or nascent moments' corresponding to the quantitatively finite lengths to be subsequently obtained by integration. However, to be as realistic as possible in a like situation, we have to admit that the current elements do have at least some finite lateral extensions, measured by *finite areas* deciding the flux of particles representing the currents *per se*. We shall return later on, to consider this issue a little closer.

However, Grassmann insists that even as linear finite elements, the current elements are not properly used, *i.e*. according to experience. For, this experience proves that the behavior of the closed currents with respect to each other, or with respect to parts of currents as reflected in a formula like (5.1.1) 'should be completely preserved', on which, in fact, Grassmann concedes. However, he notices something more:

> Therefore, without first making any *arbitrary assumption*, I proceed from that of eliminating the arbitrariness of Ampère's hypothesis, assuming, as it must be done, that this hypothesis, inasmuch as it has been tested so far, *i.e*. inasmuch as it refers to the attractions that closed currents exert upon other currents or parts of currents, *should be completely preserved*. At first it is easy to deduce all the phenomena connected with the problem just defined, if one knows the effect which an *angular current*, *i.e*. an infinite current flowing through an angle, has *on a current element whose center lies in the plane of the angle*. For, first, I can regard every closed or non-closed current as composed of such current elements and, secondly, I can regard *every closed current as a polygon through which the current flows*, and this polygon as composed of angular currents which form its angles, whereby I only take from experience the assumption that *equally strong opposing currents flowing through the same conductor cancel each other out* [(Grassmann, 1845); *our rendering and emphasis*; see also (Tricker, 1965), pp. 203 – 204 and (Maxwell, 1873), *loc. cit. ante*]

Thus, since the experience also shows that 'equally strong opposing currents flowing through the same conductor, cancel each other out', this must, again, be taken as a not quite so arbitrary assumption. This observation was apparently decisive for Grassmann, so that he settled for a force formula in the case of *coplanar currents*, which, in the notations from equation (5.1.6) would involve a monomial containing a factor that, in his very notations, would read:

$$\frac{ab}{r^2}\sin(\varepsilon) \tag{5.1.7}$$

Such a factor would indicate a possible vector product involved in the force between two contiguous currents making an angle ε between them. Then a force expression can be constructed between two current elements in a general position in space, with the assistance of *Biot-Savart law*, discovered just about the time of Ampère discovery (Biot & Savart, 1820). The story goes only in modern termes, for Grassmann himself never wrote it but only in the old notations used here in the equations (5.1.6) and (5.1.7). We follow this story in the terms it is described by Edmund Whittaker, for this description fits the Ampère's own idea of the most general Newtonian force, as presented by us right above [see (Whittaker, 1910), p.91]. Moreover, such a presentation demonstrates that from among the three properties of the Newtonian forces listed by us above, their *centrality* – the arbitrary assumption incriminated by Grassmann in the excerpt above – has to be carefully used, if not given up altogether.

Fact is, that in the perception existing at the times of Ampère – *i.e.* that perception according to which wires exist only one-dimensionally, *as paths to be followed by the particles of the electric currents* – his expression of the force, as given here in equation (5.1.5), satisfies the third principle of the Newton from the classical dynamics. This can be seen from the fact that the bilinear form serving as coefficient for the magnitude of Newtonian force in (5.1.5) is symmetric in $\boldsymbol{dl}$ and $\boldsymbol{dl'}$. The Grassmann's observation can be taken as showing that this bilinear form – imposed *mathematically* by the idea of central forces, as the equation (5.1.4) shows – is not, in fact, the most general one that will satisfy the principle of an 'Ampèrian generalization', as it were, of the Newtonian forces. It is easy to see that the most general bilinear form, symmetric in $\boldsymbol{dl}$ and $\boldsymbol{dl'}$, can be written as *a vectorial sum*:

$$A[(\hat{\boldsymbol{r}}\cdot\boldsymbol{dl'})\boldsymbol{dl}+(\hat{\boldsymbol{r}}\cdot\boldsymbol{dl})\boldsymbol{dl'}]+B(\boldsymbol{dl}\cdot\boldsymbol{dl'})\hat{\boldsymbol{r}} \tag{5.1.8}$$

not just as a scalar, where A and B, are two arbitrary homogeneous functions of the components of $\boldsymbol{r}$. This expression avoids the term $(\hat{\boldsymbol{r}}\cdot\boldsymbol{dl'})(\hat{\boldsymbol{r}}\cdot\boldsymbol{dl})$, which might be in position to arouse issues of the nature of those signaled by Grassmann for Ampère's force expression. Notice, however, that such an expression does not mean central forces, but only in the cases where the constant A vanishes. Generally, however, the expression from equation (5.1.8) can be physically validated in modern mathematical terms. Indeed, the force realizing the action of the current element $I\boldsymbol{dl}$ on $I'\boldsymbol{dl'}$ located at $\boldsymbol{r}$ with respect to it, should be given by a force normal to the this second current element:

$$d^2\boldsymbol{F}_{dl\to dl'}=I'\boldsymbol{dl'}\times d\boldsymbol{B}(\boldsymbol{r}) \tag{5.1.9}$$

where $d\boldsymbol{B}(\boldsymbol{r})$ is the intensity of the field created by the current element $I\boldsymbol{dl}$ at the location $\boldsymbol{r}$ of $I'\boldsymbol{dl'}$. According to Biot-Savart law, this field intensity is given by

$$d\boldsymbol{B}(\boldsymbol{r})=\frac{\mu_0}{4\pi r^2}I\boldsymbol{dl}\times\hat{\boldsymbol{r}} \tag{5.1.10}$$

so that the force that the current element $I\boldsymbol{dl}$ develops on the curent element $I'\boldsymbol{dl'}$ will be given by

$$d^2\boldsymbol{F}_{dl\to dl'}=\frac{\mu_0 I\cdot I'}{4\pi r^2}\boldsymbol{dl'}\times(\boldsymbol{dl}\times\hat{\boldsymbol{r}}) \tag{5.1.11}$$

In view of a known vector algebra identity, this force can be written as

$$d^2\boldsymbol{F}_{dl\to dl'} = \frac{\mu_0 I\cdot I'}{4\pi r^2}[(\hat{\boldsymbol{r}}\cdot d\boldsymbol{l}')d\boldsymbol{l} - (d\boldsymbol{l}\cdot d\boldsymbol{l}')\hat{\boldsymbol{r}}] \tag{5.1.12}$$

which is the modern expression of the force provided by Grassmann. The term from equation (5.1.7) can be recognized in a strange position revealed by the equation (5.1.11). In view of the 'circular identity' – or Jacobi identity – of a triple vector product:

$$d\boldsymbol{l}'\times(d\boldsymbol{l}\times\hat{\boldsymbol{r}}) + d\boldsymbol{l}\times(\hat{\boldsymbol{r}}\times d\boldsymbol{l}') + \hat{\boldsymbol{r}}\times(d\boldsymbol{l}'\times d\boldsymbol{l}) = \boldsymbol{0}$$

the Grassmann force satisfies the identity:

$$d^2\boldsymbol{F}_{dl\to dl'} - d^2\boldsymbol{F}_{dl'\to dl} = \frac{\mu_0 I\cdot I'}{4\pi r^2}\hat{\boldsymbol{r}}\times(d\boldsymbol{l}\times d\boldsymbol{l}') \tag{5.1.13}$$

Therefore the Grassmann's force does not satisfy the third principle of dynamics: the difference between the force developed by $I d\boldsymbol{l}$ on $I' d\boldsymbol{l}'$ and the force developed by $I' d\boldsymbol{l}'$ on $I d\boldsymbol{l}$ is proportional to the factor from equation (5.1.7) indeed, but it also depends on the position where we calculate this difference with respect to the plane of the two elements. Inasmuch as, in space, the two current elements are, as a rule, hardly in the same plane, the difference (5.1.13) is typically a nonzero vector. However, if the two current elements are parts of a *Grassmann angular current* as defined in the excerpt right above, only the component of force perpendicular to the plane of the two elements satisfies the third principle of dynamics. The other components of force are quite… strange, in terms of classical mechanics!

The force that, in Grassmannian terms, would satisfy the third principle of dynamics, should be of the form given in the expression from equation (5.1.8). In view of equation (5.1.12) that expression can be realized as the *average*

$$d^2\boldsymbol{F}_{dl\leftrightarrow dl'} \equiv \frac{1}{2}[d^2\boldsymbol{F}_{dl\to dl'} + d^2\boldsymbol{F}_{dl'\to dl}] \tag{5.1.14}$$

for $A = 1/2$ and $B = -1$. However, this expression of the force cannot be *physically* justified, so that the fact remains that the third principle of dynamics needs to be used strictly within the conditions of existence of situations that led to the concept of Newtonian central forces. The problem to be solved then, is how can we generalize such a condition.

Grassmann's critique is referring to the one-dimensional *geometry* of a current element, without recognizing the *physics* involved in this advent of electrodynamics. Had it recognize this physics, then it would start with the observation that there are two material components involved here: the current *per se*, as a *flux of material particles*, and the wire as a *material conductor* of this flux of particles, and these material components are independent of each other. In the context of this instalment of our work, we can say that the first such 'declaration of freedom', which came in the form involving the motion and the geometry, was triggered by Ampère's expression of forces. It belongs to François-Alexandre Reynard, and we reproduced it before, in the introduction to Chapter 4. Its argument stands exclusively on *statics*, and is based on the observation that even though the forces between finite currents can carry a hint of noncentrality, this is strictly due to the relative position of the two current elements as Grassman's formula based on the Biot-Savart physical law of magnetic field clearly shows [see equation (5.1.12) above]. Thus, the force between currents being innately *noncentral*, Ampère was compelled *to choose* an expression coping with the idea of centrality of forces. The reason of such a choice should be clear, according to our discussion thus far in the present work. Namely, we do not know too much about the action at distance, over what was bestowed to us by Newton; and this is referring to the action at distance between *material points*, *i.e.* objects having no space extension. However, on the occasion of discovery of the electric current, we face an action at distance between *finite material objects*. In need of calculation of forces, we have to rely upon what is already known, and thus we *invent* current elements having no relevant – that is, no finite – length. However, they may have finite *lateral extension*, and this speaks in the favor of the kind of independence between geometry and motion, which within the expression of Reynard, gains a new level of discernment for what we need to properly understand.

Putting it into words, if the force to be calculated is central anyway, why not choose a right way to calculate it, while still keeping the physics of motion of charges in its right place?! This fact will be, by itself, able to put the kinds of force of action at distance – attraction or repulsion – in connection with that 'being of the reason' which is the motion. However, while Reynard targets an improvement of Ampère's theory by keeping its original spirit – *i.e.* by limiting the analysis to a formula for static forces – the fact remains that he is referring to a *medium* transmitting the action at distance. In this case, the statics involves *stresses and strains of the medium*, and just about the time of Reynard's work, the scientific community started realizing the strange properties of such a medium, which interest us especially (Beltrami, 1886).

Meanwhile, our purpose here is best served by noticing that, guided by the theory of the Newtonian innate forces of the fundamental components of matter, – the "molecules of matter" – Reynard insists on the fact that Ampère has, in fact, *made a choice*, and this is actually the point at issue: *the interaction of elementary currents cannot be assimilated with a Newtonian force*. However, while being aware that the explanation of the existing facts require the hypothesis of a medium for the description of such an interaction, Reynard did not go outside the area of statics in its classical form, as he was referring exclusively to forces. And so it comes that the report of the Commission of Mathematical Physics of the French Academy (Bertrand, 1869) concentrates on the force formulas he proposed, stating that they can, in fact, be found to Gauss in his *Nachlass* (Gauss, 1833), and even to Ampère, 'for those who know how to see them'. Then, two Academy meetings later, Joseph Bertrand comes again with a short notice, reminding the fact that Joseph Liouville drew attention on the issue raised by Reynard even from the times of Ampère (Liouville, 1831). Exactly to what the Liouville's note is referring, is quite an interesting fact by itself, from natural philosophical point of view, and can be found in the original work of Liouville just cited. For now, though, we are only interested in his conclusion. Quoting:

> Therefore, rigorously, one could assume that two Voltaic elements act upon each other by *an action* composed as follows: 1° *a force* directed along the line passing through their middles and represented by Mr. Ampère's formula; 2° *four other forces* similar to those coming from *two magnetic molecules*, forces whose resultant *is not generally along the line joining* the two bodies [(Liouville, 1831); *our translation and emphasis*].

This excerpt, even if not conceptually complete or, more to the point, even if not exact, gives one a taste of the difference between the *action at distance* and *the force*: the action at distance may be realized not by a single force, as in the case of central forces, but by *a set of forces*. This is the general idea of the new philosophy of forces initiated by Ampère in order to complete the classical Newtonian theory of forces. In a word, the action a distance may be realized even by *noncentral forces* [see (Mazilu, 2020a), §8, especially the equation (46)]. Be it as it may, Reynard was indeed right: from the action at distance Ampère actually *chose* what, according to the classical view of forces, is deemed as experimentally noticeable. His main point, though, should be of essence for the physics to come: the current and the wire *should be considered physically different material things*, independent of one another, and the motion explaining the current *is mainly a figment of imagination*. These details are not recognized altogether even today in physics, in spite of the fact that they are always implicit – sometimes even explicit – in each and every one of its theoretical achievements.

Reynard's work allows us, therefore, to discern the point where physics involved in electrodynamics took its turn towards the modern form it has today. This is the *problem of motion*: theoretically, the motion can be real, *viz*. observable, or just as imaginary as the material point it characterizes. The choice was, then as well as today, that the motion should be *observable*. No possibility of considering the motion as a figment of imagination is ever contemplated in physics, in spite of the fact that most of the essential motions we can describe are only *assumed* realities. This is, in our opinion, a real handicap for physics, in particular for the physics of brain and heart, to take two obvious examples of branches in need of careful elaboration. In order to understand where we are heading with such an observation, notice that, in the case we are presently interested, we need to admit as having to deal with two kinds of motion involved in the definition of a current element, connected to the two material components: a motion of the particles in the current *inside the wire*, which seems to be the only one recognized by Reynard under that qualification of "thing immaterial, a being of reason", and a *motion of the wire itself*, by which we have, in fact, the only possibility to *observe* the electrodynamic phenomena. Therefore, this last kind of motion cannot qualify as a figment of imagination, for it is manifestly observable. Speaking for the physics as we have it today, one can say that it has, at that historical moment of birth of electrodynamics, settled for the observability of motion, and did not even consider the possibility of being introduced by 'reason' as Reynard puts it. The path taken by the mainstream physics is best illustrated by the words of Gauss from a 1846 letter to Wilhelm Weber. Quoting, therefore:

> Perhaps I will be able to delve a little bit more into these things, from which I had strayed away, until you have pleased me with a visit at the end of April or the beginning of May, for you gave me hope. No doubt, I would have made my investigations public long ago, had it not been for the fact that at the time I interrupted my preoccupation on them, I was missing what I considered to be the actual keystone, namely *the derivation of the additional forces* (which are to be added to the mutual effect of *stationary parts of electricity in case they are in relative motion*) due to *the non-instantaneous, but time-propagating* (in a similar way to *light*) effect…
>
> *Nil actum reputans si quid superesset agendum*
>
> I did want to succeed in this at that time; but as much as I remember, I left the study, not without the hope that I might succeed later, although – if I recall it correctly – with the personal conviction that it was first necessary *to have some practical idea on how the propagation occurs* [(Gauss, 1845); *our rendering and Italics*].

The Latin adage – in a suggestive translation: *Nothing has been done, if something remains yet to be done* – explaines that remarkable feature of the Gauss' scientific ethics, from the species of which we can hardly see something today: in any *public* productions one should be as thorough as it gets, and if it is not possible a right conclusion then better not make anything public. For, inherently, such productions are never *absolutely* thorough.

Notice the phrase that can be taken as an important conclusion regarding the object of physics involved in the electrodynamics of Ampère's times: 'stationary parts of electricity'. The whole physics of the parts of electricity was actually described in those times by the principles involved in the *statics of forces* between what is referred to by Gauss as 'stationary parts of electricity'. The 'additional forces' are then referring to such parts but in *motion*, and this is the way taken then by the future physics. This physics was, indeed, built towards the end of the 19th century and the beginning of the 20th century, on the particular foundations of electrodynamics, that suggestively took the form of 'electrodynamics of *moving bodies*', which culminated with the special relativity. Along this line, the contradiction with the third principle of dynamics flared up again (Poincaré, 1900), precisely as a neglect of that "thing immaterial, being of reason" left behind [(Mazilu, Agop, Mercheș, 2020), Chapter 1].

Another point here is the realization – which, according to Gauss' words in the above excerpt, may have been inspired by Wilhelm Weber – that the forces must have those *additions* due to the *relative* motion of the current elements. In other words, the kinematic forces are entirely different from their statical counterparts. The term 'relative motion', was understood by Gauss, and is still understood today – in spite of the discovery of electromagnetic waves – as a clear consequence of the *identification* of current elements with the currents of particles in motion, describing these currents inside the wires. Again, this very identification or, more to the point, the lack of its *proper recognition* by the natural philosophy, in the way it was signaled by François-Alexandre Reynard, marks the whole physics, as we have it today. We *recognize* it: let us see where it leads, and what the consequences may be for what we think is a *proper* physics, to be applied in the worlds like the brain and the heart, for instance.

5.2 Newtonian Forces in Continua: Poisson's Equation

The Siméon Denis Poisson's discovery essentials can be appropriated for physical purposes as they are aptly summarized in the last section of the first chapter of his own work on the attraction of homogeneous ellipsoids, communicated in 1833 to the French Academy [(Poisson, 1833), §1.8]. For obvious reasons, we prefer to display them in modern notations, and these essentials amount to the following: the function

$$V(\boldsymbol{r}_a)=\iiint\frac{dM(\boldsymbol{r})}{[(\boldsymbol{r}-\boldsymbol{r}_a)^2]^{1/2}} \tag{5.2.1}$$

correlating the space range of positions described by vectors $\boldsymbol{r}$, where the 'mass element' dM generating the attraction force is located, with the position $\boldsymbol{r}_a$, where the *attracted* point is located, produces the Newtonian central forces, having the magnitude inversely proportional with the squared distance between the two points. As it is, the formula (5.2.1) gives, according to classical precepts, an *acceleration* due to that force, by the gradient formula

$$\boldsymbol{g}(\boldsymbol{r}_a)=\nabla_a V(\boldsymbol{r}_a) \tag{5.2.2}$$

where the index of gradient operator shows that the operation involves coordinates referring to the position $\boldsymbol{r}_a$. As we shall repurpose this relation for tasks clearly outside of those classical, it will be convenient to take the vector field $\boldsymbol{g}(\boldsymbol{r})$ as a *field intensity*, much in the way it is taken in general relativity [(Einstein, 2004), pp. 59ff]. As it is, the integral in (5.2.1) extends over the three-dimensional range of coordinates (x, y, z) of the locations, components of the position vector $\boldsymbol{r}$ of material points from the matter of the body generating the attractive force correlated with the intensity $\boldsymbol{g}$. On the other hand, the attracted material point has the coordinates (a, b, c), considered, likewise, as the components of $\boldsymbol{r}_a$. The function $V(\boldsymbol{r}_a)$ thus defined satisfies what came to be known ever since as *Poisson's equation*:

$$\varDelta_a V(\boldsymbol{r}_a) = 4\pi\rho(\boldsymbol{r}_a) \tag{5.2.3}$$

provided the location ($\boldsymbol{r}_a$) is in the matter of attractive body, while, if the location ($\boldsymbol{r}_a$) is outside the attractive body, it satisfies the *Laplace's equation*:

$$\varDelta_a V(\boldsymbol{r}_a) = 0 \tag{5.2.4}$$

Here $\varDelta$ is the Laplace operator in space, and ρ is the density of matter. These last two results were communicated by Poisson as such even as early as 1812 to the *Société Philomatique de Paris* (Poisson, 1812). However, only implicit among the outcomes of that old communication, were two equations that became explicit in the 1833 work cited above, and which, later on, even started meaning very much for the physics at large: they became the basis of assigning *action properties to matter*. More to the point, they represent, mathematically, a sweeping idea of the natural philosophy: the *density* of matter is connected to the *divergence* of the field it creates, or vice versa, remaining to decide which one between the two concepts – matter and field – involved in these statements should be a primary concept in the natural philosophy. To wit, we have:

$$\nabla_a \cdot \boldsymbol{g}(\boldsymbol{r}_a) = 4\pi\rho(\boldsymbol{r}_a) \tag{5.2.5}$$

in the matter of the attractive body, and

$$\nabla_a \cdot \boldsymbol{g}(\boldsymbol{r}_a) = 0 \tag{5.2.6}$$

outside it. In order to properly understand these equations, we have to appeal to the second principle of the classical dynamics for the Newtonian gravitational law, and – *avant la lettre*, as it were, in view of what we intend to present here – to what came to be known by some scholars as the *Gauss theorem*. The history and the different mathematical aspects of this theorem are quite entangled, and we hope of not doing too much harm to the historical truth by choosing Gauss' name as connected to it, for the *Princeps Mathematicorum* was one of its main users and promoters [see, for a pertinent historical review of the issue at hand, (Katz, 1979)].

As Gauss insisted *in extenso* later on (Gauss, 1842), our concern here should be the origin of equation (5.2.1), inasmuch as it is the same for three types of Newtonian forces – corresponding to the three physical features of the matter – that can describe a Hertz material particle (see Chapter 3, §3.1 and the introduction to Chapter 4). This origin can be most clearly revealed – with all its ups and downs, in need to be closely considered once – as follows: the attractive Newtonian original force generated by a gravitational source located in a fixed point is

$$\boldsymbol{F}(\boldsymbol{r}) = -\frac{GMm}{r^2}\hat{\boldsymbol{r}} \tag{5.2.7}$$

Here M is the mass of the *attracting* material point, supposed, for now, to be located in the origin of a Cartesian reference frame, m is the mass of the *attracted* material point assumed at location $\boldsymbol{r}$ in that reference frame, and G is the *gravitational constant.* The minus sign in the right hand side of (5.2.7) shows that the force is oriented contrary to the position vector, and consequently it is attractive. Obviously, this force can be written as

$$\boldsymbol{F}(\boldsymbol{r}) \equiv m\boldsymbol{g}(\boldsymbol{r}) = m\nabla V(\boldsymbol{r}), \quad V(\boldsymbol{r}) = \frac{GM}{r} \tag{5.2.8}$$

Therefore the force is a continuous function of the position in space, except the position where it originates, and can very well be the characteristic of a continuum. If we calculate *its flux through a sphere* of radius r, according to the formula

$$\oiint \hat{\boldsymbol{a}} \cdot \boldsymbol{g}(\boldsymbol{r})(dA) \tag{5.2.9}$$

where $\hat{\boldsymbol{a}}$ is the unit normal to sphere, orienting its elementary *area dA*, we get an interesting result. As the unit normal to the sphere is just the unit vector orienting the position vector, and $dA = r^2 sin\theta(d\theta)(d\varphi)$ we have

$$\oiint \hat{\boldsymbol{a}} \cdot \boldsymbol{g}(\boldsymbol{r})(dA) = 4\pi GM \tag{5.2.10}$$

Therefore the flux of the gravitational force is, *up to a universal constant*, the mass of material point creating the force. Now, the mass of the source of field can be written in the form of a triple integral involving the Newtonian density of matter

$$M \triangleq \iiint \rho(\boldsymbol{r}) dVol \tag{5.2.11}$$

where $\rho(\boldsymbol{r})$ denotes the density of matter at the location $\boldsymbol{r}$ and *dVol* is the volume element of space at the very same location. Inserting equation (5.2.11) into (5.2.10) gives

$$\oiint \hat{\boldsymbol{a}} \cdot \boldsymbol{g}(\boldsymbol{r})(dA) = 4\pi G \iiint \rho(\boldsymbol{r}) dVol$$

from which, using equation (5.2.8) and the *Gauss theorem* for the left hand side of this equation, we finally get

$$\iiint [\nabla^2 V(\boldsymbol{r}) - 4\pi G\rho(\boldsymbol{r})] dVol = 0 \tag{5.2.12}$$

This equation can tell us most, in conditions that heuristically matter for the knowledge at large, *i.e*. for a space continuously filled with matter, with the continuity described by the density of this matter. These are usually appropriate conditions of continuity and universality, *i.e*. as long as the physics is involved here, they should be *cosmological conditions* actually. Thus, the integral (5.2.12) is referring to a whole universe where the matter exists continuously. In such conditions if the integrand is zero, we have the *Poisson equation* for the potential as a local condition to be satisfied:

$$\nabla^2 V(\boldsymbol{r}) = 4\pi G\rho(\boldsymbol{r}) \tag{5.2.13}$$

Up to formal adjustments involving some constants and notations, this is the equation (5.2.3), which came to be taken as the fundamental equation of the mechanics in a continuum approach.

The way equation (5.2.13) is obtained, formally legitimates formula (5.2.1) usually taken for granted, at least up to a certain point in the history of scientific understanding following the Poisson's own work. As we have already mentioned above, Gauss has a detailed study of this equation, starting with its rigorous demonstration and use in discussing the forces associated everywhere with the presence of matter [see (Gauss, 1842), especially §§IX, X, and XI]. We shall refer, along our analysis, to many of the aspects touched by the *Princeps Mathematicorum* in the memoir just cited, as well as in some other places of his extended mathematical works. For now, however, we just notice that Gauss himself insisted all along that work from 1842 in appropriating, out of the Newtonian expression of the forces, all it can give for the benefit of a *theory of fields*. This remark prompted some specific observations from our part. Portions of these were already used in developing the mathematical tools we need here [see (Mazilu, 2019, 2020)]. For now we find appropriate enough only a few further such observations, with specific target.

Insofar as the theory of fields will be repurposed here for the specific task of describing the matter in general, *i.e*. occasionally independently of motion, we need to restrain the heuristical point of view that allows one to pass from the global equation (5.2.12) to the local version (5.2.13). The classical example for such repurposing remains, of course, forever Einstein with his general relativity. However, in view of our previous discussions along this work, we need to proceed differently, more along the classical lines of Poisson and Gauss, involving the local field in the form of a Newtonian force. Namely, it is critical to take notice that the formula (5.2.1) depends on the fact that mass M can be expressed as in equation (5.2.11). Therefore (5.2.1) is based on the idea of mathematical *continuity of matter*. This ideal condition is, nonetheless, never satisfied for the real matter. However, even more important is the fact that the formula (5.2.1) depends also on the mass m assigned to the attracted point, and this is an inescapable condition when we work with forces, for there is no other way to work like this, but that prearranged by the classical dynamics.

In order to reveal the real importance of a proper description of the concept of continuity of matter, let us recall that the electric *static* force between two point charges has the same Newtonian space behavior:

$$\boldsymbol{F}(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon}\frac{Qq}{r^2}\hat{\boldsymbol{r}} \tag{5.2.14}$$

Here ε is the permittivity of space containing matter, and Q, q are the charges of the two particles between which the force acts. The sign of this force depends on the relative sign of the two charges. Now, the classical dynamical habit describes the motion of the charge q in the field of charge Q *via* a dynamics generated by the force (5.2.14). This means that the formula (5.2.8) gives here an *acceleration* that depends on the two charges as well as on the inertial mass of the attracted charged particle:

$$\boldsymbol{g}(\boldsymbol{r}) = \frac{q}{m}\frac{1}{4\pi\varepsilon}\frac{Q}{r^2}\hat{\boldsymbol{r}} \tag{5.2.15}$$

The trouble is that here m is clearly the *inertial mass*, inasmuch as it was introduced by the prescriptions falling under the heading of second principle of classical *dynamics*, while in the case of force from equation (5.2.7) it is, just as clearly, a *statical gravitational mass*. This is a well-known issue, which generated the crisis of mass identity at the beginning of the last century, that in turn led, among others, to the theory of general relativity, and to Einstein's new, *viz*. non-Newtonian, natural philosophy.

But this is not all of it, and what is currently missing could prove instrumental for a theoretical physics involving matter independently of motion, where the *concept of memory* is essential, and needs to be properly captured by mathematical tools. Notice that in order to write the equation (5.2.1) for the case (5.2.15), we need to assume the space continuity of Q, so that the density to be used here should be the *density of charge*. Obviously, this is not the same as the density of mass, for it concerns the charge, so that the Poisson's equation (5.2.3) is referring now to the density of charge. The question however stands: even if the two densities differ from one another – which is quite conceivable, it can happen all the time, and can even be explained according to the mathematical idea of function: the two densities are just *different functions* having the same domain – do the charge and mass fill *the same space*? But, most importantly, this dichotomy of space meant, mathematically, the clear difference between the two natural philosophical approaches that Gauss, based on the philosophy outlined right above for constructing the potential, once termed as '*geometria situs* and *geometria magnitudinis*'[see (Gauss, 1833); there is an English rendering of this beautiful fragment, due to David Delphenich]. As we have mentioned before, this problem got through in the wave mechanics as the difference between the *ordinary space* and the *coordinate space* (Darwin, 1927).

It is at this juncture that, in our opinion, one needs, according to Gauss' mathematical philosophy, to put things in order methodically. To wit, it seems to us that in order to be able to build a 'geometria situs' it is imperiously necessary to have a reliable 'geometria magnitudinis', in order to know precisely how this geometry changes with an incidental location it describes. In order to properly understand this statement, consider a continuum with the continuity described by the mass density. This mass density needs to be defined with respect to the inertial mass, exactly as Newton claimed. Indeed, in order to interpret this continuum, we need ensembles of Hertz material particles, and these are described by gravitational mass and charges, so that they may assume states of equilibrium at will. These states of equilibrium are described by *static* Newtonian forces (see §3.1 above), and we should not forget that such forces are, in the classical Newtonian view, the expression of an *eternal* motion: *the Kepler motion*. For once, therefore, this is the expression of an *eternal memory* of the universe around us, imprinted in its structure, and we have no doubt that it should be extended, for instance, on a physical theory of brain, where the concept of memory gets an entirely new meaning. In our opinion this is the main reason why the brain should be modelled as a universe (Mazilu, 2019, 2020). However, in order to build up such an extension, one needs a reliable geometry, and in order to create this reliable geometry, one needs – again, in our opinion – to get rid of the Newtonian philosophy of forces in the problems of field, inasmuch as this is based on a special formula for force and, as such, it is not a universal characteristic of the concept. More to the point, no memory should be eternal – as the facts indeed prove it for the case of *real* Kepler motion – and this should be the essential characteristic of a general theory of the universe. This is, in fact, the general point of view revealed by Einstein's natural philosophy. Let us explain the issue at hand in more detail.

A *universal* characteristic of the Newtonian forces is, as we said, their *centrality*. It seems that it is this feature the one that implies the existence of a potential – *conservativity*, the key to Poisson's equation and Gauss' considerations – but the things are not quite so simple. In fact, the centrality can be achieved with forces that may not be conservative according to the Newtonian definition of forces. However, these two conditions are always in effect if the forces have a magnitude *that depends exclusively on the distance* between the interacting bodies. And the forces having the magnitude inversely proportional to the square of the distance between bodies are of this kind. The universal conditions they satisfy are, as we already mentioned, the *Helmholtz conditions*, which seem more appropriate for our present specific purposes. For the field intensity $\boldsymbol{g}$, they sound:

$$\nabla \cdot \boldsymbol{g}(\boldsymbol{r}) = 0, \quad \nabla \times \boldsymbol{g}(\boldsymbol{r}) = \boldsymbol{0} \tag{5.2.16}$$

These conditions are derived from the practice with Newtonian forces: the first of them represents, according to a 'Poissonian experience' as it were, the field of acceleration in the absence of matter, while the second shows that in a motion described according to Newtonian dynamics at a constant potential of the field, this field is acting along a closed moving path of a material point. They lead directly to the Laplace equation *in space* just as they should do, obviously, but on the way of proof of this statement much more is revealed.

Indeed, if we apply the curl operation to the second of equations (5.2.16) we get:

$$\boldsymbol{0} = \nabla \times [\nabla \times \boldsymbol{g}(\boldsymbol{r})] \equiv \nabla[\nabla \cdot \boldsymbol{g}(\boldsymbol{r})] - \nabla^2 \boldsymbol{g}(\boldsymbol{r})$$

Now, only if the first equation from (5.2.16) is valid for the region of space where we are considering this relation, then the field intensity vector is a solution of Laplace equation, for, otherwise, we have in general:

$$\nabla^2 \boldsymbol{g}(\boldsymbol{r}) = \nabla[\nabla \cdot \boldsymbol{g}(\boldsymbol{r})] \tag{5.2.17}$$

This shows that in the space filled with matter the field intensity is connected with the *variation of density*. In the spirit of a Hertzian natural philosophy, the origin of this property rests with *the torsion* of such space, a fact that was first noticed only in later times by David Delphenich, for the case of density associated with the wave function

(Delphenich, 2013). As we shall see here, the torsion is quite a general property of the space filled with matter, and it should be intimately connected to the physical properties of a de Broglie wave surface characterizing the matter. Meanwhile, though, it is worth showing where we believe this path of thinking leads us. And, luckily we should say, there is a particular manner to unravel the final point of this path, right within classical theory, that can be helpful to us in many ways, inasmuch as it has a touch of universality; and the universality here is judged by a particular case of possibility of calculating the force, intimately connected with the mathematical idea of analyticity. As a matter of fact, it inspired us to get into a general idea of analyticity, that may be further intimately connected to that of continuity of matter, as described by Newtonian density.

Start with the fact that, in a *two-dimensional case*, the Helmholtz conditions (5.2.16), referring to the vector $\boldsymbol{g}$, assumed to have the components (g_1, g_2) say, in an Euclidean reference frame, can be taken as Cauchy-Riemann conditions for the vector of components $(g_2, -g_1)$ perpendicular to $\boldsymbol{g}$ in their common plane. As well-known, these conditions characterize the *analiticity of the potential function* that provides $\boldsymbol{g}$ by the operation of gradient. Specifically, the *Cauchy-Riemann conditions* in the plane, are conditions of existence of the Cauchy integral of the complex function corresponding to a two-dimensional vector gradient, and this integral provides the value of the potential in a point, given its values on a curve enclosing that point. One can seek a generalization for the multi-dimensional case of the Cauchy integral that would be quite appropriate in building a physics of the ensembles of Hertz material particles serving to interpretation for instance, which, in fact, we need here. Even more to the point, we need a generalization of the two-dimensional Cauchy integral for *the three-dimensional case*, and such a formula was produced almost a century ago (Fulton & Rainich, 1932). For a general field satisfying the Helmholtz's conditions (5.2.16) this general formula is:

$$4\pi \boldsymbol{g}(\boldsymbol{0}) = -\oiint \left\{ \frac{\boldsymbol{r}\cdot\boldsymbol{g}(\boldsymbol{r})}{r^3} \right\} (d\boldsymbol{S}) + \oiint \left\{ \boldsymbol{g}(\boldsymbol{r})(d\boldsymbol{S}\cdot\frac{\boldsymbol{r}}{r^3}) + \frac{\boldsymbol{r}}{r^3}[d\boldsymbol{S}\cdot\boldsymbol{g}(\boldsymbol{r})] \right\} \tag{5.2.18}$$

where the surface of integration encloses the position of origin, where the field intensity is calculated. This formula gives the field $\boldsymbol{g}(\boldsymbol{r})$ as *calculated in a point* – in this specific case the origin – from the *values of this field intensity on a surface* surrounding that point. It has the particularly resourceful mathematical property of giving a null intensity of field if the point where we calculate the force is outside the surface. The formula (5.2.18) – which will be called the *Fulton-Rainich formula* from now on, if occasion may occur – has a special significance from the point of view of interpretation process.

In order to understand this significance, we need the *general transport theorem* referring to physical characteristics defined by their densities. Such a definition involves a differential form, defining a density, which in turn involves another differential form: that of the *elementary volume*. Fact is that, for an arbitrary differential form, Ω say, the *transport theorem* sounds (Flanders, 1973)

$$\frac{d}{dt}\int_{\boldsymbol{\Phi}_t(Vol)} \boldsymbol{\Omega} = \int_{\partial\boldsymbol{\Phi}_t(Vol)} \boldsymbol{v}\cdot\boldsymbol{\Omega} + \int_{\boldsymbol{\Phi}_t(Vol)} \boldsymbol{v}\cdot(d\wedge\boldsymbol{\Omega}); \quad \boldsymbol{v} \triangleq \frac{d}{dt}\boldsymbol{\Phi}_t(\boldsymbol{r}), \quad \boldsymbol{\Phi}_0(\boldsymbol{r}) \equiv \boldsymbol{r} \tag{5.2.19}$$

where the integral may be simple, double or triple, as the case may happen to occur. The application $\boldsymbol{\Phi}_t$ represents the evolution of interpretive ensemble having initially the extension *Vol*, and $\partial\boldsymbol{\Phi}_t$ is the boundary of this extension at the time *t*. The dot product means here *the projection of the differential form* involved in the product, *along the vector* involved in that product [see (Betounes, 1983), for a few pertinent and instructive examples of such projections, involved in the mathematical theory of classical electrodynamics]. The last two identities in (5.2.19) give the meaning of velocity field of evolution and of initial condition of evolution. It is to be expressly noticed that the field of velocities is referring to an ensemble of *contemporary positions*, for which the time is a characteristic parameter.

Consider now a differential 3-form, representing an *elementary physical quantity*, related to a continuum by its Newtonian density connected to a certain interpretation, *i.e.* given by a formula involving the elementary *oriented* volume:

$$\boldsymbol{\Omega} \equiv X \cdot (dx^1 \wedge dx^2 \wedge dx^3) \tag{5.2.20}$$

where (x^k) is a system of coordinates connected to the interpretation in question. Then, according to transport theorem transcribed here in equation (5.2.19), we must have:

$$\frac{d}{dt}\int_{\boldsymbol{\Phi}_t(Vol)} X \cdot (dx^1 \wedge dx^2 \wedge dx^3) = \int_{\partial \boldsymbol{\Phi}_t(Vol)} X(\boldsymbol{v} \cdot d\boldsymbol{S})$$

because the second term in (5.2.19) is null for a 3-form in the three-dimensional manifold. With this result in hand, we can solve, for instance, the classical *Reynolds' problem*: calculate *the time variation of the volume average of a certain vector*. Such a time variation is given by an integral like the left hand side of the equation above. In the case of a vector field, we have three replicas of the integral, each one of them referring to one of the components of the field. Therefore we obviously can define a vector volume average using the integral of that vector field, and further on, we can define the *rate of variation of that mean*, using the transport theorem. Therefore we can write

$$\frac{d}{dt}\int_{\boldsymbol{\Phi}_t(Vol)} \boldsymbol{X} \cdot d(Vol) = \int_{\partial \boldsymbol{\Phi}_t(Vol)} \boldsymbol{X} \cdot (\boldsymbol{v} \cdot d\boldsymbol{S}) \tag{5.2.21}$$

which shows that the rate of variation of a vector volume average in a point, is dictated by the flux of the vector through a surface surrounding that point.

Now, if the vector $\boldsymbol{X}$ is our field $\boldsymbol{g}(\boldsymbol{r})$ from equation (5.2.18), the terms in square bracket under integral sign are of the form given in (5.2.21). Specifically, the first term under the curly bracket of the second integral of equation (5.2.18) can be written as

$$\left.\frac{d}{dt}\right|_N \int_{\boldsymbol{\Phi}_t(Vol)} \boldsymbol{g}(\boldsymbol{r}) \cdot d(Vol)$$

where the lower index of the time differentiation operator, indicates that the time variation is defined according to equation (5.2.21), *via* a *rate of variation of the surface* given by the 'Newtonian' velocity field:

$$\boldsymbol{v} \triangleq \frac{d\boldsymbol{r}}{dt} = \Gamma_1 \frac{\boldsymbol{r}}{r^3} \tag{5.2.22}$$

Here, obviously, $\boldsymbol{r}$ describes a position on the surface. The constant Γ_1 is necessary for a dimensional agreement, and should have the dimensions of a time rate of a volume (L^3T^{-1}). This way the Newtonian force – or, to be more precise, a vector proportional to it – has the clear interpretation of the rate of variation of the position itself, when this position is on the surface delimiting the volume where we are calculating the average of our force. Insofar as this position is on a surface surrounding the origin, we shall have, of course, to elaborate further on the variation of that surface.

Likewise, the second term in the curly bracket of the second term of (5.2.18) can be construed as the variation of the volume mean of Newtonian force itself:

$$\left.\frac{d}{dt}\right|_g \int_{\Phi_t(Vol)} \frac{\boldsymbol{r}}{r^3}\cdot d(Vol)$$

provided the variation of the surface delimiting the space region serving for average is calculated with the aid of the force field whose value is supposed to be calculated in origin:

$$\boldsymbol{v} \triangleq \frac{d\boldsymbol{r}}{dt} = \Gamma_2 \boldsymbol{g}(\boldsymbol{r}) \tag{5.2.23}$$

where $\boldsymbol{r}$ belongs to the surface, naturally. Here Γ_2 is another constant of dimensional adjustment, having the dimensions of a time (T). This approach provides a natural way to replace the second principle of dynamics by the theory of surfaces, in concordance with *Wigner's dynamical principle* [see (Mazilu, Agop, & Mercheș, 2019), Chapter 2], having as expression one of the equations (5.2.22) and (5.2.23). This replacement is analogous to the Bertrand's replacement of the Kepler laws with the requirement on the form of orbit (see §4.1, above). Along the same line, the first term in equation (5.2.18) is a special kind of *average of the virial* of force to be calculated in origin, involving the components of *solid angle* vector. On this we will have to insist longer later on, as we go along with our work.

For now, we notice only that, in view of the transport theorem referring to differential forms – *a priori* necessary, in our opinion, for a genuine Newtonian theory of the physical magnitudes – the Fulton-Rainich formula (5.2.18) has a very suggestive reading expressed in the equation (5.2.18), which now can be written as:

$$4\pi \boldsymbol{g}(\boldsymbol{0}) + \oiint \left\{\frac{\boldsymbol{r}\cdot\boldsymbol{g}(\boldsymbol{r})}{r^3}\right\}(d\boldsymbol{S}) = \left.\frac{d}{dt}\right|_N \iiint_{\Phi_t(Vol)} \boldsymbol{g}(\boldsymbol{r})d(Vol) + \left.\frac{d}{dt}\right|_g \iiint_{\Phi_t(Vol)} (\frac{\boldsymbol{r}}{r^3})d(Vol) \tag{5.2.24}$$

Reading, therefore, the information from this equation: the *value of intensity of a force field in a point* in space is given by an average of the virial of the field over the solid angle around that point taken as reference, to which we have to add two further contributions. One of these contributions is generated by the time rate of the average of the force field, where the time is defined by Newtonian force field, while the other contribution is defined by time rate of the average of the Newtonian field, with the time defined by the very force field to be calculated. According to this view – a *field-theoretical view*, we should say – no force can be *pointwise* calculated independently, for its value in a given position depends not only on its values in virtually *any position* in space, but it also depends on the *values of the Newtonian force* as well; this last part of our statement can very well be taken as a characteristic of the universality of Newtonian force fields. The whole statement, however, has an interesting connotation that needs to be revealed right away, inasmuch as it touches the concept of memory but from another angle, which, in the case of brain, for instance, is a crucial differentia of the concept.

The point is that, as we already mentioned above, the forces invented by Newton are an *expression of the memory of a physical universe*. It is instrumental, indeed, in view of the modern trends in theoretical physics, to take notice of the fact that Newton invented these forces based on a clear *idea of measurement* [see (Mazilu, Agop & Mercheș, 2020), Chapter 5; (Mazilu & Porumbreanu, 2018), Chapter IV]. To wit, the Corollary III of the Proposition VII from the first book of *Principia* [(Newton, 1974); p. 51], produces the force pulling a material point in motion toward a certain point from space, based on the force pulling that very material point towards *any other point* in space, and some geometrical characteristics of the trajectory of its motion, as we have shown in §4.1 above. The only necessary condition, in order to do this, is that the two points to which the force is directed, *have to be in the plane of that trajectory*. This condition is always satisfied in the case of the classical Kepler motion which, at the historical moment of Newton's existence, gave a *scientific understanding* of the celestial matter. The very Newton's creation of the system of classical dynamics can then be seen as a *change of paradigm*, as it were, in order to bring *the present*, apparently represented by the forces of human experience, in the description of permanency, apparently represented by the Kepler motion of matter. It is for this purpose that Newton introduced the *concept of inertia* in the very heart of scientific understanding. Later on, the inertia was definitively instated as a force in matter, by the *d'Alembert's principle*. Kepler motion can then be seen as an actual *expression of the memory of the universe*, imprinted in its very structure, as we have asserted a few times before in this work. In what follows from the present work, we take this conclusion as a *methodological principle* for our theory of memory.

According to this view on memory, the formula (5.2.24) asks for a specific connection between geometrical concept of surface, and that of inertia. Although the formula requires an *a priori* arbitrary surface, our terestrial experience indicates for inertia – and therefore for the memory – a sort of description analogous to what is nowadays known in the differential geometry as a *foliation*. In order to get a rough idea of what this case is all about, assume that we need to evaluate the field intensity $\boldsymbol{g}(\boldsymbol{r})$ in the case of Earth's interior, and use for this the formula (5.2.24). $\boldsymbol{g}(\boldsymbol{r})$ is the gravitational acceleration field answering to the problem of free fall on Earth, and then the formula should replace the d'Alembert principle. To wit, the value of the field intensity inside Earth, for instance – where the man has no access whatsoever – can be decided, on one hand, by the field of Newtonian gravitation outside the surface of Earth, in a region far enough from its surface to have, indeed, a valid 'approximation' of the situation in which the Newtonian force is valid (for instance for the Moon, which would then be considered as the depository of Earth's inertia!). On the other hand, we need a term with the time controlled by the gravitational acceleration, decided by a 'Galilean region', close enough to the surface of Earth.

Coming back to our current subject concerning the electricity, the analysis right above shows how byzantine could be the very problem of interpretation in general, if one wants to involve the physics in the issues regarding the space or the matter. Gauss himself has comprehensively understood the intricacies of the subject, to which he dedicated quite a significant amount of work, partly published [see (Gauss, 1842); see also the volume V of *Werke* (Gauss, 1867)], partly in the form of unpublished notes [see (Gauss, 1833) for a remarkable instance of such a *Nachlass* from the *Princeps Mathematicorum*]. Like many researchers before us and, surely, a lot more after us, we shall take this work of Gauss as a guide in judging what we have to achieve with our own endeavor.

5.3 The Contribution of Bernhard Riemann

Bernhard Riemann's contribution to the birth of electrodynamics is somewhat controversial [(Bottazzini & Tazzioli, 1995); one can also rely for details on Detlef Laugwitz's comprehensive monograph on the life and work of Riemann (Laugwitz, 2008)]. If we were to judge by his own attitude, though, Riemann should not even be mentioned as first-hand participating to electrodynamics, for he did not make public his ideas – except for the course of lectures, obviously – and when he published them, he withdrew the paper right away. His ideas, however, were saved for our knowledge by his students, like Karl Hattendorff, who was awarded a doctorate in mathematics under Riemann's supervision at Göttingen in 1862, and habilitated for private lecturer in 1864, and who edited, among others, the course of lectures on electrodynamics held by Riemann at Göttingen in 1861 (Riemann, 1876). An essential part of this work was translated into English (White, 1977). We are interested here in what we think is one of the most remarkable moments of electrodynamics (Riemann, 1867, 1976, 1985), presented in 1858 to the Göttingen Royal Society, but immediately withdrawn, as we said. The reasons of withdrawing are not known first-hand but, sure enough, there are both technical (Clausius, 1868), as well as natural philosophical details (Betti, 1868, 1985) that may have caused the attitude of Riemann towards this work [for documentation purposes one can consult the work of Umberto Bottazzini and Rossana Tazzioli, quite informative on the very subject we are interested in here, even though it covers a larger area (Bottazzini & Tazzioli, 1995); also very helpful is the monograph of Detlef Laugwitz already cited above (Laugwitz, 2008)]. Luckily enough for us today, as well as for the future of physics, we should say, the work has been saved, both in original, as well as quite a few foreign translations and analyses.

Our reason to consider Riemann as a protagonist in constructing the electrodynamics is quite simple: if his teacher and friend, Gauss, only mentioned that he stopped short of finding a 'practical idea of how the propagation occurs', Riemann continued with the mathematical description of the most remarkable such practical idea: the propagation of light, only mentioned by Gauss, and the propagation of heat in solids. First, we have, in the work of Riemann, the first instance of what later came to be known as the Klein-Gordon equation, which was used by Louis de Broglie to characterize his concept of physical ray [see Chapter 2, §2.1, equations (2.1.1) and (2.1.4)]. However, this is not all of it, and what seems to us by far more important to be taken into consideration is a solution of the *problem of mass* that predates the definition of Hertz reproduced by us in the introduction of Chapter 4 here, by some four decades. Let us, therefore, reproduce in English the Riemann's own essential words, and then discuss the whole meaning of his endeavor. Thus, quoting:

> I took the liberty to communicate to the Royal Society a remark which brings the theory of electricity and magnetism into a *close connection with the theories of light and radiating heat*. I have found that the electrodynamic effects of galvanic current can be explained if one assumes that the effect of an *electric mass* on other masses does not occur *instantaneously*, but *propagates to these with a constant speed* (equal, within observational errors, to the speed of light). The differential equation for the propagation of electric force becomes, by this assumption, *the same as the equation of propagation of light and radiating heat* [(Riemann, 1867); *our rendering and Italics here*; see (Riemann, 1867, 1985) and (White, 1977), pp. 295 – 300 for translations of the whole article; also (Laugwitz, 2008), pp. 261 – 262 has a translation of this very excerpt].

Riemann's approach has been criticized from different points of view, on which we have no room of insisting in the present work, but only partially; the interested reader is referred to the trustworthy literature already cited above, for due details in case they are needed. However, from our point of view expressed all along the present enterprise, Riemann's work from which we excerpted the introductory fragment right above was, in spite of the possible forced technicalities and perhaps some misplaced natural philosophical conclusions, right in the place it should be, for no other place seems to be more adequate for its content. Quoting, again, further:

> According to the existing view about electrostatic action, the potential function U of arbitrarily distributed electrical masses, when ρ is their density at points (x, y, z), is defined by the condition
>
> $$\frac{d^2U}{dx^2}+\frac{d^2U}{dy^2}+\frac{d^2U}{dz^2}-4\pi\rho=0 \tag{5.3.1}$$

and by the condition that U is continuous and constant *at infinite distance* from the *acting masses*. A particular integral of the equation

$$\frac{d^2U}{dx^2}+\frac{d^2U}{dy^2}+\frac{d^2U}{dz^2}=0 \tag{5.3.2}$$

continuous everywhere except the point (x',y',z'), is

$$\frac{f(t)}{r} \tag{5.3.3}$$

and this is the potential function generated by the point (x',y',z'), *if there is the mass* $-f(t)$ *at the time* t located in that point.

Instead of this, I now assume that the potential function U is determined by the condition

$$\frac{d^2U}{dt^2}-\alpha^2\left(\frac{d^2U}{dx^2}+\frac{d^2U}{dy^2}+\frac{d^2U}{dz^2}\right)+4\pi\alpha^2\rho=0 \tag{5.3.4}$$

so that the potential function generated by the point (x',y',z'), *if the mass* $-f(t)$ *is located in it at the time* t, becomes

$$\frac{f(t-r/\alpha)}{r} \tag{5.3.5}$$

[(Riemann, 1867); *our rendering and Italics*; see also English translations (Riemann, 1867, 1985) and (White, 1977), pp. 295 – 300].

The formula (5.3.5) is usually seen as the precursor of the idea of *retarded potentials* in electrodynamics, of which a first specimen occurred just about the time when Riemann's work was published, *i.e.* about a decade after it was first presented to the Academy (Lorenz, 1867). We did not bring here the above excerpt in order to discuss issues of priority though, but rather in order to notice the procedure, which fits harmoniously in a general natural philosophy.

Indeed, it is quite clear that what Riemann had in mind, was not so much the mathematical procedure, that came to be known later as 'retardation', already mentioned, as much as he wanted to characterize the 'acting mass' in connection with its location. We have noticed before that the Ampère's era regarding the forces stands in the words of Joseph Liouville, that can be simply summarized by the observation that the Newtonian forces at large have to be non-central forces. The Ampère procedure of calculating the forces is in fact *a choice* from a set of possibilities, as Reynard observed. One can even say that the problem of Newtonian forces was doomed from the very beginning to an uncertainty, a fact best expressed toward the end of the 19th century by Henri Poincaré on the occasion of the analysis of Hertz's mechanics. After an analysis of the possibilities of defining the forces, and their logical ways of failure, the great natural philosopher concludes:

We are left therefore with nothing, and our efforts were unfruitful; we are compelled to adopt the following definition, which is nothing else but a confession of incapability: *the masses are coefficients, convenient to introduce in calculations*.

We will be able to redo the whole Mechanics, by attributing to all the masses different values. This new Mechanics will not be in contradiction with experience nor will it be contradicting the general principles of Dynamics (the principle of inertia, the proportionality of the forces with masses and with accelerations, equality of action and reaction, rectilinear and uniform motion of the center of gravity, the principle of areas).

Only, the equations of this new Mechanics will be *less simple*. Let's understand this well: only the first terms will be simpler, i.e. the ones we already know from experience; it would be possible that, by altering the masses by small quantities, the complete equation neither gain nor drop anything from their simplicity. [(Poincaré, 1897); *our translation, original emphasis*]

Riemann was well aware of these problems, prompted to criticality during the Ampère moment of human knowledge by the necessity of characterization of the concept of *acting masses* in the case of dynamic electricity. No better proof for this observation can be offered than the precautions took by Riemann in posing the problem to be solved by the calculation of those 'additions' to static forces, as mentioned by Gauss, his illustrious predecessor. One can say that on this issue, Riemann went back to Newton, and took the solution of the problem from the very point it was left by Newton himself, exactly like Ampère before him. Quoting, again:

Let S and S' be *two conductors traversed by constant voltaic currents but not moving towards each other*; let ϵ be an *electric mass element* in the conductor S that at the time t is located in the point (x,y,z); let ϵ' be an *electric mass element* of S' that at the time t' is in the point (x',y',z'). As regards the *motions of the electric mass elements*, which in each *conductor element* are opposite for the negative and the positive electricity, I assume that at *every time moment* they are so distributed that the sums

$$\sum \in f(x,y,z), \quad \sum \in' f(x',y',z') \tag{5.3.6}$$

extended over all the electric mass elements in the conductor can be neglected as compared with the same sums extended only over the positively electrical, or only over the negatively electrical mass elements, *as long as the function f and its differential quotients are continuous*. [(Riemann, 1867); *our English rendering and emphasis; compare to the other existing translations*]

To us it was quite clear that Riemann realized how important is to distinguish between 'electric mass elements' (*Massentheilchen*) and 'conductor elements' (*Leitertheilchen*). It is easy to see that in this excerpt he defines in any moment a static state of the elements involved in the expression of Ampère force, based on the freedom of flow of charges existing in the conductor through which the flow proceeds. Such a state is conditional on the neutrality of the conductor at every time moment, whose expression involves an indeterminate function in the expression of the force – therefore of the potential – which has to satisfy general requirement of continuity and differentiability. In the equation proposed by Riemann such a function is quite arbitrary but depends on time, suggesting that the mass itself should depend on time. The position dependence is then eliminated, and remains an uncertainty in the theory. However, it can be brought to bear again by Riemann's prescription given in equation (5.3.5). This is why we take Riemann's prescription as a necessity of defining *simultaneous mass elements*, of the species that started being only lately noticed in theoretical physics, in connection with the relativity prescriptions of non-local character [see (Amelino-Camelia, Freidel, Kowalski-Glikman, & Smolin, 2011), especially their suggestive Figure 2]. If we may be allowed, a theorist of Riemann's astuteness, might have sensed here a contradiction lurking, between such a concept of mass and the Newtonian definition of density, and this may have determined him to withdraw the paper from the Secretariat of Academy. Suffice it to notice that the Newtonian concept of density misses one necessary *differentia*, brought in only later by the general relativity in a kind of Hertz-type natural philosophy: this is the mathematical *cardinality*, imposed by any physical structure of a general relativistic construction, which became essential in Einstein's phenomenological account of physics, leading to the general relativity [see (Mazilu, 2020b)].

Actually, there are two occurrences that biased us into such a view at this juncture. First, comes what came to be known as the Klein-Gordon equation, proposed by Riemann in the form (5.3.4) above. The Reynard-type of independence between the charge flow and the conductor supporting it, allows us to think of the *current elements* as *parts of a de Broglie physical ray* (see, §2.1), for which this equation is instrumental, provided a special connection exists between its solutions and the density of the Hertz particles describing the electric fluid [see de Broglie's equation (2.1.14) here]. Then, again, the solutions in question cannot be quite arbitrary from a functional point of view, but they have to be of an *exponential type* [equations (2.1.2) or (2.1.3) in §2.1]. This is that type of solution that introduces a kind of coordinates specially requested by the Ampère's generalization [equation (5.1.4) above] of the Newtonian central forces: the *Shpilker coordinates* [see (Mazilu, 2020a), §9, equations (56) & (57)]. According to their original presentation, these coordinates establish a reference frame necessary in the very definition of Newtonian forces. But there is a more profound connection: the cycles necessary for the definition of the Shpilker coordinates [*loc. cit. ante*, equation (54)] are naturally provided by the *geometrical theory of charges* (*loc. cit. ante*, §7; see also §§4.3 and 4.4 in the present work). We shall have to insist later on this important issue that occupied the theoretical physicists for quite a while at the beginning of the 20th century.

The second occurrence that biased us into judging Riemann's as one of the greatest contributions of all times to the construction of modern electrodynamics, involves the conclusion of the Berry-Klein gauging procedure, which we connected at the end of §4.2 with the name of Ivan Vsevolodovich Meshchersky: *the very definition of the mass*. Recall the Definition 2 of Hertz's list quoted by us in the introduction of Chapter 4:

> The number of material particles *in any space*, compared with the number of material particles in some chosen space *at a fixed time*, is called *mass* contained in the first space.

Riemann's was, historically speaking, the first and the only attempt in history to define the mass coefficients in the spirit given later forth by Poincaré in the excerpt we just gave above, from the work discussing the very Hertz's production:

> … we are compelled to adopt the following definition, which is nothing else but a confession of incapability: *the masses are coefficients, convenient to introduce in calculations*.

Riemann devised a way to deal with these coefficients that can be tied up with at least two of the future great theories of physics of the 20th century. The first of these is the wave mechanics: the equation (5.3.4) indicates a de Broglie's wave-theoretical approach *avant la lettre*, as it were. Then, we have the contention that the function $f(t)$, representing the mass at time t, that *becomes* $f(t-r/\alpha)$, thus depending on the position. For a constant t, we have therefore an ensemble of values of mass, corresponding to the positions it occupies: *simultaneous positions*, as needed by Hertz's definition. It should be quite significant that this idea has come to light *via* electrodynamics, for it was the electrodynamics – although in a Maxwellian take, as it were – that determined Einstein a half a century later, to define the *condition of simultaneity* that instated the special relativity (Einstein, 1905a). Undoubtedly, the time was not yet ripe for a statistical interpretation, but the signs that this should be the case have occurred ever since the publication of Riemann's work.

5.4 Enrico Betti's Adjustment and Rudolf Clausius' Criticism

By the unavoidable course of life, Riemann was very close to an Italian school of mathematics, mainly represented, on the subject we are interested here, by two notable Italian mathematicians from the University of Pisa: Enrico Betti and Eugenio Beltrami. Their names appear alongside that of James Clerk Maxwell, sometimes even on issues of priority, based on a Riemann kind of modern electrodynamics against Maxwell's, issues to which we cannot subscribe though. Fact is, however, that Enrico Betti contributed to reinforcing the Riemann's electrodynamics based on the principle just presented above, while Eugenio Beltrami addressed a few direct critiques to Maxwell's ideas which, in our opinion, are far from raising questions of priority. Quite the contrary, Beltrami's critiques, properly exploited, simply show that Maxwell ideas are destined to *sustain* Riemann's electrodynamics, rather than contradicting his ideas. But we postpone a specific discussion of Beltrami's contributions for later, in a further instalment of the present work, and for now we focus only on one of Betti's contributions to electrodynamics, insofar as it is connected with Gauss' ideas, as these are illustrated in the letter to Weber from which excerpted the text above. On this occasion we intend to show, on one hand, how Riemann's concept of electrodynamics was usually perceived at the time it got in the open, and, on the other hand, that the ways of perception were by no means exhausted. Also, Betti brought in a tangible idea of *phase* in one of its modern connotations, specifically the one that led to the concept of frequency in the physical optics in the times of Fresnel. Quoting, therefore:

> In 1858 Riemann presented a paper at the Academy of Science from Gottingen, published, after his death, in the number six of the Poggendorff's Annalen for 1867, where he deduced the potential of two constant *closed currents* acting upon each other, admitting that *the action of electricity propagates in space with a constant velocity equal to that of light*, and assuming that *the current consists of the motion of two electricities*, positive and negative, *travelling simultaneously through the wire in opposite directions* and, moreover, that the sums of the products of positive and negative electricities by a function of the coordinates of the points of wire, are negligible when compared with the sums of the positive electricity alone, or of the negative electricity alone, multiplied by the same function. This concept of *electric current*, completely ideal, *is hardly in agreement with what is known about it*, and it seems that Riemann himself was not satisfied, thus withdrawing the article from the Secretariat of Academy, and renouncing to publish it later on. In this context, it seems to me that it is not without importance to show how *the electrodynamic actions can be explained by means of their propagation in time*, considering that *the action of dynamic electricity takes indeed place according to Newton's law for static electricity*, without being based on that concept though, but assuming instead that *the current consists of a periodic polarization of the elements of the wire*, which is more in agreement with all known facts [(Betti, 1868), *our rendition and Italics*; *see also a previous English translation* (Betti, 1985)].

By comparison with Riemann's initial view, referring to an instantaneous statics, necessary in order to build a state of the element of current, Betti shows that the idea of current as a flow is usually perceived as 'the motion of two electricities traveling simultaneously in opposite directions'. It is not quite so clear that he succeeded in distinguishing between the 'mass element' and 'conductor element' as neatly as Riemann did, but apparently he targets this last one, understood by him as 'element of wire' gaining periodic polarity during the transport of charge, "more in harmony with all known facts". From the development of the original work, however, it seems that Betti shared the contemporaneous idea of a Grassmann current element, whereby the *path of charge and the wire are identified*. But let us follow closely the Betti's own work (Betti, 1868), which we reproduce in some detail, for it contains that variant of Ampère's theory used by Riemann himself and referring to *potential* not to *force*. That theory too, incorporates the essential take of the epoch in matters electrodynamic: the trajectory of motion of electricity is identified with the conductor. This means that when speaking of 'element of current' one would understand in fact that the path of motion of a charge is identified with the conductor it traverses, a conception to be found in the same mathematical form in the Ampère's fundamental work (Ampère, 1823) and in the Gauss' *Nachlass*, the part concerning electrodynamics (Gauss, 1868).

Betti starts with the observation that the interaction potential, describing the reciprocal action between two electric loops, which we reproduce here after him and, implicitly, after Riemann's original work, should be, in modern notations

$$V = I \cdot I' \oint_l \oint_{l'} \frac{\cos(d\boldsymbol{l}, d\boldsymbol{l}')}{r} ds \cdot ds' \tag{5.4.1}$$

where ds is the element of length of the path $\boldsymbol{l}(t)$, while ds' is the elementary length of the path $\boldsymbol{l}'(t)$. Now Betti notices (*loc. cit. ante*, §II), that in view of the definition $\boldsymbol{r}^2 \equiv (\boldsymbol{l} - \boldsymbol{l}')^2$ the expression (5.4.1) can be written as

$$V = -\frac{I \cdot I'}{2} \oint_l \oint_{l'} \frac{d^2(r^2)}{dsds'} \cdot \frac{ds \cdot ds'}{r} \tag{5.4.2}$$

and by a few manipulations based on a partial integration using the property of cyclicity of the two paths of current, can be brought to the form

$$V = -\frac{I \cdot I'}{2} \oint_l \oint_{l'} \left\{ r^2 \cdot \frac{d^2}{dsds'} \left(\frac{1}{r} \right) \right\} ds \cdot ds' \tag{5.4.3}$$

It is on this expression of the potential that Betti goes on an attempt to prove that, within reasonable assumptions, it equals the interaction energy between the two loops.

The formula (5.4.3) carries over into mathematics the essential property of the Newtonian potential of being an energy of interaction: for two 'active masses' m and m', *whatever they are*, it is proportional to $(m{\cdot}m'/r)$. Of course, in view of the motion of charges through wires, those active masses have to be some functions of time, and Betti takes them in the form

$$m = e \cdot \phi(t) \cdot ds, \quad m' = e' \cdot \phi(t) \cdot ds' \tag{5.4.4}$$

where $\phi(t)$ is a periodic function of time – a *phase*, as he calls it – having the period T, say. Then the energy of interaction between two loops of current, can be expressed by an integral in which we introduce the Ampère's current elements:

$$W = (e \cdot e') \int_0^T \oint_l \oint_{l'} \left\{ \phi(t) \cdot \phi(t' - r/c) \cdot \frac{d^2}{dsds'} \left(\frac{1}{r} \right) \right\} ds \cdot ds' \tag{5.4.5}$$

extended in time over a period. Inside this period of time the two active masses are connected at arbitrary times t and $t' = t + \tau$, while (r/c) allows for the propagation of the force between the two active masses, according to the ideas of Gauss and Riemann. Then, by a Taylor expansion to the second order

$$\phi(t' - r/c) = \phi(t) + (\tau - r/c)\dot{\phi}(t) + \frac{1}{2}(\tau - r/c)^2 \ddot{\phi}(t) + \cdots \tag{5.4.6}$$

and exploiting the mathematical properties of the phase $\phi(t)$, one can get the formula (5.4.3) where the notations:

$$\int_0^T \left(\dot{\phi}(t)\right)^2 dt \equiv a^2, \quad a \cdot e \equiv I, \quad a \cdot e' \equiv I' \tag{5.4.7}$$

are used. Whence the conclusion:

> Therefore, the *electrodynamic actions* can be explained, assuming that *they propagate in space with a speed equal to that of light*, that they *are exercised according to Newton's law like the electrostatic actions*, that the *currents consist of a kind of polarization of their elements, periodically variable*, that *the law of variation is the same in all currents*, and that *the duration of the period is small even compared to the time it takes the action to propagate to the distance unit.* [(Betti, 1868); *our translation and emphasis*]

Of course, the existence of a periodic function expressing the active mass in electrodynamic phenomena may be assumed *a priori*, with no reference whatsoever to a Klein-Gordon type equation. However, the great merit of Riemann's approach is that such a reference suggests two important physical things. First of all, the periodical properties are referring to the *active mass* of the current elements – as suggested by the original Poisson's equation – and, secondly, that the measure of these elements – the active mass *per se* – depends on the *close environment*, by that possibility of retardation offered by Riemann's proposal of amendment according to equation (5.3.4). In a way, the active mass is what nowadays we know as a *collective* physical magnitude.

From this perspective, and having the advantage of a hindsight given by the Berry-Klein gauging procedure, we can see in Enrico Betti's amendment to Riemann's proposal the roots of future wave mechanics. Namely, the typical periodic structure in physics is the harmonic oscillator and, as we have shown above, the harmonic oscillator is *by default* the free particle in a Berry-Klein gauged world. But then, the active mass properties are to be *statistically* described as transition properties between the close and the far environments of the free particles inside the wire, like in the case of the classical damped harmonic oscillator [see §4.3, equation (4.3.14) ff]. At the risk of getting way ahead of ourselves in these matters, we may be allowed to suggest now an approach according to the Katz's natural philosophy of a world of charge. This should be taken by now as just an indication of a future mathematical procedure to be developed as we go along with this work.

The basis of approach is the Wagner's theorem expressed through the equations (4.3.20) and (4.3.21). The active masses are to be considered as Katz's charges: referred to some Berry-Klein gauged times, each free particle in the world of charge is submitted to two oscillatory motions with different periods, due to electric and magnetic interactions. Then the free Berry-Klein particle, should be described by the following two undamped oscillator equation, in two different times corresponding to the two different split angles (see §4.4):

$$\ddot{X}(t,\tau) + \omega_1^2 X(t,\tau) = 0, \quad X''(t,\tau) + \omega_2^2 X(t,\tau) = 0 \tag{5.4.8}$$

The dots mean derivative on time t, while the accents mean derivative on time τ. The problem then occurs: what relationship should exist between the two times, so that equations (5.4.8) are concurrently valid for a given free particle? An answer is offered by a *rational linear relation* between the ratios of the fundamental solutions of the two equations. In order to prove that this can be a certified answer, we shall use a special property of the ratio of two independent solutions of the second order linear differential equation: the frequency is given by the *Schwarzian derivative*, defined as in equation (4.4.10) of that ratio. For the two equations (5.4.8) we have

$$\{k_1,t\}=2\omega_1^2,\quad \{k_2,\tau\}=2\omega_2^2 \tag{5.4.9}$$

where the following notations are used for the ratios of the two fundamental solutions of the equations:

$$k_1(t)=e^{2i\omega_1 t},\quad k_2(\tau)=e^{2i\omega_2\tau} \tag{5.4.10}$$

Now, using the chain property of the Schwarzian derivative, expressed by one of the two possibilities of transition between the two times, for instance by

$$\{u,t\}=\dot{\tau}^2\{u,\tau\}+\{\tau,t\} \tag{5.4.11}$$

we can consider the two functions k_1 and k_2 as two specimens of one and the same function k of two time variables t and τ, satisfying the two different conditions from equation (5.4.9). As a consequence, therefore, we can very well set out and try to determine a connection between the two times, such that the following equation is valid:

$$2\omega_2^2\dot{\tau}^2+\{\tau,t\}=2\omega_1^2 \tag{5.4.12}$$

This is a nonlinear equation for $\tau(t)$, that can be integrated in general terms, according to the standard scheme suggested in the equation (4.4.12). First, notice that

$$\{\tau,t\}=-2\sqrt{\dot{\tau}}\ddot{\xi},\quad \xi(t)\equiv\left(\sqrt{\dot{\tau}}\right)^{-1} \tag{5.4.13}$$

so that the equation (5.4.12) becomes

$$\ddot{\xi}+\omega_1^2\xi=\frac{\omega_2^2}{\xi^3} \tag{5.4.14}$$

and has the immediate first integral

$$\dot{\xi}^2+\omega_1^2\xi^2+\frac{\omega_2^2}{\xi^2}=const \tag{5.4.15}$$

For completeness, we might add that the theory goes just as well with the roles of t and τ switched. So, if instead of the pair (ξ, t) we consider the pair (η, τ), and instead of (5.4.13) we have

$$\{t,\tau\}=-2\sqrt{t'}\eta'',\quad \eta(\tau)\equiv\left(\sqrt{t'}\right)^{-1} \tag{5.4.16}$$

then the equivalent of equation (5.4.14) is

$$\eta''+\omega_2^2\eta=\frac{\omega_1^2}{\eta^3} \tag{5.4.17}$$

while the equivalent of equation (5.4.15) is

$$(\eta')^2+\omega_2^2\eta^2+\frac{\omega_1^2}{\eta^2}=const \tag{5.4.18}$$

Then, according to Wagner's theorem we can see the first of the equations (5.4.8) as a result of a Berry-Klein gauging through the pair (X, ξ), where the equation (4.3.21) is valid in the form:

$$\ddot{X}-\frac{X}{\xi}\left(\ddot{\xi}-\frac{\omega_2^2}{\xi^3}\right)=0 \tag{5.4.19}$$

Thus, the first equation (5.4.8) is valid provided the equation (5.4.14) is valid. The same connection acts for the pair (X, η), in explaining the second of the equations (5.4.8) in terms of equation (5.4.17). Therefore, the periodic properties of the Betti's active masses might be explained as gauge properties of the free particles, in a special gauging theory of a Berry-Klein type, independently of the equation proposed by Riemann. However, as we said, with such proposal we are getting way ahead of ourselves in many respects. But, we presented it at this juncture anyway, for two major reasons. First of all, in order to show that the linear rational transformation, connected to the presence of Scwarzian derivative, is a very attractive feature in describing a phase space in the form of $\mathfrak{sl}(2,R)$ Lie algebra, as discussed before (see Chapter 3 above, §§3.3 and 3.4). However, we are missing its meaning, and we are thereby pledged to provide such a meaning. Mention should be made that this meaning can only be given in the world of charge. Secondly, and this is actually the main reason, connected to this, there is a significant historical case, helping us in settling the ideas with respect to *quantization*: we shall describe this case in the closing conclusions of the present instalment of our work.

Related to the double periodicity property above, we need to mention here the critique of Rudolf Clausius, addressed to this approach of electrodynamics. Clausius assigns to Gauss the whole idea of electrodynamics (see the excerpt above from Gauss' letter to Weber), and analyzes three works that take this idea to its completion: Carl Neumann's work on the foundations of electrodynamics (Neumann, 1868), and Riemann's and Betti's works just cited above. All three works gave him a taste of deficiency, and he set out to draw attention to the scientific world of the reasons of this deficiency. Quoting:

> I have read these works with the greatest interest, but I must confess that I am not satisfied by them, and I believe that the importance of the subject and the well-deserved respect in which those authors are held in the scientific world will justify publishing my concerns. If three excellent mathematicians, using different methods of investigation, arrive at a result that is essentially in agreement, this seems to be a guarantee of the correctness of the investigations, *and perhaps some physicists will be determined to consider the matter as settled*. Under these circumstances, an open statement of the conflicting concerns can only be useful, encouraging further investigations, *perhaps carried out from other points of view*. [(Clausius, 1868); *our rendition and Italics*]

Clausius' analysis is an example of analysis carried out in the ordinary space, based on kinematics of the classical material point. There is no room for statistics in this approach and, as we see it, the condition of definition of the active masses of Riemann should be essentially based on statistics. Also, as we said, the analysis is conducted in ordinary space, while it needs to be done in a coordinate space. So, whereas we have to recognize the *meticulousness* of Claudius' argument, it is more important, though, to recognize the overwhelming *heuristical* importance of the Riemann's and Betti's suggestions, to say nothing of the Neumann's arguments. However, in connection with Betti's suggestion, Claudius has one valid observation that, properly elaborated, may be able to offer the "mathematical principles of a natural philosophy of charges', as it were.

Indeed, Clausius draws attention of the fact that the Taylor expansion of the phase, properly cut as in equation (5.4.6) is instrumental for Betti's argument. However, with a periodic phase, this cut may not be available. And Clausius gives the simple example of a periodic function, which we transcribe here in the current notations in order to see more clearly what his observation suggests: $\phi(t) = sin(\omega_1 t)$. A Taylor series of $\phi(t)$, involving the period $(\tau - r/c)$ as in equation (5.4.6) can be written in the form:

$$\phi(t' - r/c) = \sin(\omega_1 t) + 2\pi\frac{\omega_1}{\omega_2}\cos(\omega_1 t) - \frac{(2\pi)^2}{1\cdot 2}\left(\frac{\omega_1}{\omega_2}\right)^2 \sin(\omega_1 t) - \frac{(2\pi)^3}{1\cdot 2\cdot 3}\left(\frac{\omega_1}{\omega_2}\right)^3 \cos(\omega_1 t) + \cdots \qquad (5.4.20)$$

where $\omega_2 \equiv 2\pi/(\tau - r/c)$. Now, the Clausius' argument becomes clear: *it is not always possible to cut such a series to the second term* unconditionally: everything depends on the ratio of the two pulsations. And, in this respect, Betti's suggestion is really showing up of being in deficit any way you look at it, be it classical or statistical. A few further improvements are needed if we want to keep the Betti's suggestion, which is indeed very valuable from a heuristical point of view.

The first improvement presented to our wits along the previous line of discussion is to get rid of the condition of periodicity: according to the idea of definition of frequency by the instantaneous frequency, for instance (see §4.4), the phase needs to be a linear rational function of time. In this case, however, it becomes problematic if we can accept the suggestion of Riemann in order to be able to characterize the active mass. While, as we shall see in the present work, the idea of this solution is quite tractable with a proper group theoretical analysis of the concept of physical phase, due notice must be taken, however, of a straightforward time-statistical theory having a remarkable geometrical, therefore kinematical, representation. To wit, let us recall that a free Berry-Klein particle – which is the best canditate, if not the only one in fact, serving for interpretation in the case of electrodynamic phenomena – has a frequency defined by the Schwarzian equation:

$$\{u,t\} = 2\omega^2 \tag{5.4.21}$$

The solution of this equation is

$$u(t) = u_0 + \omega \tan(\omega t + \varphi) \tag{5.4.22}$$

which is plainly a periodic function, just as Betti claimed. It has a statistical meaning though: the mean of a family of distributions parameterized by the values of *t*, having the quadratic variance function given by

$$\dot{u}(t) = \frac{\omega^2}{\cos^2(\omega t + \varphi)} \tag{5.4.23}$$

Then we need to take notice, for further reference, that the pair (u, v) defined by

$$u(t) = u_0 + \omega \tan(\omega t + \varphi), \quad v(t) = \frac{\omega}{\cos(\omega t + \varphi)} \tag{5.4.24}$$

i.e. by the *mean* and *standard deviation* of the family of distributions indexed by *t*, are points along the geodesics of a conformal two-dimensional Lorentzian metric:

$$(ds)^2 \equiv \omega^2 (dt)^2 = \frac{(du)^2 - (dv)^2}{v^2} \tag{5.4.25}$$

as one can easily verify. According to the Berry-Klein gauging procedure, the Schwarzian equation (5.4.21) can thus be taken in the sense of Louis de Broglie, as associating a frequency 'to the wave phenomenon called Hertz material particle'.

Chapter 6 Instead of Conclusions: a Berry-Klein Update of Classical Dynamics

Any well-rounded work needs, naturally, some conclusions. However, since the present work is far from being 'rounded' – it obviously necessitates at least one more instalment in order to mean something towards its declared purpose – and since, on the other hand, about that 'well' we should be a little modest, so much the more as the work is not 'rounded', it seems to us that 'instead' is quite well placed in the heading of a last chapter of this instalment. Then again, since that declared purpose of the work is succintly comprised in its very general title, let us forget about the many subordinate clauses of this purpose, spread all along what we have achieved up to this point, and go directly for the foundation of a possible physics of the Louis de Broglie's doctrine in order to close the achievements thus far decently. This foundation would be, in a common opinion, which we unreservedly share, the association of a frequency with the classical material point (de Broglie, 1923), making a 'wave phenomenon' out of it, as de Broglie himself once declared (de Broglie, 1926c). We are not only ethically, as it were, indebted with such a final discussion here, but even feel like owing it to the reader – and, as it turned out along the developments in making this closing chapter, we owe it, in fact, to the proper completion of this very work – by one basic result of the Berry-Klein gauging philosophy: *the free particle in this gauging looks like the harmonic oscillator*, and consequently there is obvious support for a general contention that a free particle should have a frequency associated, by its very nature. We are tempted to say that this gauging theory is the crowning jewel of the concept of interpretation, as this one came to be along the history of our knowledge, and with good reasons, to be presented as follows.

First, judging in hindsight, this was the conclusion of Robert Hooke's general theory of light who, by phrases saying that the motion characterizing the light is 'exceedingly *quick*,… *vibrative motion*…', in fact, a 'very *short vibrating motion*' [(Hooke, 1665), pp. 55 – 56; *original emphasis*] gave actually the very first taste of what a gauging theory connected to an interpretation should be: it has to be quantitatvely described by a *speed*, by its *time quality* – uniform, accelerated, vibrative etc – and, once established its kind, by a quantity showing its *extension in space* – the amplitude in this case. From this perspective, one might say that Hooke's views were, in fact, sanctioned later, when the *diffraction* phenomenon was added to the phenomenology of light by Augustin Fresnel, on the occasion of his physical theory of light (Fresnel, 1821, 1826, 1827). The addition of a fourth phenomenon to this phenomenology (Gabor, 1948) – the *holography* (see the general story in the §2.1 above) – which was not signaled but only long after the de Broglie's original works, speaks by itself, in our opinion, in favor of consideration of the «wave phenomenon called 'material point'» from the viewpoint of the Berry-Klein gauging procedure.

On the other hand, the very definition of frequency itself – which seems mandatory for the Berry-Klein gauging procedure by its fundamental interpretative element, the Hertz material particle – can only be connected, in this theory, with the Schwarzian equation (5.4.21). It is not quite so obvious as yet (we need, in fact, to come back later to this subject) but the general solution (5.4.22), can be arranged into an exponential form like the factors from equation (5.4.10), a subject on which, again, we shall have to dwell for a while as we go along with this work. In this respect, it becomes highly significant that the modern theory of scale relativity depends on an *exponential scale factor* [(Nottale, 1992), §3]. Not only this, but for the Laurent Nottale's description of the scale transition, the choice of word 'relativity' seems to be the only right one, in view of the history of coming to being of quantization condition in connection with the electromagnetic theory of light. Let us elaborate a little further on this issue, insofar as it proves instrumental for the theoretical physics of the waves in general, and for the concept of interpretation especially.

To wit, the Riemann's equation, that came later to be known in theoretical physics of particles as the Klein-Gordon equation, extends the Ampère's generalization of the Newtonian static forces, in order to include the time. Indeed, as we have shown [see §5.1 above, especially the discussion around equation (5.1.4)], this generalization asks necessarily for a 'prefactor', if we may say so, in the traditional expression of the magnitude of central forces, and this prefactor depends on *the ratios of coordinates* or of their variations. We can say that this is a property of scale invariance of classical forces with respect to the space scale transitions of the coordinate space containing the electricity, that is to say, with the dimensions of the wires in which the electricity exists. This condition is tantamount to the possibility of definition of the *specific* Hertz material particles – the current elements – necessary in the case of electrodynamics, inasmuch as these have directional space extensions. Now, a scale transition invariance of the Riemann's equation would require not only the ratios of coordinates, but also the ratios of the coordinates and time, therefore the velocities, and this is a key point of the Nottale's scale transition theory, being precisely the point that connects it to 'relativity'.

Fact is that the relativity, as we have it today, has considered explicitly, and even amplified where possible, the two fundamental differentiae of the concept of interpretation: the *motion* and the *propagation*. The separate contemplation of each one of these physical phenomena leads to different conclusions, and these conclusions can be formulated in the following fashion [(Fock, 1959), §8 and Appendix A]. Regarding the motion, considered alone: the spacetime transformations carrying the *uniform rectilinear motion, into a uniform rectilinear motion* must be realized by *homographies* [for the idea of homography as it appears to Fock, one can consult (Appell, 1889)], *i.e.* transformations of the kind given in the equation (4.1.5) above, only not in two, but in four variables, three space coordinates and one time coordinate. Regarding the propagation, considered alone: the spacetime transformations taking *the wave fronts into wave fronts* are defined up to an arbitrary scaling, by *a conformal transformation*, *followed by a Lorentz transformation*. One can get rid of the conformal part of the transformation only by asking that *finite values of space and time coordinates should be transformed into finite values of space and time coordinates*. This condition is of an overwhelming importance in the construction of relativity, for one can prove that the conformal transformations in spacetime are intimately connected with the *accelerated motion*, and this issue transcends the precepts of special relativity (Haantjes, 1940), which, obviously, are only connected with the uniform motion. A few notices on this circumstance are worth making, mainly for the future reference in this work.

First of all, if it is to judge by its pure space counterpart, the conformal transformation is intimately associated with the inversion with respect to spheres – a geometrical gauging *avant la lettre*, as it were – and therefore can help into transferring the geometry of coordinate spaces – that can be thought of as always enclosed by a sphere in space – into the geometry of the ordinary spaces. In this respect we find highly significant that, first of all, if it is to discard the condition of finiteness and, therefore, to allow for the scale transcendence in a relativistic view of the world, the Lorentz transformation has to be applied to *already gauged coordinates*: the conformal transformation does not commute with the linear Lorentz transformation. This would mean, for instance, that in an incidental transport theorem, referring to a density for instance [as in §5.2; see especially the equation (5.2.19)] the application $\boldsymbol{\Phi}_t$ taking volumes into volumes – therefore coordinate spaces into coordinate spaces – should be, in general, a linear transformation.

We need to take notice at this juncture, of the fact that the Ampère's theory of forces is *innately* connected to a conformal transformation in space, by the very way it is constructed. That is to say that the electrodynamics was destined by its birth, as it were, to lead to relativity, as in fact just happened (Einstein, 1905a). Indeed, the Ampère's theory is based upon the idea of a metric element of space, denoted *ds* in the theory, identical with the ideal one-dimensional wire serving as conductor – which is thus reduced, for a given direction, to a line of current – and is working with this element of space by considering it dependent *only on the position in space*. This is a property sanctioned by the Euclidean expression of the finite distance in space. Such a geometrical line element cannot be but only *conformal invariant* (Bromwich, 1901): *it does not depend on the ratios of the coordinates*, but only on coordinates themselves. However, as we have seen, the forces *do depend* on the ratios of coordinates, and this dependence is, again, *innate* to the Ampère's generalization of the Newtonian forces. We would say that the property is just natural for the concept of force, inherited by the very definition of the Newtonian forces, and transmitted as such into relativity.

Indeed, Newtonian static forces – the basis of Berry-Klein gauging theory – have many properties connected with the transition of scale. It seems to us that all these properties derive from the old *Kelvin inversion theorem*, which bestows upon Newtonian forces the fundamental property of the Cartesian coordinates, of being *harmonic coordinates* [(Mazilu & Agop, 2012), Chapter 8, §5]. This means that from the point of view of Newton's measurement procedure, which instated the classical theory of forces (see introduction to Chapter 4 and §4.1 above) the Newtonian forces are the only physical, as well as mathematical entities *eligible as coordinates*: they are solutions of the Laplace equation at any space scale. From this point of view, Riemann's approach to the definition of active masses in the electrodynamic theory of electricity seems just reasonable, inasmuch as it starts from a definition coping with that of the coordinates themselves [see equations (5.3.2) and (5.3.3)]. However, the Betti's *a priori* definition of the active mass elements, simply by the periodicity condition, should be an entirely different story: it seems only by accident related to a space coordinate, inasmuch as Betti's procedure does not appeal to the Riemann's equation, which turns out to be mandatory according to Riemann's idea. And, as Clausius' critique reveals it, the periodicity property may preclude the essential property of the 'phase', which allows for a proper expression of the potential energy of interacting current loops.

The bottom line of this discussion is that the scale transition should be *a priori* described in the manner of Betti, however, not by a phase *per se*, as we understand it physically [see equations (4.4.15) ff above] but by a *Nottale exponential phase factor*. This phase factor then defines a frequency through its Schwarzian derivative with respect to time, and this frequency is, in turn, associated to a Hertz material particle serving for interpretation in a Berry-Klein gauging theory. The process of interpretation is based on the Wagner's theorem, and has a *dynamical moment involving a central force* of magnitude proportional with the *inverse cube of an adequate radial coordinate* serving for the Berry-Klein interpretation. One can say that the Berry-Klein freedom of the Hertz material particle appears exclusively due to a particular dynamics, and can be defined, based on Wagner's theorem, as follows: if there is a Berry-Klein gauging – involving, by its very definition, a radial motion for its realization (see §4.2 above) – where a free Newtonian particle is harmonically perturbed, then there is, necessarily, a gauging where it is Berry-Klein free. This means that the gauged particle belongs to an ensemble whose general radial motion in a Cartesian reference frame appears as being described by the equation of motion of a harmonic oscillator.

In order to properly understand these statements, let us relegate them, for the moment being, to what appears as the first classical attempt to what is known today as the *second quantization* of an electromagnetic field (Poincaré, 1900). With an electromagnetic theory of matter, the classical ether is naturally manifest by electromagnetic waves. This is a kind of instance where one can conclude, borrowing the words of C. G. Darwin, that without doubt *the ether is a continuum that needs to be interpreted*. Henri Poincaré saw in this the opportunity to reconcile the Lorentz theory of the electric world with the third principle of Newtonian dynamics, inasmuch as a charge in motion – like those involved in the Lorentz view of the matter – *generates an electromagnetic field*.

Then, as this field can be made responsible for a mechanical imbalance in the third principle of classical dynamics, for an overall physical balance of the Lorentz matter it is only necessary *to interpret the electromagnetic field itself as a fluid*, according to the old view of ether as matter, in order to apply the mechanical equations of continua, connected, as known, to d'Alembert's principle. Which is what he tried indeed to accomplish, by the definition of a *fictitious Poynting fluid*, as he calls the fluid destined to interpretation (Poincaré, 1900). However, there is a catch, contained in the old idea of *static equilibrium*, which is left behind by this interpretation.

The situation can be understood by going back in history, to another interpretation of a material continuum – for, according to Maxwellian electrodynamics, the electromagnetic field is a material continuum – namely that of Augustin Louis Cauchy. In this interpretation, a static material continuum is viewed as an ensemble of 'molecules', this time *a priori* 'detached' therefore, with static forces between them, whose *fluxes through any imagined plane* in this continuum are responsible for stresses. These stresses assumed the name of Cauchy ever since. Fact is that the *static case* cannot be rationally understood within framework of the classical mechanics, since an understanding, in this case, would need an interpretation of forces according to *Fulton-Rainich formula* for instance (see Chapter 5, §5.2 above), because in the static case *inertia is conspicuously missing* according to our experience. The same conclusion may go, however, for the 'fictitious' Poynting fluid of Poincaré: from the point of view of the concept of interpretation, *the electromagnetic light has the same speed in any direction and everywhere* – it is a conformal invariant – and, therefore, some classical material points representing a continuum that would be liable to interpret this light, are at 'rest' with respect to each other. And yet, the 'electromagnetic' fluid, which Poincaré needed for his interpretation, must account for the *creation* and *destruction* of *energy*, phenomena which then bring in the *concept of inertia*. This fact was instituted in physics by Einstein in 1905, in a phenomenological argument that brought to light the celebrated relation between the energy and mass. For the moment, however, one had to insist on structural properties related to what came to be known later as *creation* and *anihilation processes*, related explicitly to energy. Quoting, therefore, Poincaré:

> In order to *define the inertia of the fictitious fluid*, one must agree that the *fluid which is created* in a certain point by *the transformation of energy*, arises first *without speed* and that *it borrows its speed from the already existing fluid*; therefore, if the quantity of fluid increases but the speed remains constant, it will be nevertheless *a certain inertia to overcome*, because the new fluid borrowing speed from the old fluid, the speed of ensemble will decrease if a certain cause does not intervene in order to maintain it constant. By the same token, when *destruction of electromagnetic energy occurs*, it is necessary that the fluid, before it is destroyed, *looses its speed by imparting it to the subsisting fluid*. [(Poincaré, 1900), *our translation and Italics*]

Therefore, according to Poincaré, the inertia should be, at least partially, allocated to some *transport properties*, for which the Poynting's theorem (Poynting, 1884) offers the basis, as an *equation of continuity*. It is this fact that makes the theory of Lorentz 'souple', and for this very reason it is held by Poincaré as the most flexible among all of the theories of the electromagnetic matter: the *possibility of interpretation*, in spite of the 'fictitious' character of the fluid used in accomplishing this interpretation. In fact, according to Hertz's natural philosophy (see Chapter 4 above) every fluid involved in a process of interpretation should be 'fictitious', if it is to have a *static equilibrium instance*. For such an instance exists only instantaneously, i.e. at an infrafinite time scale if it is to admit the scale transitions as part of our physics.

It is at this statistical juncture that the Berry-Klein gauging procedure come in handy, provided that, according to a Poincaré's view of the electromagnetic matter, the inverse cubic forces necessary in the formulation of Wagner's theorem *have a reality* that can be associated with some transport properties of the ensemble serving for interpretation. We show in this finale of the present instalment of our work, that this is, indeed, the case, provided *the Bohr's quantization is allowed to enter the stage*. In other words, we reiterate the old idea contained in the Kirchhoff's laws of radiation, that the equilibrium radiation in a coordinate space, like the Wien-Lummer 'hohlraum' serving to the study of radiation, needs matter for its physical realization. However, that matter needs to have a given physical structure, in order to be able to insure the possibility of those transport properties invoked by Poincaré in the description of his 'fictitious' fluid. This idea is not new at all: it has already been clearly expressed more than a century ago by Sir Joseph James Thomson, as an alternative to *energy quantization*. As it turns out, it has to be taken into consideration from this new perspective too. Quoting:

> It is usual to regard Planck's equation ($W = h\nu$, a/n) as showing that *radiant energy is molecular in its structure*. I have suggested… that the same result would follow *without any such assumption as to the character of radiant energy*, if the mechanism in the atom by which the radiant energy is transformed to kinetic energy, is such as to require the transference to the mechanism of a definite amount of energy, sufficient, for example to rupture some system before the transference can take place; that in fact *Planck's relation depends on the properties of atom, the agent by which the energy is transformed*, rather than upon the existence of a structure in the energy itself. [(Thomson, 1913), *our emphasis*]

Our contention is simply that here the atom in question is well represented by Rutherford's planetary model, indeed, as in fact was, historically speaking, sanctioned by the physics of last century. However, this does not mean at all that the 'radiant energy is *not* molecular in its structure', as Thomson contended. True, this may seems a kind of King Solomon's judgment: *everybody is right*, but here we can bring the positive argument that follows, in order to prove that such is, indeed, the case even logically, with outstandingly trustworthy testimonies. No need to cut the baby to pieces in order to reveal the truth!

6.1 A Proper Framing of some Historical Facts

The name of Sir J. J. Thomson is conspicuously missing from the specialty literature regarding the Ermakov-Pinney equation. While in connection with the mathematics of this equation, people have touched just about every aspect of the problem triggered by the existence of this equation, including the historical moment of its true 1874 birth [see (Morris & Leach, 2017), and the literature cited there], when it comes to the physics of quantization procedure connected to Ermakov-Pinney equation one cannot pronounce the same. Beyond the fact that it is associated by Harold Lewis' work with a generalization of the Planck's constant – defined by the ratio between the energy and the frequency of light – there does not seem to be a struggle to discover any other historical occasions to cite this equation. And yet, there is an occasion connected with the very Planck's constant in need of generalization, and not just any generalization but the one naturally in keeping with the electromagnetic nature of that constant, as suggested by the Poincaré's idea. To wit, J. J. Thomson proposed this equation as early as 1910, and judging by the works that led to this proposal, the idea may have been published somewhere even earlier, but we were not able to locate the documents (Thomson, 1910, 1913). Let us therefore, discuss in some details the Thomson's research conclusions, in order to reveal what Poincaré's idea really involves.

In order to discuss J. J. Thomson's philosophy it is best to start with his own summary, from the paper where the great physicist puts forward, apparently for the first time ever in connection with the Planck's quantization of the electromagnetic field, an Ermakov-Pinney equation, considered as a *dynamical* equation of motion (Thomson, 1910). Thus, quoting:

> In the Philosophical Magazine for August 1907, I discussed a theory of radiation from hot bodies which regarded the radiation as arising from the impact of negatively charged corpuscles with the molecules of the body; the impact starting electric pulses which collectively constitute radiation from the body. When we resolve, by Fourier's theorem, this radiation into its constituent harmonic vibrations, we find that the amount of light of any given period depends upon *the ratio of that period to the time occupied by a collision*. It was shown, moreover, that this radiation would not conform to the Second Law of Thermodynamics, unless the *time occupied by a collision* varied *inversely as the kinetic energy of the corpuscle* before it came into collision, and in addition, that the time of collision of a corpuscle moving with a given speed *must be constant and independent of the nature of the molecule against which the corpuscle collides*. I showed that *the first* of these conditions would be satisfied if *the forces exerted during the collision* between a corpuscle and a molecule *varied inversely as the cube of the distance* between them; *the second* condition will be satisfied if *the collision is regarded as taking place* not between the corpuscle and the molecule as a whole, but as *between the corpuscle and systems dispersed through the molecules*, these systems being of the same character in whatever molecules they may be found, and repelling the corpuscle with forces varying inversely as the cube of the distance between them. *Forces of this type would be exerted by electric-doublets of constant moment* with their negative ends pointing to the corpuscles.
>
> In this paper I shall consider more in detail the collision theory of radiation when the forces exerted during collision vary inversely as the cube of the distance between colliding bodies. [(Thomson, 1910); *our emphasis*]

There are a few important observations to be made on this excerpt, in order to pinpoint some fundamental issues. First, notice that, just like in the case signaled by Clausius in his critique on the Betti's contribution to Riemann's electrodynamics (see §5.4 above), the radiation produced inside a solid is, according to Thomson's theory, described by harmonic components whose amount depends on the ratio between their frequency and the inverse of the 'collision time' [equation (5.4.20)]. What we want to say is that an Ampère wire has to be assimilated with a coordinate space where the radiation is omnipresent, just like Poincaré once contended. Secondly, that period of time 'occupied by the collision' is a *statistic* decided by the inverse of the 'kinetic energy of corpuscle just before collision'. As known, a statistic connected with the kinetic energy is the absolute temperature, but it needs to be independent of the nature of the target molecules and, besides that, it still needs a 'sample' in order to be quantitatively defined, like any statistic for that matter. This sample is here taken care of by a 'system dispersed through the molecules'. In a word, a collision is here statistically described as a *sampling* defining the local temperature, whereby the sample is simply given by those systems dispersed among the molecules. The temperature is here taken by Thomson as a *sufficient statistic* though, just like in the classical case of the ideal gas [for the concept of sufficiency see (Mazilu, Agop, & Mercheș, 2020), the discussion in Chapter 2]. In order to be well defined from physical point of view, the sample needs to be *statically defined*, whence the intervention of a omnipresent central force going inversely with the cubic distance. Thus, the Thomson's collision event can be seen as a transfer of the kinetic energy of the charged particle to a *sample* of static 'systems' in a field of forces.

For the rest everything goes just smoothly, from a purely mathematical point of view: having to do with an interpretation based on the equation

$$\ddot{r} = \frac{\kappa^2}{r^3} \tag{6.1.1}$$

this is a Berry-Klein gauging, knowing to have a transcendent conservation law (see §4.3)

$$\dot{r}^2 + \frac{\kappa^2}{r^2} = const \tag{6.1.2}$$

which is taken by Thomson as essential in carrying out his calculations. For our own future benefit, we still have to notice one important thing here: the conservation law (6.1.2) has to vary on the *ensemble of Thomson dispersed systems*. This would mean that, in this case, the sufficiency of statistic is not quite absolute as in the case of the classical ideal gas: the temperature for an individual system does not make sense. If we submit the quantity (6.1.2) to a limited time variation expressed by the invariance of the time integral of (6.1.2), then the following equation of motion is obtained:

$$\ddot{r} = -\frac{\kappa^2}{r^3} \tag{6.1.3}$$

We read this equation as saying that the force between elements pertaining to different Thomson dispersed systems are attractive central forces, going with the inverse cubic power. They are characteristic to what we would like to call an 'inverse Hubble law', giving the radial velocity, as it were, by the equation coming out from the vanishing of the de Alfaro-Fubini-Furlan Lagrangian [see equation (4.3.3)], which gives the velocity

$$\dot{r} = \pm\frac{\kappa}{r} \quad \therefore \quad \ddot{r} = -\frac{\kappa^2}{r^3} \tag{6.1.4}$$

The 'Hubble law' proper describes in the very same manner an *inverted harmonic oscillator* [see equation (4.2.20)], whose equation of motion is given by the stationary fluctuations of the energy. That is, the null Lagrangian of a corresponding straight oscillator defines the velocity by a 'Hubble law':

$$\dot{r} = \pm\lambda r \quad \therefore \quad \ddot{r} = \lambda^2 r \tag{6.1.5}$$

which is useful in tunneling problems of the kind of Thomson's problem above.

However, there is a major problem here: one cannot deny the presence, at least at a certain scale, of the central Newtonian inverse quadratic force, if not for anything, at least for the form of Coulomb forces. In this respect the Thomson's argument is that we can by no means assume that the forces due to charges inside the atom would have the same character as the forces between charges we can control. Some kind of average may intervene to give the Coulomb forces as means over some ensembles. More precisely a system of forces composed of:

> (1) A radial repulsive force, varying inversely as the cube of distance from the centre, diffused throughout the whole of the atom, combined with
>
> (2) A radial attractive force, varying inversely as the square of the distance from the centre, *confined to a limited number of radial tubes in the atom* [(Thomson, 1913); *original emphasis*]

all over the coordinate space encompassed by an atom will do the job. The main point of these two hypotheses, is that in the regions delimited by those radial tubes there are positions of equilibrium of the two forces, and that equilibrium is stable, for there is an elastic restoring force. Indeed, the dynamical radial equation of motion of a corpuscle of mass *m* and charge *e*, perturbed radially by an amount *q* from the equilibrium radial coordinate r_0 is

$$m\ddot{q} = \frac{Ce}{(r_0+q)^3} - \frac{Ae}{(r_0+q)^2}, \quad r_0 = \frac{C}{A} \tag{6.1.6}$$

showing that, in the first order, the equation of *q* is that of a harmonic oscillator:

$$\ddot{q} + \omega^2 q = 0, \quad \omega^2 \equiv \frac{Ce}{mr_0^4} \tag{6.1.7}$$

It is this last result that stirred up one of the most important fundamental issues of the last century physics, the one that led to the very creation of quantum mechanics: *extending the quantization from the field of electromagnetic radiation, as done initially by Planck, to the matter in general*. Thomson's approach to this issue may be taken as fundamental from theoretical point of view, and goes along the following lines.

It is only natural to assume that any real perturbation – for instance a collision event – of an electric corpuscle located at the equilibrium distance in an atom described by the above fields of forces, will lead to a harmonic motion described by equation (6.1.7). Then, Thomson's contention is that the frequency of this harmonic *disturbance* is to be found in the electromagnetic radiation emited on the occasion of a collision. According to Planck's quantization procedure, this radiation has an energy amounting to:

$$w \equiv \hbar\omega = \frac{\hbar}{r_0^2}\sqrt{\frac{Ce}{m}} \tag{6.1.8}$$

The specific dependence of this energy on the equilibrium position inside Thomson's tubes, indicates that it cannot originate but in the field of universal inverse cube forces, vhose potential energy would be, according to its classical definition:

$$W \triangleq \int_{r_0}^{\infty} \frac{Ce}{r^3} dr = \frac{1}{2}\frac{Ce}{r_0^2} \tag{6.1.9}$$

Then the constant *C* of the universal third inverse power forces can be evaluated by the conservation law of energy, from which Thomson gets

$$C = \frac{4\hbar^2}{em} \tag{6.1.10}$$

which is about 3.05×10^{-19} SI units.

At this juncture it is worth mentioning the difference between the *three* major quantization procedures for the case of matter, in order to put things in a somewhat logical order. The *first* comes Arthur Erich Haas' quantization procedure from 1910, which uses one single field of forces in atom – the inverse quadratic forces – and, therefore, is referring to the potential energy calculated with this field instead of (6.1.9) [see (Schidlof, 1911); see also (Hermann, 1965)]. Thus Haas' quantization condition is usually taken as giving the measure of extension of the atomic space: it gives the so-called Bohr's radius as

$$r_0 = \frac{4\pi\varepsilon_0\hbar^2}{e^2 m} \tag{6.1.11}$$

where ε_0 is the vacuum permittivity. Occasionally this procedure was characterized as giving a universal structure of the Planck's constant itself (Schidlof, 1911). It is significant, in connection to this theory of Schidlof, that it assumes a space structure of the Planck's resonator, much in the spirit of Thomson's proposal for the atomic structure. The second quantization procedure on matter, is that well-known and largely publicized one, of Niels Bohr, on which we will have to dwell at some length in this very chapter, in order to settle the ideas about the Berry-Klein gauging procedure; so, we only mention it here, just for the order of things connected with this topic. Finally there is the Louis de Broglie's quantization procedure – the subject-matter of this very chapter – given by us previously in the equation (2.1.4), and expressing the fact that the Planck's quantum is connected to the energy of matter, even for the simplest if its formations. It is our contention that all these ideas are to be clarified in a Berry-Klein gauging theory, as it has been initiated by Sir J. J. Thomson at the beginning of the last century.

Continuing on with the Thomson's theory, one has to notice that there is not any reason to assume the validity of the equation (6.1.9), while knowing that the Newtonian field of force is also present in the space of atom. However the Thomson's tubes of flux, assumed to exist in that space, save the day. Quoting:

> If a corpuscle at P were *inside one of the tubes of attractive force* inside the atom, it could be removed to an infinite distance: (1) *by moving it gradually outwards and keeping it inside the tube the whole way*. If the attractive force on unit charge at a distance r from the centre is A/r^2, the work required to remove the corpuscle in this way from r to an infinite distance is Ae/r. The corpuscle could however be moved to an infinite distance in another way (2) *by moving it sideways out of the tube at P and then moving it outside the tube to an infinite distance*; this later process will absorb no work as the attractive force vanishes outside the tube. By the conservation of energy, the work required must be the same, whether we adopt the process (1) or (2); hence *the work required to move the corpuscle sideways* out of tube at P, must be equal to Ae/r. [(Thomson, 1913); *our emphasis*]

In what is left from this very section we will be occupied with that 'motion sideways' that seems to be counterintuitive, even paradoxical we should say, in a central field. It is our belief that, by and large, this was the main reason for the fact that the gauging theory of Thomson's was so easily dismissed in the favor of Bohr's theory that sanctioned the planetary model.

The geometrical reference frame of unit vectors ($\hat{\boldsymbol{e}}_r$, $\hat{\boldsymbol{e}}_\theta$, $\hat{\boldsymbol{e}}_\varphi$), associated with the spherical coordinate system in an *ordinary space*, as in §2.3 above, is not the unique reference frame that can be used in the physics of space, especially when this space is a *coordinate space*. More to the point, it has a correspondent in any coordinate space, provided a gauging is enforced. To wit, speaking of the inverse cubic central force, Thomson mentions the fact that these forces are generated by 'electric-doublets of constant moment', and this speaks plainly of a kind of Ampère generalization, if we may say so, of the pairs of material particles in a coordinate space: both ends of the pair – or the dipole – are freely variable in space. The spherical coordinate system is just a particular case, where one of the ends is fixed, offering a static stance necessary to the process of interpretation. It is not too hard to see, in this case, that the length of a dipole can be taken as a *gauging length*, and this length generate an infinite number of Cartesian reference frames on the coordinate space in question. First, we can think of a fictitious Cartesian reference frame in which the gauge length can be decomposed in three components, according to classical geometrical habit. One can say that a gauge length defines an ∞^2 such reference frames, depending continuously on two parameters, according to the freedom given by equation $r^2 = x^2+y^2+z^2$, where r is the gauge length. As one can see, there are just two free parameters on which our family of Cartesian reference frames depends, for a fixed gauge length, which, in a 'static stance' described in spherical coordinates, are taken as spherical polar angles of longitude and colatitude. However, just to keep things in perspective, we need to stress that these parameters are, in general, two arbitrary real ones. The whole point here is that the general theory of gauging has to account for this fact.

In order to do this we first define a generic reference frame independent of any geometry, but still retaining a geometrical spirit. We present here a case of 'second degree' vectors in Cartesian coordinates, that can be very well taken as a Cartesian reference frame in the general case, for a reference length geometrically described as a distance r, and therefore for an arbitrary gauge length. Thus, we take three unit vectors in the matrix form:

$$\hat{\boldsymbol{e}}_1 = \begin{pmatrix} 2\dfrac{x^2}{r^2}-1 \\ 2\dfrac{xy}{r^2} \\ 2\dfrac{xz}{r^2} \end{pmatrix}, \quad \hat{\boldsymbol{e}}_2 = \begin{pmatrix} 2\dfrac{xy}{r^2} \\ 2\dfrac{y^2}{r^2}-1 \\ 2\dfrac{yz}{r^2} \end{pmatrix}, \quad \hat{\boldsymbol{e}}_1 = \begin{pmatrix} 2\dfrac{xz}{r^2} \\ 2\dfrac{yz}{r^2} \\ 2\dfrac{z^2}{r^2}-1 \end{pmatrix} \tag{6.1.12}$$

where (x, y, z) are *any* three real numbers satisfying the quadratic condition $r^2 = \sum x^2$. These vectors form, indeed, an orthonormal reference frame, as one can easily verify. A description of the possible origin and realm of this reference frame should be salutary for what we have to say further in this work.

The classical Poisson equation was, from the very beginning (Poisson, 1812) taken to mean the preponderance of matter over the field. Both concepts where naturally brought to human knowledge, with this apparently natural 'dominance', if we may say so, by the Newtonian theory of forces (see §5.2 above). There is, however, a notorious case whereby the Poisson's equation is 'rustled', as it were, from the above-mentioned classical habit of *defining the field* when the *density of matter* is given, and used *into defining the density* when *the field is given*, which is one main point in the Louis de Broglie's physical doctrine (see §2.4). Considering this doctrine, we are compelled to find such an idea significant even from another point of view, namely because it is a construction that served to build an image of the electric ether based on *considerations of statics*. Indeed, the essence of the problem of ether, as we already presented it up to this point, is well represented by the so-called *Maxwell stress system*, described by Clerk Maxwell in the Chapters 4 and especially 5 of the first volume of his classical treatise (Maxwell 1873). Even though this system is mostly cited as an example of the failure to describe the ether as what is classically conceived as an isotropic incompressible medium [see especially (Love 1944), where the system of Maxwell stresses is presented from different mechanical perspectives in various places of the book], we think that it is still in position to straighten up some of our modern physical concepts, especially that of *static ensemble of equilibrium*, so necessary to any theory of interpretation. As we said, Maxwell himself did not seem to have used his system of stresses, as defined this way, very much. In hindsight, this appears to have happened mostly because he seems to have been carried away by the electromagnetic image of the light, whereby the dynamics appears to be the essential working ingredient. By the same token, however, the subsequent neglect of the Maxwell stress system in physics may have been due to a deeper, objective reason, that can be delegated to the necessity of interpretation in physics. We will turn back to this important issue later in this work.

Maxwell's problem was to find the stresses induced by the action of forces in ether, in order to explain the omnipresent gravitational and electric forces. The attraction was represented in those times, just as it is today, by the Newton's forces, which proved also to be valid for electricity, as Charles Coulomb would have long shown. Maxwell did not take into consideration this property directly, but first translated it into a problem *involving the continua*: finding the stresses *statically equivalent* with a system of forces in general. Notice that these stresses had also to face, later on, the fact that the matter does not seem to be dragged by ether, which was proved experimentally toward the end of the 19th century. This circumstance too, may have participated to ignoring the case as inessential, inasmuch as neither the gravitation, for instance, nor the electric action could be consequently explained as drag forces. This conclusion was even reinforced by Henri Poincaré, who specifically showed that the electric matter of Lorentz is in default with respect to classical dynamics, inasmuch as it does not obey the classical principle of action and reaction (Poincaré, 1900). He even pushed this property into describing the forces of gravitation, and the forces of cohesion of matter in general, thus inventing the so-called *Poincaré stresses* (Poincaré, 1906).

The mathematics of a force generated by matter was in those times, and still is today for that matter, expressed by the Poisson equation, which we rewrite here in the form:

$$\nabla^2 U(x,y,z) = 4\pi\rho(x,y,z) \tag{6.1.13}$$

In this equation $U(x, y, z)$ is the potential of the forces in the medium of density $\rho(x, y, z)$. If this medium is electrically active, then ρ is the density of electricity and U is an electric potential. Maxwell apparently took equation (6.1.13) as *defining the density* of the medium, rather than the potential, for the following good reason: he proved that the equation of equilibrium of a system of stresses is satisfied with the volumetric forces corresponding to a matter with density given by (6.1.13). Indeed, the equation of equilibrium of a continuous stress system in general (Love, 1944) which, in its simplest form, asserts that the divergence of the second order stress tensor, $\boldsymbol{t}$ say, is given by the density of volume forces $\boldsymbol{f}$:

$$\nabla \cdot \boldsymbol{t} + \boldsymbol{f} = \boldsymbol{0} \tag{6.1.14}$$

when specifically applied to the stress tensor $\boldsymbol{t}$ defined by the matrix

$$\begin{pmatrix} \frac{1}{8\pi}\left[2\left(\frac{\partial U}{\partial x}\right)^2 - (\nabla U)^2\right] & \frac{1}{4\pi}\frac{\partial U}{\partial x}\frac{\partial U}{\partial y} & \frac{1}{4\pi}\frac{\partial U}{\partial x}\frac{\partial U}{\partial z} \\ \frac{1}{4\pi}\frac{\partial U}{\partial x}\frac{\partial U}{\partial y} & \frac{1}{8\pi}\left[2\left(\frac{\partial U}{\partial y}\right)^2 - (\nabla U)^2\right] & \frac{1}{4\pi}\frac{\partial U}{\partial y}\frac{\partial U}{\partial z} \\ \frac{1}{4\pi}\frac{\partial U}{\partial x}\frac{\partial U}{\partial z} & \frac{1}{4\pi}\frac{\partial U}{\partial y}\frac{\partial U}{\partial z} & \frac{1}{8\pi}\left[2\left(\frac{\partial U}{\partial z}\right)^2 - (\nabla U)^2\right] \end{pmatrix} \tag{6.1.15}$$

is identically satisfied for a force density $\boldsymbol{f}$ given by

$$\boldsymbol{f} \triangleq \frac{1}{4\pi}(\nabla^2 U)\cdot\nabla U \tag{6.1.16}$$

In other words, this stress system is *statically equivalent* with the system of volume forces of the matter having a density given by Poisson's equation. Thus the gravitation, for instance, can be conceived as a tension due to these stresses through ether, and likewise the electric force. The Poincaré conclusion about Lorentz material system can be taken as showing that such a system of stresses is insufficient to do the job they are called for, no matter of the system of forces taken into consideration.

Now, if we replace the gradient components from the matrix (6.1.15) by *any* three real numbers, and then normalize the matrix to its trace, we get an orthogonal matrix whose columns are the vectors from equation (6.1.12). It is in this sense that we declare that such a reference frame can be connected with any gauge length, we might say, in a heuristic sense only. There are things to be observed though, if we are to be thorough, but we postpone the discussion for later in this work. For now, we just concentrate on the system of the three orthogonal and normalized vectors given in equation (6.1.12), and their connection with Thomson's ideas as presented here. Thinking of a dipole length as of a gauge length, we are led to the theory of Ampère generalization of the Newtonian forces, where this length, denoted r and considered the Euclidean distance between two moving point charges, plays an essential part in calculation of the forces (see especially §5.4 above). Let us treat it independently, as a *linear potential* in the Maxwell theory above. First we have

$$\nabla U = \frac{\boldsymbol{r}}{r} \quad \therefore \quad \nabla^2 U = \frac{2}{r}$$

and therefore the field of static forces (6.1.16) is given by the intensity

$$\boldsymbol{f} = \frac{1}{2\pi}\frac{\boldsymbol{r}}{r^2} \tag{6.1.17}$$

and correspond to a Maxwell tensor

$$\boldsymbol{t} = \frac{1}{8\pi}\begin{pmatrix} 2\left(\dfrac{x}{r}\right)^2 - 1 & 2\dfrac{xy}{r^2} & 2\dfrac{xz}{r^2} \\ 2\dfrac{xy}{r^2} & 2\left(\dfrac{y}{r}\right)^2 - 1 & 2\dfrac{yz}{r^2} \\ 2\dfrac{xz}{r^2} & 2\dfrac{yz}{r^2} & 2\left(\dfrac{z}{r}\right)^2 - 1 \end{pmatrix} \tag{6.1.18}$$

These two last equations suggest an interesting story, which we need to unravel along this work. Going a little ahead of us here, that story can be summarized as follows.

Let us notice that the columns of the matrix from equation (6.1.18), are the vectors from equation (6.1.12). Therefore the matrix is orthogonal. It represents a *Maxwell continuum*, once 'criticized' by Eugenio Beltrami as being unrealistic according to human experience (Beltrami, 1886). We should even add more on the Beltrami's conclusions: standing on the words of Poincaré (see §3.2 above) this continuum is fictitious according to human experience. However, in a note of disagreement with the great scholar, we have to add that, far from being a drawback, this quality is, in fact, its 'strength', so to speak. As a matter of fact, it cannot have any physical reality without an interpretation, and the key to its interpretation stays in the static forces (6.1.17) equivalent to this tensor, in the sense described above.

To wit, we need to notice that, in an Euclidean reference frame – which, by the way, cannot be but fictitious here – the vector field from equation (6.1.17) is a radial inversion of the very position vector with respect to a sphere of radius $(2\pi)^{-1/2}$. Now, *if a Berry-Klein gauging is available*, we can say even more about this inversion: by assigning a *realistic, therefore, physical radius* to the inversion sphere, we can make an 'elastic Berry-Klein force', as it were, out of the position vector [see equation (4.2.7)], and thus, using an inversion with respect to the sphere so defined, we have a central force like that from equation (6.1.17). This last force has a magnitude going *inversely with the distance* – not with *quadratic* or *cubic power* of distance, like the forces involve in Thomson's discussion, but with the *first power* – and this is one of the most important facts: according to Dennis Sciama, this is the static force *responsible for inertia* [for a more comprehensive story on this subject, and some of the physics around it, see (Mazilu, Agop, & Mercheș, 2020), especially the Chapter 3]. And so, our story can be wrapped up as a gauging description of the inertia property of matter, as follows.

The sphere of inversion is to be seen as *an absolute*, beyond which the matter is only imaginarily accessible: this is that distant matter which, according to Mach's principle, induces the inertia in the local matter. Using the inversion with respect to absolute as an 'induction procedure', so to speak, this local matter can be described as a Maxwell continuum, in a geometry to be constructed by means of what we would like to call *Beltrami reference frames* – in view of the fact that the great mathematician was their promoter along this line of thought – as the one given in equation (6.1.12). As we have said, this conclusion is pending on an *available* Berry-Klein gauging procedure, in order to turn the positions into 'elastic Berry-Klein forces'. According to Wagner's theorem this procedure is indeed available under the first of Thomson's hypotheses regarding the fields of forces inside the space of an atom [see equations (6.1.6) and (6.1.7) above]. Of course, this closing of our story leaves many points to be explained, which constitute just as many points of necessary clarifications to be delivered by our work. For the moment being, let us limit the discourse to the purpose intended in the beginning of the story: that counterintuitive 'sideways motion' inside a tube of common action of the two fields of forces involved in Sir J. J. Thomson's natural philosophy.

6.2 On the J. J. Thomson's Tube, and Sideways Motions

Notice that in a Beltrami reference frame any gauge length can be represented as an Euclidean vector

$$\boldsymbol{r} = x\hat{\boldsymbol{e}}_1 + y\hat{\boldsymbol{e}}_2 + z\hat{\boldsymbol{e}}_3 \tag{6.2.1}$$

so that the reference frame is *uniquely* associated with the gauge length: there is but a single reference frame associated with any decomposition of the gauge length in the sense of algebraic relation $r^2 = \sum x^2$. In this sense, one can say that a Cartan-type parallelism [see Chapter 11 in (Mazilu, Agop, & Mercheș, 2019)] cannot be defined. And as this type of parallelism is to be recognized as the most general possible parallelism – at least from a physical point of view – one can very well infer that the parallelism is not, in general, possible in this world. The torsion though, can be still available here by the variation of density (Delphenich, 2013), and this may be related to a *gauge freedom*, if we may say so, because, as a gauge length, *r can vary independently*. Moreover, there is a definite possibility of describing this independent variation statistically, therefore maintaining the space scale, in acordance with the common experience: a statistical procedure does not change the space or time scale of things things involved in that statistics [for further reference see (Mazilu, Agop, & Mercheș, 2020), Chapter 4]. We extend this gauge freedom at the infrafinite scale, by asuming a certain independence between what we make out geometrically of the gauge length, during the gauging procedure, and the gauge length itself. And, as we geometrically make a vector out of the gauge length, this freedom would mean that the variation *dr* of the gauge length is somehow independent of the variations of the components of the vector $\boldsymbol{r}$. In an Ampère generalization of the central forces this fact should have an overwhelming importance, so that we have to elaborate on it a little bit longer.

Luckily for us, there was a case in history specifically addressed to a problem just like the one above and, moreover, even in connection with the equation which is the basis of wave mechanics as we have it today. Naturally, then, we can consider this case for guidance in our proceedings. To wit, let us quote:

> In the conventional textbook treatment of *the Schrödinger equation* for the one-electron atom, *spherical coordinates* are nearly always employed. The student is often given to believe that spherical coordinates are *the only ones* that can be used in central-field problems. It is the purpose of this note to point out a method for solving Schrödinger's equation for the one-electron atom *in rectangular coordinates*. The angular momentum eigenfunctions are found *without any reference to polar angles*. [(Fowles, 1962); *our emphasis*]

Just about for the same reasons, the purpose of the present section of our work is to show that using rectangular coordinates according to Fowles' prescription, means much more than purifying our wits of the domineering importance of spherical coordinates. It may mean, for instance, that the nature of any conservative force, other than those involved in the Thomson's construction, is 'holographic', so to speak, inasmuch as the Fowles' method brings in *the spin concept* in an analytic form, involving the idea of a physical surface (Mazilu, Ghizdovăţ, & Agop, 2016). But, let us see what is the essence of Fowles' method. Quoting, again:

> Suppose we regard the wave function *as a function of* x, y, z, *and* r, where $r^2 = x^2+y^2+z^2$. That is, whenever the combination $x^2+y^2+z^2$ occurs, it will be written r^2. [(Fowles, 1962); *our emphasis*]

In our context here, the Ampère's extension to Newtonian forces stands witness as a typical case where this prescription may be necessary. Hoping that we applied it correctly to our very case, the results are as follows: there are Frenet-Serret equations for the reference frame (6.1.12), that can be written in the form

$$d\hat{\boldsymbol{e}}_k = \sum\nolimits_j \Omega_{kj} \cdot \hat{\boldsymbol{e}}_j \tag{6.2.2}$$

where $\boldsymbol{\Omega}$ is a skew symmetric matrix, having the components

$$\Omega_{12} \triangleq \hat{\boldsymbol{e}}_2 \cdot d\hat{\boldsymbol{e}}_1 = 2\frac{xdy - ydx}{r^2}, \quad \Omega_{23} \triangleq \hat{\boldsymbol{e}}_3 \cdot d\hat{\boldsymbol{e}}_2 = 2\frac{ydz - zdy}{r^2}, \quad \Omega_{31} \triangleq \hat{\boldsymbol{e}}_1 \cdot d\hat{\boldsymbol{e}}_3 = 2\frac{zdx - xdz}{r^2} \tag{6.2.3}$$

Thus, the entries of the Frenet-Serret matrix can be assimilated as the components of the solid angle in a spherical representation. Just for the record, and also for possible later purposes, using the spherical polar angles (see §2.3), we have the following entries:

$$\begin{aligned} \Omega_{12} &= 2\sin^2\theta d\varphi, \\ \Omega_{23} &= 2\sin\varphi d\theta + \sin 2\theta \cos\varphi d\varphi \\ \Omega_{31} &= -2\cos\varphi d\theta + \sin 2\theta \sin\varphi d\varphi \end{aligned} \tag{6.2.4}$$

Now, we need to calculate the differential of a certain vector in this reference frame, just as we have done before in §2.3, for the reference frame related to spherical coordinates.

Thus, let us consider the vector force, whose components with respect to the reference frame (6.1.12) are taken as contravariant components, just for convenience should say, in view of the position of indices of the unit vectors of the frame:

$$\boldsymbol{f}(\boldsymbol{r}) \triangleq f^k(\boldsymbol{r})\hat{\boldsymbol{e}}_k \tag{6.2.5}$$

Its differential can be written as

$$d\boldsymbol{f}(\boldsymbol{r}) = Df^k(\boldsymbol{r})\hat{\boldsymbol{e}}_k, \quad Df^k \equiv df^k + f^j \boldsymbol{\Omega}_j^k \tag{6.2.6}$$

where the summation convention over dummy indices of different variances enters the game. This, as it can be seen on our formula, needed a sudden change in the position of one of the indices of the entries of matrix $\boldsymbol{\Omega}$, with the following explanation: if we have to use a general formalism for this geometry, then we need to forget about the Cartesian notation, and adopt an indicial one. According to this notation, a position vector can be written as in the equation (6.2.5), just like any vector, so that the coordinates are to be considered, even if just conventionally for now, as its contravariant components in a Beltrami reference frame. Thus, we have a formula entirely analogous to (6.2.5):

$$\boldsymbol{r} \triangleq x^k \hat{\boldsymbol{e}}_k \tag{6.2.7}$$

and therefore, instead of (6.2.6):

$$d\boldsymbol{r} = (Dx^k)\hat{\boldsymbol{e}}_k, \quad Dx^k \equiv dx^k + x^j \boldsymbol{\Omega}_j^k \tag{6.2.8}$$

Calculating the components of this vector, based on equation (6.2.3) results in

$$Dx^k \equiv -dx^k + 2x^k \frac{dr}{r} \tag{6.2.9}$$

but – and this is very important here – with the observance of Fowles' rule, and the following consideration of the entries of the matrix $\boldsymbol{\Omega}$:

$$\boldsymbol{\Omega}_j^k \equiv 2\frac{x^k dx_j - x_j dx^k}{r^2} \tag{6.2.10}$$

As a *mnemonic* rule, therefore: for each position, any pair of the two kinds of coordinates (x^k, x_j) can be interpreted as a pair of coordinates on a surface, say a *de Broglie surface*, at the finite scale. Corespondingly, at the infrafinite scale, we have the pair of differentials (dx^k, dx_j). Then the equation (6.2.10) represents the dot product of the finite pair of coordinates thus defined, with the infrafinite pair (dx_j, $-dx^k$), which can be written as an expression involving the fundamental skew-symmetric two-dimensional matrix:

$$\boldsymbol{\Omega}_j^k = \frac{2}{r^2}\begin{pmatrix} x^k & x_j \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} dx^k \\ dx_j \end{pmatrix} \tag{6.2.11}$$

The upper and lower indices of this second order tensor, are the indices of the two coordinates in the pair. The problem of definition of the covariant components *remains in suspension* for now, because, in general, a metric may not be available, in order to follow the classical route.

Now, if the differential forms (6.2.9) are the fundamental differential forms in this world, then the *covariant* components of a force vector, which are known to be generally defined by the elementary mechanical work, are specifically defined by the differential form:

$$DW = f_k(\boldsymbol{r})(Dx^k) \tag{6.2.12}$$

respecting the Fowles' prescription. The notation DW signifies that, just like in the case of coordinates, the elementary variation is, in general, not an exact differential. Consequently, in view of (6.2.9), this differential form can be expanded as

$$DW = -f_k(\boldsymbol{r})dx^k + 2[x^k f_k(\boldsymbol{r})]\frac{dr}{r} \tag{6.2.13}$$

which shows that this differential form is not an exact differential even if the covariant force vector is conservative, as in the classical case. Indeed, assuming that the covariant force derives from a potential depending only on the coordinates x^k, but not on the sum of their squares, we can write

$$f_k(\boldsymbol{r}) = \frac{\partial U(x^1, x^2, x^3)}{\partial x^k} \quad \therefore \quad DW = -dU + 2\left(x^k \frac{\partial U}{\partial x^k} \right) \frac{dr}{r} \tag{6.2.14}$$

showing clearly in what conditions DW is an exact differential, by comparison with the classical case. As one can see, everything depends here on the *projection of the conservative force along the position vector*. In particular, we can say that if the force acts perpendicularly to the position vector, the potential is a homogeneous function of degree zero, and we have the case where the Ampère's forces, as decribed by us previously, can enter the stage in the manner presented by us. Of course, we shall need some details from our theory.

For the moment, notice that everything depends here on the coefficient of the differential $d(\ln r)$ from equation (6.2.13). It is the celebrated *virial of forces*, and was introduced to theoretical physics by Rudolf Clausius in the attempt of connecting the fields of forces, within the molecular ensembles, with thermodynamics, *via* statistics [(Clausius, 1870); the two articles to be found under this heading in the literature listed here are practically the same regarding their content. However, the French version is particularly suggestive as to what virial means along the lines of the present section of our work: "a quantity analogous to potential"]. That attempt was, in fact, one of the classical cases of interpretation, where, like in the later Lorentz case (see §6.1 above), the continuum to be interpreted is conspicuously missing. And here is the virial, again, appearing naturally we should say, and most significantly, in close connection with a 'Maxwell continuum', as described above.

In the general case, represented by equation (6.2.13), with no reference whatsoever to potential, one can merely say that the differential form DW can only be *closed* if its exterior differential is zero: $d\wedge(DW) = 0$. The existence of a potential is, in many cases, too much to ask. In quite general conditions, further respecting the Fowles' prescription, this equation comes down to

$$df_k(\boldsymbol{r}) \wedge Dx^k = 0 \tag{6.2.15}$$

whence, according to Cartan's Lemma we need to have

$$df_k(\boldsymbol{r}) = \Lambda_{kj} dx^j, \quad x^k df_k(\boldsymbol{r}) = \Lambda \frac{dr}{r} \tag{6.2.16}$$

for some conveniently chosen symmetric matrix $\boldsymbol{\Lambda}$ and function Λ. Obviously, these two cannot be quite independent of each other, for we must have

$$\Lambda_{kj} x^k dx^j = \Lambda \frac{dr}{r} \tag{6.2.17}$$

Just to settle our ideas, choosing $\boldsymbol{\Lambda}$ as a matrix of constants, and Λ as a constant, will give r^2 *as a Gaussian*, and if the matrix $\boldsymbol{\Lambda}$ is positively defined and the constant Λ is positive, we shall have

$$\frac{1}{r^2} \propto \exp\left(-\frac{\Lambda_{kj}}{\Lambda} x^k x^j \right) \tag{6.2.18}$$

thus giving us a proper Gaussian distribution to be used in the Berry-Klein gauging procedure. In view of the fact that in the capacity of a gauging length r2 may be taken as the variance of a distribution, this observation is essential: the variance can be taken as a probability distribution. One regular case is that of a Cauchy distribution that can be taken as the variance of a family of distributions with quadratic variance function. The best classical example, however, remains the Planck law of radiation: physically, it is a distribution of the ratio between frequency and temperature in the spectrum; statistically, it represents the mean of a family of distributions with quadratic variance distributions (Mazilu, 2010). In general though, the equation (6.2.17) suggests a manner of lowering the indices offered by the matrix $\boldsymbol{\Lambda}$, which thus plays the part of a metric tensor. But, before launching ourselves on the way of mathematical speculations, let us see some facts.

There are reasons to believe that in the interpretation of matter, in general, the absolute temperature is no more a *sufficient statistic* as in the classical case of the ideal gas [for a detailed story of the facts leading to this conclusion, see (Mazilu, Agop, & Mercheș, 2020), the discussion in Chapter 2]. This fact is also obvious in Thomson's theory, as we have noticed before. The idea is that in view of the Planck quantization procedure, which turns out to be a necessity, some forces *must exist* inside any material physical structure – even ideal gases – *whose virial is constant*. Therefore, no interpretation is apparently possible, for no ensembles of classically free particles exist. All we can hope under a Planck quantization, is the existence of a gauging procedure based on a 'partial freedom', as it were, like the Berry-Klein gauging procedure, which asks only for a kind of… 'radial freedom'. As it happens, the condition of constant virial for such forces, means just that, provided the Fowles' prescription is followed. Indeed, assume the *physical* situation where we calculate the virial in a *fixed* reference frame, in a coordinate space. If this quantity is constant in the space described within this reference frame, we must have:

$$f_k(\boldsymbol{r})dx^k + x^k df_k(\boldsymbol{r}) = 0 \quad \therefore \quad dW + \Lambda\frac{dr}{r} = 0 \tag{6.2.19}$$

Here we have used the definition of the elementary mechanical work dW, and the second of the conditions (6.2.16) which is a necessary condition that the differential form (6.2.13) should be closed. Under the constant virial, DW should therefore read

$$DW = K\frac{dr}{r}, \quad K \equiv \Lambda + 2C \tag{6.2.20}$$

where C is the constant value of the virial. Therefore, within the Fowles' prescription, the constant virial forces should be *logarithmic* forces having the magnitude that goes inversely with the distance between particles. If these forces are conservative and central too, they *do not depend separately on coordinates, and have no radial component* in any reference frame. In other words, any particle under these forces is 'radially free', and therefore the Berry-Klein gauging procedure works. Let us prove this.

Any force *central* and *conservative* can be described by an equation like

$$\nabla U(\boldsymbol{r}) = f(\boldsymbol{r})\frac{\boldsymbol{r}}{r} \quad \therefore \quad (\nabla U)^2 = [f(\boldsymbol{r})]^2 \tag{6.2.21}$$

Here $f(\boldsymbol{r})$ is the magnitude of force, while $U(\boldsymbol{r})$ is a potential function generating the vector force *via* the operation of gradient. The centrality of forces cannot be declared but in a fixed reference frame, in which the equation (6.2.21) can be solved in a 'spherical' coordinate system. Using the notations from §2.3 in transcribing this equation, it looks like

$$\left(\frac{\partial U}{\partial r}\right)^2 + \frac{1}{r^2}\left[\left(\frac{\partial U}{\partial \theta}\right)^2 + \frac{1}{\sin^2\theta}\left(\frac{\partial U}{\partial \varphi}\right)^2\right] = [f(\boldsymbol{r})]^2 \tag{6.2.22}$$

which is the expression of the magnitude of forces having the spherical components

$$f_r(\boldsymbol{r}) = \frac{\partial U}{\partial r}, \quad f_\theta(\boldsymbol{r}) = \frac{1}{r}\frac{\partial U}{\partial \theta}, \quad f_\varphi(\boldsymbol{r}) = \frac{1}{r\sin\theta}\frac{\partial U}{\partial \varphi}$$

Now, the equation (6.2.22) can be solved by sum-separation method, much in the manner of solving the classical Hamilton-Jacobi equation. Provided, of course, *the magnitude* of force depends exclusively on the distance between particles, as in the case envisioned by Newton for the inverse quadratic forces. In this case we can write (6.2.22) in the form

$$r^2\left[\left(\frac{\partial U}{\partial r}\right)^2 - f^2(r)\right] = -\left[\left(\frac{\partial U}{\partial \theta}\right)^2 + \frac{1}{\sin^2\theta}\left(\frac{\partial U}{\partial \varphi}\right)^2\right] \tag{6.2.23}$$

and assume tentatively a solution of the form

$$U(r,\theta,\varphi) = R(r) + F(\theta,\varphi) \quad \therefore \quad f_r(\boldsymbol{r}) \equiv R'(r) \tag{6.2.24}$$

where a prime means differentiation with respect to the unique independent variable, as usual. The equation (6.2.23) can have such a solution if, and only if

$$r^2\left\{[R'(r)]^2 - f^2(r)\right\} = -\left[\left(\frac{\partial F}{\partial \theta}\right)^2 + \frac{1}{\sin^2\theta}\left(\frac{\partial F}{\partial \varphi}\right)^2\right] \equiv -\beta^2 \tag{6.2.25}$$

where β is a real constant. Thus we must have

$$r^2\left\{[R'(r)]^2 - f^2(r)\right\} = -\beta^2, \quad \left[\left(\frac{\partial F}{\partial \theta}\right)^2 + \frac{1}{\sin^2\theta}\left(\frac{\partial F}{\partial \varphi}\right)^2\right] \equiv \beta^2 \tag{6.2.26}$$

The conclusion to be drawn here, is that for *a field of central forces* having the magnitude going as the inverse distance between particles the forces *can have no radial component*. Indeed, in this case, if

$$f(r) = \pm\frac{\beta}{r} \quad \therefore \quad f_r(r) \equiv R'(r) = 0 \tag{6.2.27}$$

The particles in such a field are 'radially free' with respect to each other, and the Berry-Klein gauging procedure may be applied. This property defines free particles, indeed, but *not from classical point of view*, inasmuch as, the other two components of the force are possibly nonzero. Indeed, the other components of these forces can be further calculated in exactly the same manner of separation of the variables. That is, trying for the second of the equations from (6.2.26) a solution of the form:

$$F(\theta,\varphi) = \Theta(\theta) + \Phi(\varphi) \quad \therefore \quad f_\theta = r^{-1}\Theta'(\theta) \;\&\; f_\varphi = (r\sin\theta)^{-1}\Phi'(\varphi) \tag{6.2.28}$$

we have only forces acting 'sideways':

$$f_\theta(\boldsymbol{r}) = \pm\frac{1}{r}\sqrt{\beta^2 - \frac{\gamma^2}{\sin^2\theta}}, \quad f_\varphi(\boldsymbol{r}) = \pm\frac{\gamma}{r\sin\theta} \tag{6.2.29}$$

Now, we may not know, at this point, too much about the covariant coordinates of position, but we certainly can define some contravariant coordinates in a Beltrami reference frame, as involved in equation (6.2.7). True, the spherical coordinates define some other vectors (6.1.12) instead of those involved in the original notation from §2.3 but, just for settling the ideas, we preserve those notations. We have here the possibility of defining the contravariant coordinates using the *physical condition of constant virial*, if we make out of a Berry-Klein gauge length a geometrical position according to Maxwell prescription suggested in equation (6.1.15), whereby the vector (6.2.7) that has the form

$$\boldsymbol{r} = (\alpha r)\hat{\boldsymbol{e}}_r \pm \alpha r\sqrt{\beta^2 - \frac{\gamma^2}{\sin^2\theta}}\hat{\boldsymbol{e}}_\theta \pm \frac{\alpha r\gamma}{\sin\theta}\hat{\boldsymbol{e}}_\varphi \tag{6.2.30}$$

where α is a constant. The corresponding virial of these forces is, indeed, a constant

$$\boldsymbol{f}\cdot\boldsymbol{r} = \alpha(1+\beta^2) \tag{6.2.31}$$

Their elementary mechanical work is all invested in the 'sideways displacements'

$$d\boldsymbol{r} = \pm\alpha r\left\{\sqrt{\beta^2 - \frac{\gamma^2}{\sin^2\theta}}\left(\frac{dr}{r} + \frac{\gamma^2\tan\theta d\theta}{\beta^2\sin^2\theta - \gamma^2}\right)\hat{\boldsymbol{e}}_\theta + \frac{\gamma}{\sin\theta}\left(\frac{dr}{r} - \tan\theta d\theta\right)\hat{\boldsymbol{e}}_\varphi\right\} \tag{6.2.32}$$

and amounts to

$$\boldsymbol{f}\cdot d\boldsymbol{r} = \alpha\beta^2\frac{dr}{r} \tag{6.2.33}$$

which, of course, is an exact differential. This can count as Thomson's 'motion sideways', so Thomson's general theory has no cracks… provided we can say something about those central forces with magnitude going with the inverse length. This remains as one of our next chores in this work.

6.3 The Statical Condition on Particle Freedom

Mention should be made that Bohr's quantization prescription had, in fact, the possibility to explain the fundamentals of Thomson's theory, but only 'for those who know how to see them', to use one of Joseph Bertrand's expressions. And there were, indeed, very many of those at the time! To wit, we chose in the context here two works of Edwin Bidwell Wilson (Wilson, 1919, 1924), which stand – among a great many others for that matter – witness to the fact that the Newtonian spirit was not completely overwhelmed within the avalanche of new facts, awakened by the association of the light with the atomic structure, as the physics envisioned it at the end of 19th century. Incidentally, these works justify the 'Thomson's spirit' as it were, having a direct message for the triumphant quantization of any kind today. In order to make our point here, we follow closely the second of the works cited above. This work is explicitly aligned to a Newtonian natural philosophy. Not only Wilson lists the facts that should lead to the expression of force responsible for them but, most importantly, he has a critical philosophy regarding what should be considered as fact, and what should be taken, in any way, as product of our imagination. This is something to reckon with, mostly today, when such a distinction hardly seems to matter. Let us, therefore, follow closely Wilson's own idea from the beginning to its conclusions.

Wilson recognizes that *the only* experimental fact we have at our disposal in order to describe the classical hydrogen atom was the so-called Balmer's series formula, extracted from observational spectroscopic facts:

$$\nu = R(1/n_1^2 - 1/n_2^2) \tag{6.3.1}$$

Here ν is the frequency of the observed light, and R denotes *Rydberg's constant*. This formula was able to offer the closest account for the hydrogen frequencies in experimental observations. Wilson's problem was a typical problem of classical Newtonian philosophy: *the Balmer's formula is simply a new fact to be added to the classical phenomenology that led to Kepler planetary model*. Then the question is: how can we amend the planetary model, in order to account for this new fact of experience? And Wilson considered the Bohr's 'amendments' (Bohr, 1913) which have to do only with a… speculative experience, if we may say so, along the following lines.

Assuming that the hydrogen atom is a classical Keplerian system with *circular orbit* and the center of force in the center of the orbit, its physical description is pending on two equations:

$$E = U + mr^2\omega^2 / 2, \quad mr\omega^2 = -F = dU / dr \tag{6.3.2}$$

defining the total energy and the centrifugal force. Notice that, assuming that we are allowed to identify the energy with the Hamiltonian, this one has already the Berry-Klein gauged form, with an elastic force defined by the centrifugal force from the second equation (6.3.2). As we see it, this is the reason that the Bohr's theory was so successful in explaining the experimental facts represented by hydrogen spectra: *it is a true scale transition theory*. It could not be interpreted as such due to a fact that passed, by and large, unnoticed, but the quantization according to Bohr's rules brings it to the fore. We realize now that, mathematically speaking, this is not the only way to exhibit it, but the fact is that quantization contributes to it, and the classical way of thinking is got to survive by this. The limited way in which it survived is the only one that shows that something may have been forgotten in the procedure.

Indeed, it is time to notice what Wilson himself noticed in his time, namely that Bohr has *two* quantization hypotheses, not only one, and these hypotheses are the well-known and often-reproduced relations, ever since the Bohr's work from 1913 made its appearance:

$$mr^2\omega = n \cdot \hbar, \quad E_2 - E_1 = 2\pi\hbar\nu \tag{6.3.3}$$

One of them is for the kinetic moment and the other for energy. These pure mind creations are the only things helping to build a classical theory, and Wilson did it in the way we will describe here. He inserts, as a last warning observation, even in a footnote, not in the main text, the comment that his theory might not mean too much, but:

> There is some advantage in replacing *the hypothetical Coulomb law* by an *experimental fact*. (*our emphasis*, *n/a*)

Truth be told, all the advantage should be there but, for a right physical understanding these two ingredients will have nevertheless to live together. The problem is to realize that the Coulomb law serves to interpretation, and the interpretation needs a moment of definition of the ensemble serving to its completion. This is the statics' Newtonian definition *involving only forces* in a material particle constitution of the ensemble.

Wilson starts by trying to obtain an equation for the potential, because it is directly connected to forces. First he notices that, when combining the second of Bohr hypotheses (6.3.3) with the experimental condition (6.3.1), a certain "conservation law" emerges, in the form

$$E_1 + \frac{2\pi\hbar R}{n_1^2} = E_2 + \frac{2\pi\hbar R}{n_2^2} \tag{6.3.4}$$

Further on, when combining this equation of conservation with the definition of energy from equation (6.3.2), and one recalls along that the potential energy is actually defined up to an additive constant, the result is that we have to consider the equation

$$U + \frac{mr^2\omega^2}{2} + \frac{2\pi\hbar R}{n^2} = 0 \tag{6.3.5}$$

as fundamental for the 'quantized' hydrogen atom. This equation is then transformed by Wilson in a *Clairaut-type equation*, using the definition (6.3.2) of force and the quantization hypothesis for kinetic moment from equation (6.3.3). It is

$$U = uU' + \frac{A}{U'}; \quad A \equiv \frac{\pi\hbar^3 R}{m} \tag{6.3.6}$$

with prime denoting the derivative with respect to independent variable $u \equiv r^{-2}$. The *general solution* of this equation is function of a constant representing U', *i.e.* the very force of the model, which may be obtained as follows: differentiating equation (6.3.6) with respect to u gives

$$\left(u - \frac{A}{(U')^2}\right) \cdot U'' = 0 \tag{6.3.7}$$

showing that the regular solution satisfies $U'' = 0$, therefore the derivative U' *should be a constant*. Thus, denoting this constant with C, the equation (6.3.6) gives

$$U(u) = Cu + \frac{A}{C} \tag{6.3.8}$$

This gives a force going *inversely proportional to the third power of distance*:

$$F(r) \equiv -U'(r) = -U'(u) \cdot u'(r) = \frac{2C}{r^3}$$

The constant C can be calculated as the centrifugal force using, again, the definition of force from equation (6.3.2) and the first quantization assumption from (6.3.3), so that the final result is

$$F(r) = \frac{\hbar^2}{m} \frac{n^2}{r^3} \tag{6.3.9}$$

Wilson gets therefore the important result that the experimental constraint (6.3.1), combined with classical dynamical precepts, and with the two quantum Bohr hypotheses, *leads to the quantization of the magnitude of force*. The magnitude of force is therefore quantized – *not the orbit*! However, this quantized force is not the Coulomb force – the quintessence of the planetary model of atom – but a force of magnitude inversely proportional with the cube of distance between the moving electron and the nucleus, *i.e.* the *force required by J. J. Thomson's theory*. That is, as long as the moving electron and the nucleus are conceived as material points, and we assume that the nucleus is just a force center.

Nevertheless, the force from equation (6.3.9) is not the only solution of the Clairaut's equation (6.3.6). In case $U'' \neq 0$ in equation (6.3.7), the so-called *singular solution* is obtained by eliminating U' between original equation (6.3.6) and the second condition coming out of (6.3.7). The result is:

$$\begin{aligned} U &= uU' + \frac{A}{U'} \\ u - \frac{A}{(U')^2} &= 0 \end{aligned} \quad \Rightarrow \quad U(r) = \frac{2\sqrt{A}}{r} \tag{6.3.10}$$

with the constant A given in equation (6.3.6). Obviously, this is the potential giving Newtonian forces – or, in the case in point, Coulomb forces – inversely proportional with the square of distance, as it was to be expected actually, due to the explicit scale invariance of the classical equations of motion for the corresponding Kepler problem. This force will therefore include the Planck's constant in its final expression, but its value will not be quantized like the previous one with magnitude inversely proportional with the third power of distance. In point of fact we have:

$$F(r) = \frac{2\sqrt{A}}{r^2} \tag{6.3.11}$$

where A is the constant defined in equation (6.3.6) above. It is, again, requested by Thomson in this theory, but there is a catch: while, according to Thomson, there are cases where the two forces coexist, the Bohr's hypotheses make them *mutually exclusive*, according to mathematical procedure.

We have therefore the following result: adding light to the phenomenology of classical planetary model, however not in a purely classical way, but *via* the Bohr's quantization prescriptions *in connection* to a classical dynamics, leads to two kinds of central forces acting in this model. They have the magnitudes given by equations (6.3.9) and (6.3.11). In the first case – a central force of magnitude going inversely with the third power of distance – the *magnitude of force is quantized*, while in the second case – the genuine Newtonian case of forces going inversely with the second power of distance – *the magnitude is not quantized*. According to the mathematical procedure leading to such a conclusion, these two cases are mutually exclusive: if the force with magnitude (6.3.9) acts in the model, the one with the magnitude (6.3.11) is not effective, and *vice versa*. So, the planetary model works in such a way that the revolving electron, being in a state having the acting force on it given by the expression from equation (6.3.9), cannot be concurrently in a state in which the acting force on the electron is given by the expression from equation (6.3.11), and *vice versa*, of course. This, apparently precludes the Thomson's treatment of the atom… but only aparently! The hypothesis of Thomson's tubes (see §6.1, above) saves the day, as it were, but one can just guess that Wilson's treatment shows that there is more to learn from this historical moment than is apparent.

One has now a deeper meaning of the Wagner's theorem (see §4.3): it sanctions, as it were, *a Berry-Klein update of the classical Newtonian dynamics*. Indeed, a Berry-Klein free particle is a classical (radial) harmonic oscillator. In a world of Berry-Klein particles under the cubic force (6.3.9), this is a classical free particle having two or three degrees of freedom [(Eliezer & Gray, 1976); see also §§2.3 and 4.4 above]. The Wagner's theorem allows one to avoid this 'dimensional reduction', and go directly for a gauging: for *any* Berry-Klein particles there is always a gauge in which *they are free Berry-Klein particles*. That is, the Thomson's theory is valid without the assumption of Newtonian inverse quadratic forces, whose existence, therefore, must have a deeper meaning. From this point of view, the inverse cubic forces are like constant potential forces of the classical dynamics, and consequently one can say that the quantization condition in the material world is not just a lucky chance. As, in fact, it was not just a lucky chance when it occurred for the first time to Max Planck in the case of light. In this respect it becomes highly significant for the knowledge at large that when the quantization condition was first applied to the world of matter [see (Haas, 1910); we did not have a chance to consult these works; see, however, the discussions occasioned by the reports of Max Planck and Arnold Sommerfeld at the First Solvay Conference (Langevin & de Broglie, 1912)], it was referring to the potential energy of an atom (Schidlof, 1911), and that was chosen to be a Thomson's atom, not a Rutherford one (Haas, 1929). The inappropriate assumption that the Thomson's atom should be an exclusively *static* structure – just as inappropriate as the assumption that Rutherford's atom should be an exclusively *dynamical* structure – combined, further, with some equally inappropriate assumptions on the side of light physics, have precluded a right application of the quantization condition in the case of the world of matter. Let us, therefore, go into some depth with ideas along these lines, in the hope to find a middle line in the contemporary natural philosophy.

6.4 A Description of the Two States of Hertz Particle

Should we consider the classical problem of finding the force toward the center of a circular orbit, acting on the material point that moves along that orbit, the answer would be unambiguously that from equation (6.3.11). One can therefore see that the quantization conditions add a lot to the class of forces that might be compatible with them. While aware that these quantization conditions are simply assumptions, the results of Edwin Wilson deserve, nevertheless, a closer attention. They allow indeed some unexpected connections, giving a clear message to the modern formal quantization. If we ask how can someone characterize the state of a Hertz material particle under force, the answer is quite simple according to classical dynamical practice: by the *Bertrand's principle* (see §4.1 above). In other words, as Newton already did it, by *observing the orbit*.

Until we will study the case from the most general point of view, let us just notice, once again, that the two forces from the equations (6.3.9) and (6.3.11) are but special cases of *central forces*, from the species invented by Newton. First, their magnitude depends exclusively on distance; secondly, considered by themselves, they lead to special trajectories of the material points upon which they act. The force from equation (6.3.9) generates trajectories having the shape of logarithmic spirals (like the arms of galaxies). This does not mean too much from the perspective of the classical planetary model of atom. Indeed, intuitively, a spiralling electron would lead to the very same conclusions as the exaustion of energy – the avoidance of which enforced the quantization condition upon model – when referred to the construction of the atom *per se*: sooner or later such a construction is destined to crumble. Therefore the classical planetary atom will collapse anyway, and thus the model should be necessarily replaced, being unworthy of consideration. However, this does not mean the end of classical path to microcosm, because it turns out that the *shape of the orbit* is the one which should be essential to the argument, and this is exactly the point of a principle like that of Joseph Bertrand, which seems to be in the natural order of historical facts. The argument in question concerns at least two points of view, to be described as follows.

First, the classical shape of orbit is given by Binet's equation (2.3.23), which we reproduce here, for convenience, in the form:

$$\dot{a}^2u^2\left(\frac{d^2u}{d\phi^2}+u\right)=f(\boldsymbol{r});\quad u\cdot r=1 \tag{6.4.1}$$

Here r and ϕ are the polar coordinates of the plane of motion, and $\mathring{a}$ is the area constant that settles the time for that motion. Now, in the case of the force from equation (6.3.9) this equation reduces to

$$\frac{d^2u}{d\phi^2}+\left(1-\frac{2\sqrt{A}}{\dot{a}^2}\right)u=0 \tag{6.4.2}$$

The resulting orbit is either a *trigonometric spiral*, that can be put in the form

$$r(\phi)=\frac{r_0}{\cos(\lambda\phi+\phi_0)},\quad \lambda^2\triangleq 1-\frac{2\sqrt{A}}{\dot{a}^2} \tag{6.4.3}$$

or a *logarithmic spiral* that can be put in the form

$$r(\phi) = \frac{r_0}{\cosh(\lambda\phi + \phi_0)}, \quad \lambda^2 \triangleq \frac{2\sqrt{A}}{\dot{a}^2} - 1 \tag{6.4.4}$$

where λ is always considered here as a real number.

By the same token, the Binet's equation corresponding to a force like that from equation (6.3.11) can be integrated, thus showing that such a force generates an elliptic orbit – the well known classical result. The parameters of that orbit – the semiaxes, the eccentricity, the orientation of the orbit in its plane, and such – are determined by the initial conditions of the motion. For the sake of completeness here, we write the equation of the orbit in the form

$$\frac{\dot{a}}{r} = \frac{2\sqrt{A}}{\dot{a}} + w_1 \cos\phi + w_2 \cos\phi \tag{6.4.5}$$

where w_1 and w_2 are the components of an initial velocity of the electron.

Nothing new over classical results until now, and were it not for the question of which one of these orbits correspond to the quiet, stable atom, the classical dynamics would not be touched by anything. Indeed, only when we come to the fact that the atom must *produce radiation* – the idea that occurred in connection with inclusion of light in the phenomenology of matter – trouble enters the stage, because, according to the classical logic based on energetical arguments, this would lead to inevitable collapse of the atom. If the atom exists, and is indeed the classical structure we think it should be, then there should exist orbits along which the electron does not emit any light. Which one of the two orbits above satisfies this requirement?

So we come to the second point of view: continuing along the classical dynamical line of reasoning, we can discuss the stability in terms of the emitted radiation, as representing a new experimental fact, to be added to those already contained in the original Newtonian arsenal helping in deducing the magnitude of force. As the radiation depends on the orbit itself, the problem is to find that electronic orbit along which, if the electron moves in the realm of atom, the radiation is not produced – the stable or "radiationless orbit" in Wilson's very expression. He has shown (Wilson, 1919) that the radiationless orbits *are not ellipses*, but they should be sought for among *the logarithmic spirals*. Indeed there is a class of spiralling electronic orbits along which the moving electron does not emit radiation. Wilson judges the situation of the "radiationless orbits" by what is known today as the *radiative power* of a nonuniformly moving charge [for a modern account of the electrodynamics involved here see (Jackson, 1998), especially Chapter 16]. This power is null, therefore the charge does not emit radiation, in cases where the second time derivative of velocity is perpendicular to the velocity vector itself. In the Cartesian coordinates, x, y say, of the plane of motion, this condition amounts to the differential equation

$$\dot{x} \cdot \dddot{x} + \dot{y} \cdot \dddot{y} = 0 \tag{6.4.6}$$

where the dot over a symbol represents one time derivative, as usual. According to this condition, an electron spiralling toward the center of force can, in some instances, be radiationless. Indeed, if the equations of motion of the electron are

$$x(t) = Ae^{-\lambda t}\cos(\mu t + \mu_0), \quad y(t) = Ae^{-\lambda t}\sin(\mu t + \mu_0) \tag{6.4.7}$$

where λ and μ are two real constants, then the dot product from equation (6.4.6) is

$$\dot{x} \cdot \dddot{x} + \dot{y} \cdot \dddot{y} = A^2(\lambda^4 - \mu^4)e^{-2\lambda t} \tag{6.4.8}$$

and becomes unconditionally zero for the cases where the ratio between λ and μ is one of the biquadratic roots of unity.

Assuming therefore a ‘fresh start’ in the classical theory of atomic forces, as given by the radiationless condition derived from equation (6.4.8), we are bound to discover a particular form of equations (6.4.7), *viz.*

$$x(t) = Ae^{-\varepsilon\mu t}\cos(\mu t + \mu_0), \quad y(t) = Ae^{-\varepsilon\mu t}\sin(\mu t + \mu_0) \tag{6.4.9}$$

where $\varepsilon^4 = 1$. This is, therefore, the classical expression of the trajectory of electronic motion in a ‘stable’ Rutherford atom. Now if, in a Newtonian spirit, we want to find what are the forces corresponding to these orbits, we just have to use the classical Binet formula (6.4.1) in the reverse (Whittaker, 1917). First eliminate the time from equation (6.4.9) in order to obtain the dependent variable u from equation (6.4.1) as a function of the polar angle ϕ. We thus have

$$u(\theta) = Be^{-\varepsilon\phi}, \quad B \equiv (Ae^{\varepsilon\mu_0})^{-1} \tag{6.4.10}$$

Inserting this result into equation (6.4.1) itself gives the force responsible for determining the radiationless orbits of the electron

$$f(\boldsymbol{r}) = \frac{\dot{a}^2 B(1+\varepsilon^2)}{r^3} \tag{6.4.11}$$

In view of the fact that ε is the quartic root of unity, ε^2 should be the *square root* of unity, and therefore the equation (6.4.11) gives quite a comprehensive result, even according to the very classical rules of natural philosophy. It shows that there are *two distinct radiationless cases*. The first case occurs for $\varepsilon^2 = -1$, when the force is unconditionally zero, which is obviously the case of a free-moving electron, either plunging into the nucleus or erupting out of the nucleus. According to the laws of classical dynamics this electron has no acceleration, and therefore cannot produce radiation. A second case occurs for $\varepsilon^2 = 1$, which is the proper case corresponding to spiralling orbits. Here too, there are two distinct possibilities for that force, inasmuch as $\varepsilon = \pm 1$: one is for the invard spiralling, the other for outward spiralling electron.

This result shows that the atom does not emit radiation if the electron moves uniformly in a straight line with respect to the nucleus, toward or away from it, and also if it moves inward or outward along some spiral orbits. Truth be told, this conclusion is totally in agreement with that based on the exaustion of energy, as we have already noticed. However, while by the exaustion of energy we are led to conclude the atomic collapse unconditionally, and we cannot accept it *due to the shape* of the model’s orbit – a closed plane curve – here the very premise of atomic structure is an open orbit, and the atom would be no more ‘planetary’. Therefore the non-radiating condition leads to a premise different from the one which introduced the Rutherford atom in the first place. We might be able to accommodate such strange things in our theory, as in point of fact we always do when it comes to our mind inventions, but we ought to have a ‘reality incentive’, so to speak, that may ease our reason along the way: can the spiralling electron be part of a stable structure?

Going a little ahead of us, we can say that, granted that the nucleus, as well as the electron, are no more material points modeled as positions, but are Hertz material points, allowed to have a space expanse, as well as an interpretation that comes only naturally with this expanse, the answer is affirmative. Accepting a scale transition *a priori* as a working hypothesis for the phenomenology at large, one can say that the structures at transfinite scale – specifically, the spiral nebulae – are expression of such a kind of stability: the stars in their journey around the galactic nucleus are observable in the ‘moments’ they are reaching the spiral arms of the structure. While there is much to be explained here, let us, however, show just a few more details for now, connected with the two forces under scrutiny, in order to ease our understanding into this interpretation.

6.5 The Two Forces in a Short History

The force varying inversely with the cube of distance has first entered the history with Newton (*Principia*, Book I, Proposition IX). He certainly attached a great deal of importance to this law of force, due to another fact in close connection with what we are about to say, known historically as the *problem of revolving orbits*. Only relatively recently this problem started to draw again the attention of theoreticians, mostly in the field of astrophysics (Lynden-Bell, 2006). Here, in astrophysics, the force inversely proportional with the cube of distance is quite important, being what we would like to call a *transition force*: it actually characterizes the transition between two elliptic orbits in the *same plane*, but rotated with respect to each other. Newton himself, considered it as such (*Principia*, Book I, Proposition XLIV): a force whose magnitude is the difference between the magnitude of the force by means of which an elliptic orbit is described, and the magnitude of the force by means of which is described the same orbit, but rotated with a certain angle in its plane. In modern terms [see (Lynden-Bell, 2006); see also (Whittaker, 1917) which is the work we are following here] if $r = F(\phi)$ is an orbit in the polar coordinates (r,ϕ) of its plane of motion, described under the central force of magnitude $f(\boldsymbol{r})$, then the same kind of orbit $r = F(k\phi)$, where k is any constant, is described under the force

$$f'(\boldsymbol{r}) = f(\boldsymbol{r}) + \frac{c}{r^3} \tag{6.5.1}$$

where c is a constant. This equation can be proved directly by starting from Binet's equation (6.4.1). What really changes in the case of the primed system is only the *area constant* [see especially (Whittaker, 1917), p. 83]. However, due to the classical Newtonian character of the time, this change hides and important scale invariance of a Berry-Klein nature. Let us describe this in detail. Using equation (6.4.1) in order to calculate the magnitude of forces, we have:

$$f'(\boldsymbol{r}) = (\dot{a}')^2 u^2 \left(\frac{d^2 u}{d\phi'^2} + u \right) \quad \therefore \quad f'(\boldsymbol{r}) = (\dot{a}')^2 u^3 + \frac{(\dot{a}')^2}{\dot{a}^2 k^2} \left(f(\boldsymbol{r}) - \dot{a}^2 u^3 \right) \tag{6.5.2}$$

Now, assuming the special classical invariance of the time intervals:

$$dt' = dt \quad \therefore \quad \dot{a}' = k \cdot \dot{a} \tag{6.5.3}$$

we get the equation (6.5.1) with a special choice of the constant c:

$$f'(\boldsymbol{r}) = f(\boldsymbol{r}) + \frac{\dot{a}^2 (k^2 - 1)}{r^3} \tag{6.5.4}$$

proving what we already signaled above, namely that the central force having the magnitude that goes inversely with the cube of distance is a transition force between rotating orbits. This result is often cited as the *Newton's theorem on revolving orbits*. A few methodological observations are in order, for future reference.

On (6.5.3): the Hertz material particle in question belongs to two orbits simultaneously. This is why that relation has to be understood as an invariance:

$$\frac{dt'}{r'^2} = \frac{dt}{r^2} \tag{6.5.5}$$

where, incidentally, $r' = r$. But this is not all of it.

On the centrifugal forces: it was often stated that the inverse cubic force origin is centrifugal. It is, indeed, true, but there is a twist on the story, inasmuch as, if the motion is not a circular one, a radial component enters the equation of acceleration. Indeed, we have, in general (see §2.3)

$$\boldsymbol{v}^2 = \dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta \cdot \dot{\varphi}^2) \tag{6.5.6}$$

On the other hand, using the equation (2.3.20), which gives the area constant, we have

$$\dot{\theta}^2 + \sin^2\theta \cdot \dot{\varphi}^2 \equiv (\dot{\phi})^2 = \left(\frac{\dot{a}}{r^2}\right)^2 \tag{6.5.7}$$

so that

$$\frac{\boldsymbol{v}^2}{r} = \frac{\dot{r}^2}{r} + \frac{\dot{a}^2}{r^3} \tag{6.5.8}$$

If the purely radial motion is of the *Hubble type*, this centrifugal acceleration is always of the conservative kind involved in the theorem of Berry and Klein [see §4.2, especially equation (4.2.7)], for the centrifugal acceleration is given by

$$\frac{\boldsymbol{v}^2}{r} = \varpi^2 r + \frac{\dot{a}^2}{r^3} \tag{6.5.9}$$

where ϖ is a 'Hubble constant', which is the inverse of a time. In case it is zero, the orbit is a circle, and thus we have the reason why the inverse cubic acceleration is sometimes associated with the centrifugal force. If there is a classical dynamics connected to this acceleration, then it is described by the Binet's equation

$$\dot{a}^2 u^2 \left(\frac{d^2 u}{d\phi^2} + u\right) = \frac{\varpi^2}{u} + \dot{a}^2 u^3 \quad \therefore \quad \frac{d^2 u}{d\phi^2} = \frac{\kappa^2}{u^3} \tag{6.5.10}$$

where

$$\kappa \equiv \frac{\varpi}{\dot{a}} \tag{6.5.11}$$

is a constant having the dimensions of an inverse area. One may say that this is a result in concordance with Wagner's theorem: there is a Berry-Klein gauge in which the particle is classically free, and this is given by the logarithmic force, whereby the gauge length can be obtained by an inversion:

$$l = \frac{1}{r}, \quad d\phi = \frac{dt}{r^2} \tag{6.5.12}$$

We have to notice, though, that there are *two Berry-Klein free particles* corresponding to this *classical free particle*, according to a Hubble-type law, which, in a Berry-Klein theorem can describe expansion or contraction. As one can see, the gauging procedure does not restrict the shape of trajectory to a circle.

To close these considerations, and with them the present instalment of our work: in a Berry-Klein update of the classical dynamics, the free particle is only partially free in the Newtonian sense. It is manifesting like a harmonic oscillator, and its realm of existence is the fundamental structure of the matter, which we call the atom. This update of the mechanics is pending on a Planck's quantization, so it works even in the case of light, as actually the history proves it. The issues signaled in this final chapter will be treated at length in the next instalment of the present work.

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