On Negative-Energy 4-Spinors and Masses in the Dirac Equation

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Abstract

Both algebraic equation $Det(\hat{p} - m) = 0$ and $Det(\hat{p} + m) = 0$ for $u-$ and $v-$ 4-spinors have solutions with $p_0 = \pm E_p = \pm\sqrt{\mathbf{p}^2 + m^2}$. The same is true for higher-spin equations (or they may even have more complicated dispersion relations). Meanwhile, every book considers the equality $p_0 = E_p$ for both $u-$ and $v-$ spinors of the $(1/2, 0) \oplus (0, 1/2))$ representation only, thus applying the Dirac-Feynman-Stueckelberg procedure for elimination of negative-energy solutions. The recent Ziino works (and, independently, the articles of several other authors) show that the Fock space can be doubled. We re-consider this possibility on the quantum-field level for both $s = 1/2$ and higher spin particles.

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1 Introduction.

The recent problems of superluminal neutrinos, negative-mass squared neutrinos, various schemes of oscillations including sterile neutrinos, require much attention. The problem of the lepton mass splitting (e, μ, τ) has long history. This suggests that something missed in the foundations of relativistic quantum theories. Modifications seem to be necessary in the Dirac sea concept, and in the even more sophisticated Stueckelberg concept of the backward propagation in time. The Dirac sea concept is intrinsically related to the Pauli principle. However, the Pauli principle is intrinsically connected with the Fermi statistics and the anticommutation relations of fermions. Recently, the concept of the *bi-orthonormality* has been proposed; the (anti) commutation relations and statistics are assumed to be different for *neutral* particles [1]. We propose the relevant modifications in the basics of the relativistic quantum theory below.

Next, Sakharov in 1967 [2] introduced the idea of two universes with opposite arrows of time, born from the same initial singularity (i.e. Big Bang). Next, the authors of [3, 4] constructed (within the framework of the present-day quantum field theory) negative-energy fields for spin-1/2 fermions. Currently, the predominating consensus is the dark matter (DM) and the dark energy (DE) paradigm. Numerous possible candidates have been proposed for the DM, but to the date, search for these candidates was not successful. “There is growing favor with the idea that new ideas need to be considered until an answer is found.” “A paradigm shift that allows the serious consideration of negative mass is a real possibility.” However, see [5] on the relation of inertial and gravitational masses.

The paper is composed in the following way: Introduction, General Framework, Main Results and Conclusion. In the main text the Dirac spinor formalism is given. Next, the additional idea of doubling the Fock space is presented, and a section in which alternative mathematical results are furnished.

2 The General Framework.

The Dirac equation is:

$$[i\gamma^\mu \partial_\mu - m]\Psi(x) = 0\,. \tag{1}$$

The γ^μ are the Clifford algebra matrices, $(\mu,\nu = 0,1,2,3)$:

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\,, \tag{2}$$

$g^{\mu\nu}$ is the metrics of the Minkowski space. Usually, everybody uses the following definition of the field operator [6, 7] in the pseudo-Euclidean metrics:

$$\Psi(x) = \frac{1}{(2\pi)^3}\sum_h \int \frac{d^3\mathbf{p}}{2E_p}[u_h(\mathbf{p})a_h(\mathbf{p})e^{-ip\cdot x} + v_h(\mathbf{p})b_h^\dagger(\mathbf{p})]e^{+ip\cdot x}]\,, \tag{3}$$

as given *ab initio*. After actions of the Dirac operator at $\exp(\mp ip_\mu x^\mu)$ the 4-spinors ($u-$ and $v-$) satisfy the momentum-space equations: $(\hat{p} - m)u_h(p) = 0$ and $(\hat{p} + m)v_h(p) = 0$, respectively; h is the polarization index. It is easy to prove from the characteristic equations $Det(\hat{p} \mp m) = (p_0^2 - \mathbf{p}^2 - m^2)^2 = 0$ that the solutions should satisfy the energy-momentum relation $p_0 = \pm E_p = \pm\sqrt{\mathbf{p}^2 + m^2}$.

The general scheme of construction of the field operator has been presented in [8]. In the case of the $(1/2, 0) \oplus (0, 1/2)$ representation we have:

$$\begin{aligned}
&\Psi(x) = \frac{1}{(2\pi)^3}\int d^4p\, \delta(p^2 - m^2)e^{-ip\cdot x}\Psi(p) = \\
= \;& \frac{1}{(2\pi)^3}\sum_h \int d^4p\, \delta(p_0^2 - E_p^2)e^{-ip\cdot x}u_h(p_0,\mathbf{p})a_h(p_0,\mathbf{p}) = \\
= \;& \frac{1}{(2\pi)^3}\int \frac{d^4p}{2E_p}[\delta(p_0 - E_p) + \delta(p_0 + E_p)][\theta(p_0) + \theta(-p_0)]e^{-ip\cdot x}\sum_h u_h(p)a_h(p) \\
= \;& \frac{1}{(2\pi)^3}\sum_h \int \frac{d^4p}{2E_p}[\delta(p_0 - E_p) + \delta(p_0 + E_p)]\left[\theta(p_0)u_h(p)a_h(p)e^{-ip\cdot x} + \right. \\
+ \;& \left.\theta(p_0)u_h(-p)a_h(-p)e^{+ip\cdot x}\right] = \frac{1}{(2\pi)^3}\sum_h \int \frac{d^3\mathbf{p}}{2E_p}\theta(p_0)\left[u_h(p)a_h(p)|_{p_0=E_p}e^{-i(E_pt - \mathbf{p}\cdot\mathbf{x})} + \right. \\
+ \;& \left. u_h(-p)a_h(-p)|_{p_0=E_p}e^{+i(E_pt - \mathbf{p}\cdot\mathbf{x})}\right]\,,
\end{aligned} \tag{4}$$

where $a_h, b_h^\dagger$ are the annihilation/creation operators, and in the textbook cases

$$u_h = \begin{pmatrix} \exp(+\boldsymbol{\sigma}\cdot\boldsymbol{\varphi})\phi_R^h(\mathbf{0}) \\ \exp(-\boldsymbol{\sigma}\cdot\boldsymbol{\varphi})\phi_L^h(\mathbf{0}) \end{pmatrix}\,,\; v_h(\mathbf{p}) = \gamma^5 u_h(\mathbf{p})\,, \tag{5}$$

where $cosh(\varphi) = E_p/m$, $sinh(\varphi) = |\mathbf{p}|/m$, $\hat{\varphi} = \mathbf{p}/|\mathbf{p}|$. During the calculations above we had to represent $1 = \theta(p_0) + \theta(-p_0)$ in order to get positive- and negative-frequency parts [9]. Moreover, during these calculations we did not yet assume, which equation this field operator (namely, the $u-$ spinor) satisfies, with negative- or positive- mass?[1]

In general, we should transform $u_h(-p)$ to the $v(p)$. The procedure is the following one [10]. In the Dirac case we should assume the following relation in the field operator:

$$\sum_h v_h(p) b_h^\dagger(p) = \sum_h u_h(-p) a_h(-p)\,. \tag{6}$$

We know that [6]

$$\begin{aligned}
\bar{u}_\mu(p) u_\lambda(p) &= +m\delta_{\mu\lambda}\,, &(7)\\
\bar{u}_\mu(p) u_\lambda(-p) &= 0\,, &(8)\\
\bar{v}_\mu(p) v_\lambda(p) &= -m\delta_{\mu\lambda}\,, &(9)\\
\bar{v}_\mu(p) u_\lambda(p) &= 0\,, &(10)
\end{aligned}$$

but we need $\Lambda_{\mu\lambda}(p) = \bar{v}_\mu(p) u_\lambda(-p)$. By direct calculations, we find

$$-m b_\mu^\dagger(p) = \sum_\lambda \Lambda_{\mu\lambda}(p) a_\lambda(-p)\,. \tag{11}$$

Hence, $\Lambda_{\mu\lambda} = -im(\boldsymbol{\sigma}\cdot\mathbf{n})_{\mu\lambda}$, $\mathbf{n} = \mathbf{p}/|\mathbf{p}|$, and

$$b_\mu^\dagger(p) = i\sum_\lambda (\boldsymbol{\sigma}\cdot\mathbf{n})_{\mu\lambda} a_\lambda(-p)\,. \tag{12}$$

Multiplying (6) by $\bar{u}_\mu(-p)$ we obtain

$$a_\mu(-p) = -i\sum_\lambda (\boldsymbol{\sigma}\cdot\mathbf{n})_{\mu\lambda} b_\lambda^\dagger(p)\,. \tag{13}$$

The equations are self-consistent. In the $(1,0)\oplus(0,1)$ representation the similar procedure leads to somewhat different situation:

$$a_\mu(p) = [1 - 2(\mathbf{S}\cdot\mathbf{n})^2]_{\mu\lambda} a_\lambda(-p)\,. \tag{14}$$

This signifies that in order to construct the Sankaranarayanan-Good field operator, it satisfies $[\gamma_{\mu\nu}\partial_\mu\partial_\nu - \frac{(i\partial/\partial t)}{E} m^2]\Psi(x) = 0$, we need additional postulates. For instance, one can try to construct the left- and the right-hand side of the field operator separately each other [9].

3 Main Results. Connections with Other Physical Models.

However, other ways of thinking are possible. First of all to mention, we have, in fact, $u_h(E_p, \mathbf{p})$ and $u_h(-E_p, \mathbf{p})$ originally, which satisfy the equations:

$$\left[E_p(\pm\gamma^0) - \boldsymbol{\gamma}\cdot\mathbf{p} - m\right] u_h(\pm E_p, \mathbf{p}) = 0\,. \tag{15}$$

[1] Moreover, since bispinors are, in general, complex-valued, we can even use the different basis such as $u_\alpha = column(i\,0\,0\,0)$ etc. instead of the well-accustomed one.

Due to the properties $U^\dagger\gamma^0 U = -\gamma^0$, $U^\dagger\gamma^i U = +\gamma^i$ with the unitary matrix $U = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \gamma^0\gamma^5$ in the Weyl basis,[2] we have in the negative-energy case:

$$\left[E_p\gamma^0 - \boldsymbol{\gamma}\cdot\mathbf{p} - m\right] U^\dagger u_h(-E_p,\mathbf{p}) = 0\,. \tag{16}$$

Thus, unless the unitary transformations do not change the physical content, we have that the negative-energy spinors $\gamma^5\gamma^0 u^-$ (see (16)) satisfy the accustomed "positive-energy" Dirac equation. Their explicit forms $\gamma^5\gamma^0 u^-$ are different from the textbook "positive-energy" Dirac spinors. From the first sight (just $E_p \to -E_p$) they are the following ones:

$$\tilde{u}(p) = \frac{N}{\sqrt{2m(-E_p+m)}}\begin{pmatrix} -p^+ + m \\ -p_r \\ p^- - m \\ -p_r \end{pmatrix}, \tag{17}$$

$$\tilde{\tilde{u}}(p) = \frac{N}{\sqrt{2m(-E_p+m)}}\begin{pmatrix} -p_l \\ -p^- + m \\ -p_l \\ p^+ - m \end{pmatrix}. \tag{18}$$

We use tildes because we do not yet know their polarization properties. It is not even clear, which helicity operator, $\sigma_3/2$ or $(\sigma\cdot\hat{\mathbf{p}})/2$, or some other should be used after $T-$ conjugation [4]. Next,

$$E_p = \sqrt{\mathbf{p}^2 + m^2} > 0, p_0 = \pm E_p, p^\pm = E_p \pm p_z, p_{r,l} = p_x \pm i p_y. \tag{19}$$

What about the $\tilde{v}(p) = \gamma^0 u^-$ transformed with the γ_0 matrix? They are not equal to the previous "negative-energy" 4-spinors $v_h(p) = \gamma^5 u_h(p)$? Obviously, they also do not have well-known forms of the usual $v-$ spinors in the Weyl basis, differing by phase factor and in the sign at the mass term. The normalizations of these 4-spinors are to $(\pm 2N^2)$.

Next, one can prove that the matrix

$$P = e^{i\theta}\gamma^0 = e^{i\theta}\begin{pmatrix} 0 & 1_{2\times 2} \\ 1_{2\times 2} & 0 \end{pmatrix} \tag{20}$$

can be used in the parity operator as well as in the original Weyl basis. However, if we would take the phase factor to be zero we obtain that while $u_h^+(p)$ have the eigenvalue $+1$, but $(R = (\mathbf{x} \to -\mathbf{x}, \mathbf{p} \to -\mathbf{p}))$

$$PR\tilde{u}(p) = PR\gamma^5\gamma^0 u_\uparrow(-E_p,\mathbf{p}) = -\tilde{u}(p)\,, \quad PR\tilde{\tilde{u}}(p) = PR\gamma^5\gamma^0 u_\downarrow(-E_p,\mathbf{p}) = -\tilde{\tilde{u}}(p)\,. \tag{21}$$

Perhaps, one should choose the phase factor $\theta = \pi$. Thus, we again confirmed that the relative (particle-antiparticle) intrinsic parity has physical significance only.

Similar formulations have been presented in Refs. [11], and [12]. The group-theoretical basis for such doubling has been given in the papers by Gelfand, Tsetlin and Sokolik [13], who first presented the theory in the 2-dimensional representation of the inversion group in 1956 (later called as "the

[2] The properties of the $U-$ matrix are opposite to those of $P^\dagger\gamma^0 P = +\gamma^0$, $P^\dagger\gamma^i P = -\gamma^i$ with the usual $P = \gamma^0$, thus giving $\left[-E_p\gamma^0 + \gamma\cdot\mathbf{p} - m\right] P u_h(-E_p,\mathbf{p}) = -\left[\hat{p} + m\right]\tilde{v}_?(p) = 0$. While, the relations of the spinors $v_h(E_p,\mathbf{p}) = \gamma_5 u_h(E_p,\mathbf{p})$ are well-known, it seems that the relations of the $v-$ spinors of the positive energy to $u-$ spinors of the negative energy are frequently forgotten, $\tilde{v}_?(p) = \gamma^0 u_h(-E_p,\mathbf{p})$.

Bargmann-Wightman-Wigner-type quantum field theory" in 1993). M. Markov wrote long ago *two* Dirac equations with the opposite signs at the mass term [11].

$$
\begin{aligned}
\left[i\gamma^\mu \partial_\mu - m\right] \Psi_1(x) &= 0\,, & (22)\\
\left[i\gamma^\mu \partial_\mu + m\right] \Psi_2(x) &= 0\,. & (23)
\end{aligned}
$$

In fact, he studied all properties of this relativistic quantum model (while he did not know yet the quantum field theory in 1937). Next, he added and subtracted these equations:

$$
\begin{aligned}
i\gamma^\mu \partial_\mu \varphi(x) - m\chi(x) &= 0\,, & (24)\\
i\gamma^\mu \partial_\mu \chi(x) - m\varphi(x) &= 0\,. & (25)
\end{aligned}
$$

Thus, φ and χ solutions can be presented as some superpositions of the Dirac 4-spinors $u-$ and $v-$. These equations, of course, can be identified with the equations for the Majorana-like $\lambda-$ and $\rho-$, which we presented in Ref. [15] on the basis of postulates [14]. Of course, the signs at the mass terms depend on, how do we associate the positive- or negative- frequency solutions with λ and ρ.

$$
\begin{aligned}
i\gamma^\mu \partial_\mu \lambda^S(x) - m\rho^A(x) &= 0\,, & (26)\\
i\gamma^\mu \partial_\mu \rho^A(x) - m\lambda^S(x) &= 0\,, & (27)\\
i\gamma^\mu \partial_\mu \lambda^A(x) + m\rho^S(x) &= 0\,, & (28)\\
i\gamma^\mu \partial_\mu \rho^S(x) + m\lambda^A(x) &= 0\,. & (29)
\end{aligned}
$$

Neither of them can be regarded as the Dirac equation. However, they can be written in the 8-component form as follows:

$$
\begin{aligned}
\left[i\Gamma^\mu \partial_\mu - m\right] \Psi_{(+)}(x) &= 0\,, & (30)\\
\left[i\Gamma^\mu \partial_\mu + m\right] \Psi_{(-)}(x) &= 0\,, & (31)
\end{aligned}
$$

with

$$
\Psi_{(+)}(x) = \begin{pmatrix} \rho^A(x) \\ \lambda^S(x) \end{pmatrix}, \Psi_{(-)}(x) = \begin{pmatrix} \rho^S(x) \\ \lambda^A(x) \end{pmatrix}, \quad \text{and} \quad \Gamma^\mu = \begin{pmatrix} 0 & \gamma^\mu \\ \gamma^\mu & 0 \end{pmatrix}. \tag{32}
$$

It is easy to find the corresponding projection operators, and the Feynman-Stueckelberg propagator.

This may just related to the spin-parity basis rotation (unitary transformations). However, in the previous papers I explained: the connection with the Dirac spinors has been found [15, 16]. For instance,

$$
\begin{pmatrix} \lambda^S_\uparrow(\mathbf{p}) \\ \lambda^S_\downarrow(\mathbf{p}) \\ \lambda^A_\uparrow(\mathbf{p}) \\ \lambda^A_\downarrow(\mathbf{p}) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & i & -1 & i \\ -i & 1 & -i & -1 \\ 1 & -i & -1 & -i \\ i & 1 & i & -1 \end{pmatrix} \begin{pmatrix} u_{+1/2}(\mathbf{p}) \\ u_{-1/2}(\mathbf{p}) \\ v_{+1/2}(\mathbf{p}) \\ v_{-1/2}(\mathbf{p}) \end{pmatrix}, \tag{33}
$$

provided that the 4-spinors have the same physical dimension. Thus, we can see that the two 4-spinor systems are connected by the unitary transformations, and this represents itself the rotation of the spin-parity basis. However, it is usually assumed that the $\lambda-$ and $\rho-$ spinors describe the neutral particles, meanwhile $u-$ and $v-$ spinors describe the charged particles. Kirchbach [16] found the amplitudes for neutrinoless double beta decay ($00\nu\beta$) in this scheme. It is obvious from (33) that there are some additional terms comparing with the standard formulation.

One can also re-write the above equations into the two-component forms. Thus, one obtains the Feynman-Gell-Mann [17] equations. As Markov wrote himself, he was expecting "new physics" from these equations. Barut and Ziino [12] proposed yet another model. They considered γ^5 operator as the operator of the charge conjugation. Thus, the charge-conjugated Dirac equation has the different sign comparing with the ordinary formulation:

$$[i\gamma^\mu \partial_\mu + m]\Psi^c_{BZ} = 0\,, \tag{34}$$

and the so-defined charge conjugation applies to the whole system, fermion+electromagnetic field, $e \to -e$ in the covariant derivative. The superpositions of the Ψ_{BZ} and Ψ^c_{BZ} also give us the "doubled Dirac equation", as the equations for $\lambda-$ and $\rho-$ spinors. The concept of the doubling of the Fock space has been developed in the Ziino works (cf. [13, 18]) in the framework of the quantum field theory. In their case the charge conjugate states are simultaneously the eigenstates of the chirality. Next, it is interesting to note that for the Majorana-like field operators

$$\nu^{ML}(x^\mu) = \int \frac{d^3\mathbf{p}}{(2\,\pi)^3}\frac{1}{2E_p}\sum_\eta \left[\lambda^S_\eta(p^\mu)\, a_\eta(p^\mu)\, e^{-\,i\,p\cdot x} \;+\; \lambda^A_\eta(p^\mu)\, a^\dagger_\eta(p^\mu)\, e^{+\,i\,p\cdot x}\right]\,. \tag{35}$$

we have

$$\begin{aligned}\left[\nu^{ML}(x^\mu) + \mathcal{C}\nu^{ML\,\dagger}(x^\mu)\right]/2 &= \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{2E_p}\sum_\eta \left[\begin{pmatrix} i\Theta\phi^{*\,\eta}_L(p^\mu) \\ 0 \end{pmatrix} a_\eta(p^\mu)e^{-ip\cdot x} + \right.\\ &\left. + \begin{pmatrix} 0 \\ \phi^\eta_L(p^\mu) \end{pmatrix} a^\dagger_\eta(p^\mu)e^{ip\cdot x}\right]\,, \end{aligned} \tag{36}$$

$$\begin{aligned}\left[\nu^{ML}(x^\mu) - \mathcal{C}\nu^{ML\,\dagger}(x^\mu)\right]/2 &= \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{2E_p}\sum_\eta \left[\begin{pmatrix} 0 \\ \phi^\eta_L(p^\mu) \end{pmatrix} a_\eta(p^\mu)e^{-ip\cdot x} + \right.\\ &\left. + \begin{pmatrix} -i\Theta\phi^{*\,\eta}_L(p^\mu) \\ 0 \end{pmatrix} a^\dagger_\eta(p^\mu)e^{ip\cdot x}\right]\,, \end{aligned} \tag{37}$$

which, thus, naturally lead to the Ziino-Barut scheme of massive chiral fields, Ref. [12].

Next, the relevant paper is ref. [19]. It is obvious to merge $u(\mathbf{p})$ and $v(\mathbf{p})$ spinors in one doublet of "positive energy" and $v-$ and $u-$ spinors, in another doublet of "negative energy" , as Markov and Fabbri did. However, for us it does not matter the previous notations and interactions in those papers. The point of my paper is that both $u(p_0, \mathbf{p})$ and $v(p_0, \mathbf{p})$ contains contributions to positive/negative energies, cf. [20].

4 The Conclusions.

The main points of my paper are: there are "negative-energy solutions" in that is previously considered as "positive-energy solutions" of relativistic wave equations, and vice versa. Their explicit forms have been presented in the case of spin-1/2. Both algebraic equation $Det(\hat{p} - m) = 0$ and $Det(\hat{p} + m) = 0$ for $u-$ and $v-$ 4-spinors have solutions with $p_0 = \pm E_p = \pm\sqrt{\mathbf{p}^2 + m^2}$. The same is true for higher-spin equations. Meanwhile, every book applies the Dirac-Feynman-Stueckelberg procedure for elimination of negative-energy solutions. The recent Ziino works (and, independently, the articles of several other authors) show that the Fock space can be doubled. We re-consider this possibility on the quantum-field level for both $s = 1/2$ and higher spin particles. Next, the relations to the previous works have been found. For instance, the doubling of the Fock space and the corresponding solutions of the Dirac equation obtained additional mathematical bases in this paper.

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References

[1] D. V. Ahluwalia and A. Ch. Nayak, *Int. J. Mod. Phys.* D**23**, 1430026 (2014).

[2] A. D. Sakharov. *JETP Letters*, **5**, 24 (1967).

[3] J. Giannini, *Int. J. Mod. Theor. Phys.* **8**, 1 (2019); B. Guay, ibid. **9**, 1 (2020), and references therein.

[4] N. Debergh *et al.*, *J. Phys. Commun.* **2**, 115012 (2018).

[5] A. E. Chubykalo and S. J. Vlayev, *Eur. J. Phys.* **19**, 1 (1998).

[6] L. H. Ryder, *Quantum Field Theory.* (Cambridge University Press, Cambridge, 1985).

[7] C. Itzykson and J.-B. Zuber, *Quantum Field Theory.* (McGraw-Hill Book Co., 1980).

[8] N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields.* 2nd Edition. (Nauka, Moscow, 1973).

[9] V. V. Dvoeglazov, *J. Phys. Conf. Ser.* **284**, 012024 (2011), arXiv:1008.2242.

[10] V. V. Dvoeglazov, *Hadronic J. Suppl.* **18**, 239 (2003), physics/0402094; *Int. J. Mod. Phys.* B**20**, 1317 (2006).

[11] M. Markov, *ZhETF* **7**, 579 (1937); ibid. 603; *Nucl. Phys.* **55**, 130 (1964).

[12] A. Barut and G. Ziino, *Mod. Phys. Lett.* A**8**, 1011 (1993); G. Ziino, *Int. J. Mod. Phys.* A**11**, 2081 (1996).

[13] I. M. Gelfand and M. L. Tsetlin, *ZhETF* **31**, 1107 (1956); G. A. Sokolik, *ZhETF* **33**, 1515 (1957).

[14] D. V. Ahluwalia, *Int. J. Mod. Phys.* A**11**, 1855 (1996); D. V. Ahluwalia and D. Grumiller, *JCAP* **0507**, 012 (2005), hep-th/0412080; idem., *Phys. Rev.* D**72**, 067701 (2005), hep-th/0410192.

[15] V. V. Dvoeglazov, *Int. J. Theor. Phys.* **34**, 2467 (1995); *Nuovo Cim.* **108**A, 1467 (1995); *Hadronic J.* **20**, 435 (1997); *Acta Phys. Polon.* B**29**, 619 (1998).

[16] M. Kirchbach, C. Compean and L. Noriega, *Eur. Phys. J.* A**22**, 149 (2004).

[17] R. P. Feynman and M. Gell-Mann, *Phys. Rev.* **109**, 193 (1958).

[18] V. V. Dvoeglazov, *Int. J. Theor. Phys.* **37**, 1915 (1998).

[19] L. Fabbri, *Int. J. Theor. Phys.* **53**, 1896 (2014), arXiv:1210.1146.

[20] G.-j. Ni, in *Relativity, Gravitation, Cosmology: New Developments.* (Nova Science Pub., NY, USA, 2010), p. 237, and references therein.