A Note on L^p-Convergence and Almost Everywhere Convergence

Yu-Lin Chou*

Abstract

It is a classical but relatively less well-known result that, for every given measure space and every given $1 \leq p \leq +\infty$, every sequence in L^p that converges in L^p has a subsequence converging almost everywhere. The typical proof is a byproduct of proving the completeness of L^p spaces, and hence is not necessarily "application-friendly". We give a simple, perhaps more "accessible" proof of this result for all finite measure spaces.

Keywords: almost everywhere convergence; convergence in L^p; subsequences
MSC 2020: 28A20; 60F25; 60F15

1 Introduction

There is the classical result: For every measure space and every $1 \leq p \leq +\infty$, every sequence in L^p that converges in L^p has a subsequence converging almost everywhere. In contrast with the classical result that every sequence of measurable functions converging in measure has a subsequence converging almost everywhere, the result under consideration may be relatively less well-known.

In Rudin [1], the concerned result (Theorem 3.12) follows from the given proof of the completeness of L^p spaces. Rudin's proof may be considered technical, and hence need not be as readily graspable for application-oriented purposes.

However, for finite measures spaces, a non-technical proof by basic means is available; we find it worth sharing.

2 Proof

Thoughout, we consider precisely real-valued functions.

*Yu-Lin Chou, Institute of Statistics, National Tsing Hua University, Hsinchu 30013, Taiwan, R.O.C.; Email: `y.l.chou@gapp.nthu.edu.tw`.

For clarity, what we should like to prove is the following

Theorem. *Let* $(\Omega, \mathscr{F}, \mathbb{M})$ *be a finite measure space; let* $1 \leq p \leq +\infty$*; let* $f, f_1, f_2, \dots \in L^p(\mathbb{M})$*. If* $f_n \to_{L^p} f$*, then there is some subsequence* $(f_{n_j})_{j \in \mathbb{N}}$ *of* $(f_n)_{n \in \mathbb{N}}$ *such that* $f_{n_j} \to_{a.e.} f$ *as* $j \to \infty$*.*

Proof. If (f_k) is a sequence of $\mathscr{F}$-measurable functions, then $f_k \to_{a.e.} f$ if and only if

$$\mathbb{M}(\limsup_{k \to \infty} \{|f_k - f| > l^{-1}\}) = 0 \text{ for all } l \in \mathbb{N}. \tag{1}$$

This follows directly from the definition of almost everywhere convergence. Since $\mathbb{M}$ is a finite measure by assumption, the continuity of $\mathbb{M}$ implies that (1) is equivalent to

$$\lim_{J \to \infty} \sum_{k \geq J} \mathbb{M}(|f_k - f| > l^{-1}) = 0 \text{ for all } l \in \mathbb{N},$$

which is equivalent to

$$\sum_k \mathbb{M}(|f_k - f| > l^{-1}) < +\infty \text{ for all } l \in \mathbb{N}. \tag{2}$$

With the above fact in mind, if $1 \leq p < +\infty$, then

$$\mathbb{M}(|f_n - f| > l^{-1}) = \mathbb{M}(|f_n - f|^p > l^{-p}) \leq l^p \int |f_n - f|^p \, \mathrm{d}\mathbb{M} \tag{3}$$

for all n, l. By the L^p-convergence assumption, for every $j \in \mathbb{N}$ there is some $n_j \in \mathbb{N}$ such that $(n_j)_j$ is strictly increasing and

$$\int |f_{n_j} - f|^p \, \mathrm{d}\mathbb{M} < 2^{-j};$$

and so

$$\sum_j l^p \int |f_{n_j} - f|^p \, \mathrm{d}\mathbb{M} = l^p \sum_j \int |f_{n_j} - f|^p \, \mathrm{d}\mathbb{M} < +\infty$$

for all l. Now (3) and (2) together imply that $f_{n_j} \to_{a.e.} f$.

There remains the case where $p = +\infty$. With $|f_n - f|_{L^\infty}$ denoting the L^∞-norm of $f_n - f$ for each n, we have

$$\mathbb{M}(|f_n - f| > l^{-1}) \leq \mathbb{M}(|f_n - f|_{L^\infty} > l^{-1}) \leq l \cdot |f_n - f|_{L^\infty} \cdot \mathbb{M}(\Omega) \tag{4}$$

for all n, l. The convergence assumption again implies that for every $j \in \mathbb{N}$ there is some $n_j \in \mathbb{N}$ such that $(n_j)_j$ is strictly increasing and

$$|f_{n_j} - f|_{L^\infty} < 2^{-j},$$

and hence

$$\sum_j l \cdot |f_{n_j} - f|_{L^\infty} \cdot \mathbb{M}(\Omega) = l \cdot \mathbb{M}(\Omega) \cdot \sum_j |f_{n_j} - f|_{L^\infty} < +\infty$$

for all l. It then follows from (4) and (2) that $f_{n_j} \to_{a.e.} f$; this completes the proof. □

References

[1] Rudin, W. (1987). *Real and Complex Analysis*, (international) third edition. McGraw-Hill.