On linear ordinary differential equations of second order and their general solutions

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Abstract

We have worked out a new geometric approach to linear ordinary differential equations of second order which makes it possible to obtain general solutions to infinite number of equations of this sort. No need new families of special functions and their theories arose, solutions are composed straightforwardly. In this work we present a number of particular cases of equations with their general solutions. The solutions are divided into four groups the same way one encounters in any book on special functions.

1 Introduction

Linear ordinary differential equations of second order (LODEII) are of special interest to physicists because they always appear as a result of separation main equations of classical theories. Originally they have been discovered in the form [1]

$$\frac{1}{f}\frac{\mathrm{d}}{\mathrm{d}x}\left(f\frac{\mathrm{d}y}{\mathrm{d}x}\right)+\left(k^2+\frac{m^2}{f^2}\right)=0, \tag{1}$$

where the form of the function $f(x)$ is predetermined by choice of the coordinate system. They have been solved by great mathematicians of XIX century for several special cases of the function f and thanks to them mathematical physics was developed and took the form found in numerous classical books on the subject. Each solution obtained then was a new found class of special transcendent functions, so, general solutions of an equation of this sort includes a complicated theory of a sertain kind of special functions called after the name of its discoverer. Presently, solutions found in literature on special functions as well as the method they have been obtained by, are insatisfactory.

Development of physics goes on and variety of equations physicists encounter, grows. Presently physicists need complete solutions of LODEII of general form which can be represented in the form

$$f\,\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{f}\frac{\mathrm{d}y}{\mathrm{d}x}\right)+Fy=0, \tag{2}$$

thus, contains two arbitrary functions f and F. Their form is predetermined by both choice of coordinate system and varying properties of the medium, [2, 3]. It must be pointed out that all equations (1) solved mainly in XIX century, were done by substituting power series for the desired function. Afterwards opportunities of this approach have been exhausted and as a result a belief was created that

- to solve an equation like (2) is the same as to create another theory of special functions
- all special functions are already discovered and usable theories of special functions are already known, so, it is pointless to ty to create new and more sophisticated ones
- the era of analytical solutions in classical physics is over, so, only numerical simulations will make sense in the future, or, in other words, it is pointless to spend efforts to unsolved differential equations.

This belief is completely wrong.

We have worked out a new approach to equations of the form (2) based on geometric considerations. Since these considerations cannot be expressed in textual form we do not present details. Main properties of our approach are:

- provides general solutions to the equations of this form which contain two arbitrary functions, one of which is the function $f(x)$ and another one which specifies both the function $F(x)$ and explicit form of general solution.
- our approach does not create new transcendental functions and corresponding theories

The main deficience of our method is that explicit form of the function $F(x)$ is being generated along with the solution, so, it cannot be prescribed arbitrarily. Therefore, for example, it cannot be applied to the typical quantum mechaincal problem of spectrum calculation. However, they have other useful applications. Below we present a number of examples of equation (2) along with their general solutions.

2 The method of generating equations with their complete solutions

In the next section we present examples of equations and their complete solutions generated by our method. An ordered exposition of the material requires that the equations are classified and divided into classes and subclasses that we have done as follows. First, all possible LODEII's were divided into two main classes of "typical" and "atypical" ones. Each of these two classes is divided into subclasses specified by a certain form of the function $f(x)$. After that we have a number of subclasses of equations which differ only in the form of the function $F(x)$ which, however, do not exhaust abilities of the method. Each subclass can be extended with infinite number of new examples and infinite number of new subclasses can be built by our method.

Each subclass is specified by a certain form of the function $f(x)$ whichcan bes chosen arbitrarily. For simplicity, we chose these functions such a way that the indefinite integral

$$t = \int \mathrm{d}x\, f(x) \tag{3}$$

is a function of of sufficiently simple form. Hereafter the variable t actually plays the role of the main one, in particular, the function $\rho(t)$ which can be chosen arbitrarily, now is regarded as a function of the variable t. We chose it also from the simplicity reason, so that the "phase function"

$$\varphi = \int \frac{\mathrm{d}x f(x)}{\rho^2(t)} \tag{4}$$

also is as simple as possible. We need simplicity of the expressions only because in this particular work we only demonstrate our method, but in general, the simplicity reason can be ignored that allows one to generate much more examples of solved equations.

According to the equation (10), the function φ can also be represented as a function of the variable t:

$$\varphi = \int \frac{\mathrm{d}t}{\rho^2(t)} \tag{5}$$

and as a result, the general solution appears as a function of this variable:

$$y(x, C) = \rho(t) \sin(k\varphi + C). \tag{6}$$

Now, applying the operator of the equation

$$\frac{1}{f} \frac{\mathrm{d}}{\mathrm{d}x} \left(f \frac{\mathrm{d}}{\mathrm{d}x} \right)$$

to this function and using the fact that

$$\frac{\mathrm{d}t}{\mathrm{d}x} = f,$$

we obtain finally that the function $F(x)$ in the equation (2) has the form

$$F(x) = f^2(x) \left(\frac{k^2}{\rho^4} - \frac{\rho''}{\rho} \right) \tag{7}$$

where the function $t(x)$ is to be substituted from the formula (10). All particular cases presented below are composed in this techniques. The desired solution (6) appears as a function of the variable t where t as a function of the variable x (10) is to be substituted. As a result y turns into a function of the variable x and can be substituted into the original equation (2). Completing this operation shows that the equation turns into an idenetity and thereby, that $y(x, C)$ built this way indeed is the complete solution of the equation.

3 Typical cases

3.1 Introduction

In this section we present explicit form and general solution of equations which belong to the class of "typical cases". By "typical cases" we mean class of equations (2) whose first term has the same form as that in equations known from the theory of special functions. In this work we consider only the simplest cases when the equation has the form similar to that of one-dimensional Schrödinger, Bessel and Legendre equations. Thus, in turn, these classical equations belong to the corresponding "typical cases" of equations as their particular cases.

Now, let us pass to examples of generated pairs of equations and their complete solutions. Since the function $f(x)$ is known for each particular case, no need for explicit form of the equation. It suffices to specify explicit form of the function $F(x)$. So, the pairs can be represented as $(F(x), y(x,C))$.

3.2 One-dimensional Schrödinger equation

The general form of equation is

$$y'' + F(x)y = 0. \tag{8}$$

We present only explicit form of two functions, which are $F(x)$ and $y(x,C)$. The earlier specifies explicit form of the equation (8) and the latter does its general solution. Pairs generated by our method are:

$$\begin{aligned}
F(x) &= \tfrac{1}{2}\left(k^2x^2 - \frac{3}{2x^2}\right), & y(x,C) &= \frac{1}{\sqrt{x}}\cdot\sin k\left(\frac{x^2}{2}+C\right)\\
F(x) &= -\frac{1}{x^2}\cdot\left(k^2+\tfrac{1}{4}\right) & y(x,C) &= \sqrt{x}\cdot\sin k\ln Cx\\
F(x) &= k^2x^4 - \frac{2}{x^2}, & y(x,C) &= \frac{1}{x}\cdot\sin k\left(\frac{x^3}{3}+C\right)\\
F(x) &= \frac{k}{x^4}, & y(x,C) &= x\sin\left(C-\frac{k}{x}\right)\\
F(x) &= \frac{k^2-1}{(1+x)^2}, & y(x,C) &= \sqrt{1+x^2}\sin(k\cdot\mathrm{arctan}x + C).
\end{aligned}$$

Hyperbolic version of these solutions have the form

$$\begin{aligned}
F(x) &= -\tfrac{1}{2}\left(k^2x^2 + \frac{3}{2x^2}\right), & y(x,C) &= \frac{1}{\sqrt{x}}\cdot\sinh k\left(\frac{x^2}{2}+C\right)\\
F(x) &= \frac{1}{x^2}\cdot\left(k^2-\tfrac{1}{4}\right) & y(x,C) &= \sqrt{x}\cdot\sinh k\ln Cx\\
F(x) &= -\left(k^2x^4 + \frac{2}{x^2}\right), & y(x,C) &= \frac{1}{x}\cdot\sinh k\left(\frac{x^3}{3}+C\right)\\
F(x) &= -\frac{k}{x^4}, & y(x,C) &= x\sin\left(C-\frac{k}{x}\right)\\
F(x) &= -\frac{k^2+1}{(1+x)^2}, & y(x,C) &= \sqrt{1+x^2}\sin(k\cdot\mathrm{arctan}x + C).
\end{aligned}$$

It is easy to derive the general rule which reads tha hyperbolic version is the same as the trigonometric one with the only difference in sign of k^2. Nevertheless, we provide each of them explicitly for two cases.

3.3 Bessel-like case

By Bessel-like case we mean the class of equations of the form (2) with $f(x) = k/x$ which contain the original Bessel equation as a particular case. These equations can be represented in the form

$$y'' + \frac{1}{x}y' + Fy = 0. \tag{9}$$

Again, the results are presrnted below as pairs $(F(x), y(x, C))$ where the function $F(x)$ specifies explicit form of the equation (9) and $y(x, C)$ stands for its general solution.

$$
\begin{aligned}
F(x) &= \frac{1}{x^2}\left(\frac{k^2}{x^2} - \frac{1}{2}\right), & y(x,C) &= \sqrt{x}\sin\left(C - \frac{k}{x}\right) \\
F(x) &= -\frac{1}{x^2}\left(\frac{4k^2}{x^2} + 1\right), & y(x,C) &= x\sin k\left(C - \frac{1}{2x^2}\right) \\
F(x) &= k^2x^4 - \frac{9}{4x^2}, & y(x,C) &= x^{-3/2}\sin k\left(\frac{x^3}{3} + C\right), \\
F(x) &= k^2x^2 - \frac{2}{x^2}, & y(x,C) &= \frac{1}{x}\cdot\sin k\left(\frac{x^2}{2} + C\right) \\
F(x) &= \frac{k^2 - 1}{x^2(1 + \ln^2 x)^2}, & y(x,C) &= \sqrt{1 + \ln^2 x}\cdot\sin(k\cdot\arctan\ln x + C).
\end{aligned}
$$

3.4 Legendre-like case

The original Legendre equation can be represented in the form (2) with $f = \sin x$, $F = \text{const}$. Therefore by Legendre-like case we mean equations of this form with the function $F(x)$ generated by our procedure along with the general solution. Below we use notation

$$t = \operatorname{arctanh}\cos x \tag{10}$$

and the identity

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{\sin x}. \tag{11}$$

The pairs generated are:

$$
\begin{aligned}
F(x) &= \frac{1}{\sin^2 x}\left(k^2t^2 - \frac{3}{4t^2}\right), & y(x,C) &= \frac{1}{\sqrt{t}}\sin(kt^2 + C) \\
F(x) &= \frac{1}{t^2\sin^2 x}\left(\frac{1}{4} + k^2\right), & y(x,C) &= \sqrt{t}\sin k\ln Ct \\
F(x) &= \frac{k^2}{t^4\sin^2 x}, & y(x,C) &= t\sin k\left(C - \frac{k}{t}\right) \\
F(x) &= \frac{1}{\sin^2 x}\left(k^2e^{2t} - \frac{1}{4}\right), & y(x,C) &= e^{-t/2}\sin k(e^t + C) \\
F(x) &= \frac{k^2 - 1}{(1 + t^2)\sin^2 x}, & y(x,C) &= \sqrt{1 + t^2}\sin k(\arctan t + C)
\end{aligned}
$$

Their hyperbolic version is

$$
\begin{aligned}
F(x) &= -\frac{1}{\sinh^2 x}\left(k^2t^2 + \frac{3}{4t^2}\right), & y(x,C) &= \frac{1}{\sqrt{t}}\sinh(kt^2 + C) \\
F(x) &= \frac{1}{t^2\sinh^2 x}\left(\frac{1}{4} - k^2\right), & y(x,C) &= \sqrt{t}\sinh k \ln Ct \\
F(x) &= -\frac{k^2}{t^4\sinh^2 x}, & y(x,C) &= t\sinh k\left(C - \frac{k}{t}\right) \\
F(x) &= -\frac{1}{\sinh^2 x}\left(k^2e^{2t} + \frac{1}{4}\right), & y(x,C) &= e^{-t/2}\sinh k(e^t + C) \\
F(x) &= -\frac{k^2+1}{(1+t^2)\sinh^2 x}, & y(x,C) &= \sqrt{1+t^2}\sinh k(\arctan t + C)
\end{aligned}
$$

4 Atypical cases

Unlike cases considered above, atypical ones have no fixed form of the function $f(x)$ in the equation (2). There exist infinite number of atypical cases which can be defined by the form of this function. The only requirement to it reads that this form has never appeared before. Below we consider few atypical cases specifies with some certain forms of the function $f(x)$. So, material of this section is divided into six subsections, the first of wihch outlines the general principle of composing particular examples and others present the simplest atypical cases specified by certain forms of the function $f(x)$.

4.1 The case $f(x) = \coth x$

In this case the function $f(x)$ is fixed as $f(x) = \coth x$ and we use the following notation:

$$
t = \ln\sinh x. \tag{12}
$$

As usual, it suffices to specify pairs of functions $(F(x), y(x,C))$ which are

$$
\begin{aligned}
F(x) &= f^2\left(k^2t^2 - \frac{3}{4t^2}\right), & y(x,C) &= \frac{1}{\sqrt{t}}\sin(kt^2 + C) \\
F(x) &= \frac{f^2}{2t^2}\left(k^2 + \frac{1}{4}\right), & y(x,C) &= \sqrt{t}\cdot\sin k\ln Ct \\
F(x) &= -\frac{k^2\, f^2}{t^4}, & y(x,C) &= t\sin k\left(C - \frac{1}{t}\right) \\
F(x) &= \frac{f^2(k^2-1)}{(1+t^2)^2}, & y(x,C) &= \sqrt{1+t^2}\cdot\sin(k\arctan t + C) \\
F(x) &= f^2\cdot\left(k^2t^4 - \frac{2}{t^2}\right), & y(x,C) &= \frac{1}{t}\cdot\sin k\left(\frac{t^3}{3} + C\right)
\end{aligned}
$$

4.2 The case $f(x) = \dfrac{1}{\sqrt{1+x^2}}$

In this case the variable t is

$$t = \text{arcsinh} x \tag{13}$$

and below the simplest pairs $(F(x), y(x,C)$ are presented, which specify explicit forms of the equation (2) and its general solution.

$$\begin{aligned}
F(x) = f^2\left(k^2 - \frac{1}{4}\right), &\qquad y(x,C) = \sqrt{t}\,\sin k \ln Ct; \\
F(x) = \frac{f^2k^2}{t^4}, &\qquad y(x,C) = t\cdot \sin k\left(C - \frac{1}{t}\right); \\
F(x) = f^2\cdot\left(k^2t^2 - \frac{3}{4t^2}\right), &\qquad y(x,C) = \frac{1}{\sqrt{t}}\cdot \sin k\left(\frac{t^2}{2} + C\right) \\
F(x) = f^2\cdot\left(\frac{2}{t^2} - k^2t^4\right), &\qquad y = \frac{1}{t}\sin k\left(\frac{t^3}{3} + C\right); \\
F(x) = \frac{k^2-1}{(1+t^2)^2}, &\qquad y(x,C) = \sqrt{1+t^2}\cdot \sin k(\arctan t + C).
\end{aligned}$$

Those are the simplest pairs of equations and their general solutions, more complicated ones can be composed by analogy.

4.3 The case $f(x) = \tanh x \,\textbf{sech} x$

One more atypical case which provides relatively simple solutions is specified by choosing the function $f(x)$ of this form. In this case the variable t is

$$t = \text{sech} x, \tag{14}$$

but it is convenient to express the result in the variable x. Equations and their general solutions generated by our procedure are presented below in the form of pairs $(F(x), y(x,C)$

$$\begin{aligned}
F(x) &= \tanh^2 x\left(k^2 + \frac{1}{4}\right), \\
y(x,C) &= \sqrt{\text{sech}\, x}\,\sin\ln(C\text{sech}\, x) \\
&\qquad \dots \\
F(x) &= \tanh^2 x \text{sech}^2 x \frac{k^2t^2}{4}\,\sinh^2 2x, \\
y(x,C) &= \text{sech}\, x\cdot(C - \cosh x)
\end{aligned}$$

$$F(x) = \tanh^2 x \mathrm{sech}^2 x \left(k^2 \mathrm{sech}^2 x - \frac{3}{4} \cosh^2 x \right),$$

$$y(x, C) = \sqrt{\cosh x} \cdot \sin k \left(\frac{\mathrm{sech}^2 x}{2} + C \right)$$

$$\ldots$$

$$F(x) = \tanh^2 x \mathrm{sech}^2 x (k^2 \mathrm{sech}^4 x - 2\cosh^2 x),$$

$$y(x, C) = \cosh x \, \sin k \left(\frac{\mathrm{sech}^3 x}{3} + C \right)$$

$$\ldots$$

$$F(x) = \tanh^2 x \, \mathrm{sech}^2 x \, \frac{k^2 - 1}{1 + \cosh^2 x},$$

$$y(x, C) = \sqrt{1 + \mathrm{sech}^2 x} \sin k (\arctan \mathrm{sech} x + C).$$

One can easily compose other pairs by analogy.

4.4 The case $f(x) = x + \dfrac{1}{x}$

In this subsection we simply count the pairs $(F(x), y(x, C)$. It turns out that in this case the variable t also is of no use and we express the result in functions of the variable x. So,

$$F(x) = \left(x + \frac{1}{x} \right)^2 \cdot \frac{k^2 + 1}{(x^2 - 2\ln x)^2},$$

$$y(x, C) = \sqrt{\frac{x^2}{2} + \ln x} \cdot \sin k \ln C \left(\frac{x^2}{2} + \ln x \right); \ldots$$

$$F(x) = \left(x + \frac{1}{x} \right)^2 \cdot \frac{16k^2}{(x^2 + 2\ln x)^2},$$

$$y(x, C) = \left(\frac{x^2}{2} + \ln x \right) \cdot \sin k \left(C - \frac{2}{x^2 + 2\ln x} \right);$$

$$\ldots \qquad F(x) = \frac{k^2}{4} \left(x + \frac{1}{x} \right)^2 \left(\frac{k^2}{(x^2 + 2\ln x)^4} + \frac{3}{(x^2 + 2\ln x)^2} \right),$$

$$y(x, C) = \sqrt{\frac{2}{(x^2 + 2\ln x)^2}} \sin k \left(\frac{(x^2 + 2\ln x)^2}{8} + C \right);$$

$$F(x) = \left(x + \frac{1}{x}\right)^2 \left[k^2 \left(\frac{x^2}{2} + \ln x\right)^4 - \frac{8}{(x^2 + 2\ln x)^2}\right],$$

$$y(x, C) = \frac{2}{2x^2 + \ln x} \cdot \sin k \left(\frac{(x^2 + 2\ln x)^3}{3} + C\right);$$

...

$$F(x) = 16\left(x + \frac{1}{x}\right)^2 \cdot \frac{k^2 - 1}{[4 + (x^2 + 2\ln x)^2]^2},$$

$$y = \sqrt{\frac{x^2}{2} + \ln x} \cdot \sin k \arctan\left(\frac{x^2}{2} + \ln Cx\right)$$

4.5 The case $f(x) = \frac{1}{\sqrt{x(1+x)}}$

We finalize our review of pairs of equations and their solutions with this case. The corresponding pairs $(F(x), y(x, C))$ are:

$$F(x) = \frac{1}{4x(1+x)\mathrm{arcsinh}^2\sqrt{x}} \cdot \left(k^2 - \frac{1}{4}\right),$$

$$y(x, C) = \sqrt{2\arcsin\sqrt{x}} \cdot \sin k \ln 2C \arcsin\sqrt{x}$$

...

$$F(x) = \frac{1}{x(1+x)} \cdot \frac{k^2}{4\mathrm{arcsinh}^2\sqrt{x}},$$

$$y(x, C) = 2\mathrm{arcsinh}\sqrt{x} \cdot \sin k \left(C - \frac{1}{2\mathrm{arcsinh}\sqrt{x}}\right)$$

...

$$F(x) = \frac{1}{x(1+x)} \cdot \left(4k^2\mathrm{arcsinh}^2\sqrt{x} - \frac{1}{2\arcsin\sqrt{x}}\right),$$

$$y(x, C) = \frac{1}{\sqrt{2\mathrm{arcsinh}\sqrt{x}}} \cdot \sin k \left(2\mathrm{arcsinh}^2\sqrt{x} + C\right)$$

...

$$F(x) = \frac{1}{x(1+x)} \cdot \left(16k^2\mathrm{arcsinh}^4\sqrt{x} - \frac{1}{2\mathrm{arcsinh}^2\sqrt{x}}\right),$$

...

$$y(x, C) = \frac{1}{2\mathrm{arcsinh}\sqrt{x}} \left(\frac{8}{3}\mathrm{arcsinh}^3\sqrt{x} + C\right)$$

$$
\begin{aligned}
F(x) &= \frac{1}{x(1+x)} \cdot \frac{k^2-1}{1+4\mathrm{arcsinh}^2\sqrt{x}},\\
y(x,C) &= \sqrt{1+4\mathrm{arcsinh}^2\sqrt{x}} \cdot \sin k(\arctan \mathrm{arcsinh}\sqrt{x} + C)
\end{aligned}
$$

5 Conclusion

A method for generating LODEII order along withe their complete solutions is discovered. This method allows to generate pairs of equations of this sort with known complete solutions. The procedure of generation is presented in the subsection **??**. The procedure allows to generate infinite number of solved equations of the sort. We have presented description of the procedure in the subsection **??**. Though variety of equations which can be generated this way covers the complete varitey of all possible equations, presently, it does not allow to solve an equation of the form (2) with both functions $F(x)$ and $f(x)$ specified arbitrarily. However, abilities of the method still remain to be studied. Nevertheless, some of results obtained and presented in this work can be applied to various problems of classical physics.

References

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