New Principles of Differential Equations IV

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Abstract

In previous papers, we proposed several new methods to obtain general solutions or analytical solutions of some nonlinear partial differential equations. In this paper, we will continue to propose a new effective method to obtain general solutions of certain nonlinear partial differential equations for the first time, such as nonlinear wave equation, nonlinear heat equation, nonlinear Schrödinger equation, etc. .

Keywords: Analysis methods; nonlinear partial differential equations; general solutions; analytical solutions.

Introduction

The general solution of nonlinear ordinary differential equations is a field that has been studied in depth [1], and many research results have been obtained, such as Riccati equation [2], Abel equation [3-6] and so on.

Since the birth of the discipline of partial differential equations, there are very few cases that the general solution of linear equations can be obtained [7,8], and the general solution of nonlinear PDEs is one of the most mysterious areas of mathematics in which few people have studied [8,9]. Current research directions for nonlinear PDEs are mainly:

1. Use various analysis methods to obtain exact solutions [10-14], such as exp-function method [15-17], tanh-coth expansion method [18], tanh-expansion method [19], Homogeneous balance method [20-22] and so on. Among them, the solitary wave solution is one of the most concerned focuses.

2. Use diversified numerical methods to study the problem of definite solutions [23-27].

3. Using qualitative theory to analyze the definite solution problems of various nonlinear PDEs, such as the existence [28-30], uniqueness [31,32], asymptotic behavior of solutions [33,34] and so on.

4. Exact solutions and qualitative theory of fractional nonlinear PDEs [35].

In our previous paper [36], a variety of new methods for solving PDEs were proposed. A large number of general solutions of linear PDEs were obtained for the first time, such as Laplace equation, Poisson equation, Schrödinger equation, homogeneous and non-homogeneous two-dimensional wave equations, acoustic wave equations, Helmholtz equations, heat equations, transport equations, etc., and general solutions of some nonlinear PDEs were got for the first time [37,38], such as

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In $\mathbb{R}^n$

$$a_1 u_{x_1} + a_2 u_{x_2} + \ldots + a_k u_{x_k} = A\left(u\right), \tag{1}$$

where $A(u)$ *and* $a_i = a_i(u)$ *are arbitrary known functions, the general solution of Eq. (1) is*

$$f\left(y_1, y_2, \ldots y_k, x_{k+1}, x_{k+2}, \ldots x_n\right) = 0, \tag{2}$$

where

$$y_i = c_{i1}x_1 + c_{i2}x_2 + \ldots + c_{ik}x_k - \int \frac{a_i c_{i1} + a_2 c_{i2} + \ldots + a_k c_{ik}}{A\left(u\right)} du, \tag{3}$$

$$\frac{\partial\left(y_1, y_2, \ldots y_k\right)}{\partial\left(x_1, x_2, \ldots x_k\right)} \neq 0, \tag{4}$$

c_{ij} *are constants which satisfy (4)* $(i, j \in \{1, 2, \ldots k\})$.

This paper will continue to propose a new analysis method and obtain the general solutions of many types of nonlinear PDEs very concisely and clearly for the first time.

1. General solutions of nonlinear PDEs I

Earlier we proposed Z_4 Transformation [38]:

Z_4 **Transformation.** *In the domain* D, $(D \subset \mathbb{R}^n)$, *any established mth-order PDE with n space variables* $F\left(x_1, \ldots x_k, u, u_{x_1}, \ldots u_{x_k}, u_{x_1 x_2}, \ldots\right) = 0$, *set* $y_i = y_i\left(x_1, \ldots x_k, u\right)$ *and* $f(y_1, y_2, \ldots y_k, x_{k+1}, x_{k+2}, \ldots x_n)$ *are both undetermined mth-differentiable functions* $(f, y_i \in C^m\left(D\right)$, $1 \leq i \leq k \leq n)$, $y_1, y_2, \ldots y_k$ *are independent of each other, and set* $f(y_1, y_2, \ldots y_k, x_{k+1}, x_{k+2}, \ldots x_n) = 0$, *then substitute* $u, u_{x_1}, \ldots u_{x_n}, u_{x_1 x_2}, \ldots$ *into* $F = 0$
1. In case of working out $y_i = y_i\left(x_1, \ldots x_k, u\right)$ *and* $f(y_1, y_2, \ldots y_k, x_{k+1}, x_{k+2}, \ldots x_n)$, *then* $f(y_1, y_2, \ldots y_k, x_{k+1}, x_{k+2}, \ldots x_n) = 0$ *is the solution of* $F = 0$,
2. In case of dividing out all the partial derivatives of $f(y_1, y_2, \ldots y_k, x_{k+1}, x_{k+2}, \ldots x_n)$, *also working out* $y_i = y_i\left(x_1, \ldots x_k, u\right)$, *then* $f(y_1, y_2, \ldots y_k, x_{k+1}, x_{k+2}, \ldots x_n) = 0$ *is the solution of* $F = 0$, *and* f *is an arbitrary mth-differentiable function,*
3. In case of dividing out all the partial derivatives of $f(y_1, y_2, \ldots y_k, x_{k+1}, x_{k+2}, \ldots x_n)$, *also getting* $k = 0$, *but in fact* $k \neq 0$, *then* $f(y_1, y_2, \ldots y_k, x_{k+1}, x_{k+2}, \ldots x_n) = 0$ *is not the solution of* $F = 0$, *and* f *is an arbitrary mth-differentiable function.*

Below we use Z_4 Transformation to propose theorems 1 and 2.

Theorem 1. *In* $\mathbb{R}^n$,

$$a_1 u_{x_1} + a_2 u_{x_2} + \ldots + a_k u_{x_k} = 0, (2 \leq k \leq n) \tag{5}$$

the general solution of Eq. (5) is

$$f\left(\sum_{i=1}^{k} \frac{l_i x_i}{a_i} + l_{k+1} + g\left(x_{k+1}, x_{k+2}, \ldots x_n\right)\right) = 0, \tag{6}$$

where $a_i = a_i(u)$ *are arbitrary known functions,* f *and* g *are arbitrary functions, and* l_i *are relative arbitrary constants which satisfy*

$$\sum_{i=1}^{k} l_i = 0 \tag{7}$$

Prove. By Z_4 Transformation, we set $f(v) = 0$ and

$$v = \sum_{j=1}^{k} c_j x_j + l_{k+1} + g\left(x_{k+1}, x_{k+2}, \dots x_n\right) \tag{8}$$

where $c_j = c_j(u)$ are undetermined functions, l_{k+1} is an arbitrary constant, then

$$v_{x_i} = c_i + u_{x_i} \sum_{j=1}^{k} c'_{j_u} x_j, (1 \leq i \leq k)$$

$$v_{x_p} = g_{x_p}, (k \leq p \leq n)$$

$$f_{x_i} = f'_v v_{x_i} = f'_v \left(c_i + u_{x_i} \sum_{j=1}^{k} c'_{j_u} x_j \right) = 0, (1 \leq i \leq k) \tag{9}$$

where $f'_v = \frac{df}{dv}$, $c'_{j_u} = \frac{dc_j}{du}$, so

$$c_i + u_{x_i} \sum_{j=1}^{k} c'_{j_u} x_j = 0 \Longrightarrow u_{x_i} = \frac{-c_i}{\sum_{j=1}^{k} c'_{j_u} x_j}. \tag{10}$$

Then

$$a_1 u_{x_1} + a_2 u_{x_2} + \dots + a_k u_{x_k} = \sum_{i=1}^{k} \frac{-a_i c_i}{\sum_{j=1}^{k} c'_{j_u} x_j} = \frac{-1}{\sum_{j=1}^{k} c'_{j_u} x_j} \sum_{i=1}^{k} a_i c_i = 0$$
$$\Longrightarrow \sum_{i=1}^{k} a_i c_i = 0.$$

Set

$$a_i c_i = l_i, (i = 1, 2, \dots k). \tag{11}$$

Namely

$$c_i = \frac{l_i}{a_i}$$

and

$$\sum_{i=1}^{k} a_i c_i = \sum_{i=1}^{k} l_i = 0. \tag{12}$$

So the general solution of

$$a_1 u_{x_1} + a_2 u_{x_2} + \dots + a_k u_{x_k} = 0$$

is

$$f\left(\sum_{i=1}^{k} \frac{l_i x_i}{a_i} + l_{k+1} + g\left(x_{k+1}, x_{k+2}, \dots x_n\right) \right) = 0$$

The theorem is proven. □

Theorem 2. *In $\mathbb{R}^n$, the solution of*

$$a_1 u^{(2)}_{x_1} + a_2 u^{(2)}_{x_2} + \dots + a_k u^{(2)}_{x_k} = 0, (2 \leq k \leq n) \tag{13}$$

is

$$f\left(w\left(u\right)x_1+\sum_{j=2}^{k}c_jx_j+g\left(x_{k+1},x_{k+2},\dots x_n\right)+C\right)=0, \tag{14}$$

where $a_i=a_i(u)$ *are arbitrary known functions,* $u_{x_i}^{(2)}\triangleq\frac{\partial^2u}{\partial x_i^2}$*,* f *and* g *are arbitrary functions,* c_j *are arbitrary known constants* $(1\le i\le k, 2\le j\le k)$*,* C *is an arbitrary constant, and* $w(u)$ *satisfies*

$$\left(a_1w^2+A\left(u\right)\right)w_u''-2a_1w\left(w_u'\right)^2=0 \tag{15}$$

$$A\left(u\right)=\sum_{j=2}^{k}a_jc_j^2 \tag{16}$$

Prove. By Z_4 Transformation, we set $f(v)=0$ and

$$v=\sum_{j=1}^{k}c_jx_j+g\left(x_{k+1},x_{k+2},\dots x_n\right)+C \tag{17}$$

where $c_i=c_i(u)$ are undetermined functions, C is an arbitrary constant, then

$$v_{x_i}=c_i+u_{x_i}\sum_{j=1}^{k}c_{j_u}'x_j,(1\le i\le k)$$

$$v_{x_p}=g_{x_p},(k\le p\le n)$$

$$\begin{aligned}v_{x_i}^{(2)}&=c_{i_u}'u_{x_i}+u_{x_i}^{(2)}\sum_{j=1}^{k}c_{j_u}'x_j+u_{x_i}\sum_{j=1}^{k}c_{j_u}''x_ju_{x_i}+u_{x_i}c_{i_u}'\\&=2c_{i_u}'u_{x_i}+u_{x_i}^{(2)}\sum_{j=1}^{k}c_{j_u}'x_j+u_{x_i}^2\sum_{j=1}^{k}c_{j_u}''x_j\end{aligned}$$

$$v_{x_p}^{(2)}=g_{x_p}^{(2)},(k\le p\le n)$$

Because

$$f_{x_i}=f_v'v_{x_i}=f_v'\left(c_i+u_{x_i}\sum_{j=1}^{k}c_{j_u}'x_j\right)=0,(1\le i\le k)$$

$$\begin{aligned}f_{x_ix_i}&=f_v'{v_{x_i}}^{(2)}+f_v''{v_{x_i}}^2\\&=f_v'\left(2c_{i_u}'u_{x_i}+u_{x_i}^{(2)}\sum_{j=1}^{k}c_{j_u}'x_j+u_{x_i}^2\sum_{j=1}^{k}c_{j_u}'x_j\right)+f_v''\left(c_i+u_{x_i}\sum_{j=1}^{k}c_{j_u}'x_j\right)^2=0\end{aligned}$$

where $f_v''=\frac{d^2f}{dv^2}$, $c_{j_u}''=\frac{d^2c_j}{du^2}$, so

$$c_i+u_{x_i}\sum_{j=1}^{k}c_{j_u}'x_j=0\Longrightarrow u_{x_i}=\frac{-c_i}{\sum_{j=1}^{k}c_{j_u}'x_j}$$

$$\begin{aligned}&2c_{i_u}'u_{x_i}+u_{x_i}^{(2)}\sum_{j=1}^{k}c_{j_u}'x_j+u_{x_i}^2\sum_{j=1}^{k}c_{j_u}''x_j=0\\&\Longrightarrow u_{x_i}^{(2)}=-\frac{2c_{i_u}'u_{x_i}+u_{x_i}^2\sum_{j=1}^{k}c_{j_u}''x_j}{\sum_{j=1}^{k}c_{j_u}'x_j}=\frac{2c_{i_u}'\frac{c_i}{\sum_{j=1}^{k}c_{j_u}'x_j}-\frac{c_i^2}{\left(\sum_{j=1}^{k}c_{j_u}'x_j\right)^2}\sum_{j=1}^{k}c_{j_u}''x_j}{\sum_{j=1}^{k}c_{j_u}'x_j}\end{aligned}$$

Namely

$$u_{x_i}^{(2)} = \frac{2c'_{i_u} c_i \sum_{j=1}^{k} c'_{j_u} x_j - c_i^2 \sum_{j=1}^{k} c''_{j_u} x_j}{\left(\sum_{j=1}^{k} c'_{j_u} x_j\right)^3} \tag{18}$$

Then

$$\begin{aligned}
&a_1 u_{x_1}^{(2)} + a_2 u_{x_2}^{(2)} + \ldots + a_k u_{x_k}^{(2)} \\
&= a_1 \tfrac{2c'_{1_u} c_1 \sum_{j=1}^{k} c'_{j_u} x_j - c_1^2 \sum_{j=1}^{k} c''_{j_u} x_j}{\left(\sum_{j=1}^{k} c'_{j_u} x_j\right)^3} + a_2 \tfrac{2c'_{2_u} c_2 \sum_{j=1}^{k} c'_{j_u} x_j - c_2^2 \sum_{j=1}^{k} c''_{j_u} x_j}{\left(\sum_{j=1}^{k} c'_{j_u} x_j\right)^3} + \cdots \\
&+ a_k \tfrac{2c'_{k_u} c_k \sum_{j=1}^{k} c'_{j_u} x_j - c_k^2 \sum_{j=1}^{k} c''_{j_u} x_j}{\left(\sum_{j=1}^{k} c'_{j_u} x_j\right)^3} = 0
\end{aligned}$$

We get

$$2 \sum_{j=1}^{k} c'_{j_u} x_j \sum_{s=1}^{k} a_s c_s c'_{s_u} - \sum_{j=1}^{k} c''_{j_u} x_j \sum_{s=1}^{k} a_s c_s^2 = 0 \tag{19}$$

Set

$$c_1 = w(u), c_s = const, (2 \leq s \leq k) \tag{20}$$

So

$$\sum_{j=1}^{k} c'_{j_u} x_j = w'_u x_1, \sum_{s=1}^{k} a_s c_s c'_{s_u} = a_1 w w'_u,$$

$$\sum_{j=1}^{k} c'_{j_u} x_j = w''_u x_1, \sum_{s=1}^{k} a_s c_s^2 = a_1 w^2 + \sum_{s=2}^{k} a_s c_s^2 = a_1 w^2 + A(u)$$

$$A(u) = \sum_{s=2}^{k} a_s c_s^2 = \sum_{j=2}^{k} a_j c_j^2.$$

Whereupon

$$2 \sum_{j=1}^{k} c'_{j_u} x_j \sum_{s=1}^{k} a_s c_s c'_{s_u} - \sum_{j=1}^{k} c''_{j_u} x_j \sum_{s=1}^{k} a_s c_s^2 = 2 w'_u x_1 a_1 w w'_u - w''_u x_1 \left(a_1 w^2 + A(u)\right) = 0.$$

That is

$$\left(a_1 w^2 + A(u)\right) w''_u - 2a_1 w \left(w'_u\right)^2 = 0.$$

So the solution of $a_1 u_{x_1}^{(2)} + a_2 u_{x_2}^{(2)} + \ldots + a_k u_{x_k}^{(2)} = 0$ is

$$f\left(w(u) x_1 + \sum_{j=2}^{k} c_j x_j + g(x_{k+1}, x_{k+2}, \ldots x_n) + C\right) = 0,$$

where $w(u)$ satisfies Eqs. (15, 16). □

In $\mathbb{R}^n$, according to Theorem 2, we can get the solution of

$$\frac{(\sin u + \cos u)\left(\cos^2 u - \sin u\right)}{e^{2\sin u}\left(\sin u + \cos^2 u\right)} u_{xx} + (\sin u + u)\, u_{yy} + (\cos u - u)\, u_{zz} = 0$$

is

$$f\left(e^{\sin u} x + y + z + C\right) = 0.$$

2. General solutions of nonlinear PDEs II

Below we propose a new method which can effectively obtain general solutions or analytic solutions of some nonlinear PDEs.

Method 2. *In the domain D, $(D \subset \mathbb{R}^n)$, any established mth-order PDE with n space variables $F(x_1, \ldots x_n, u, u_{x_1}, \ldots u_{x_n}, u_{x_1x_2}, \ldots) = 0$, set $v = v(x_1, \ldots x_n)$ and $u = f(v)$ are all undetermined m-order derivable functions $(u, v \in C^m(D))$, then substitute $u = f(v)$ and its partial derivatives into $F = 0$ to obtain the mixed differential equation $F^d_\partial(x_1, \ldots x_n, f, f'_v, f''_v, \ldots v_{x_1}, \ldots v_{x_n}, v_{x_1x_2}, \ldots) = 0$, and let $H(x_1, \ldots x_n, v, v_{x_1}, \ldots v_{x_n}, v_{x_1x_2}, \ldots) = 0$, then*

In case of working out $v = v(x_1, \ldots x_n)$ of $H(x_1, \ldots x_n, v, v_{x_1}, \ldots v_{x_n}, v_{x_1x_2}, \ldots) = 0$, and $F^d_\partial(x_1, \ldots x_n, f, f'_v, f''_v, \ldots v_{x_1}, \ldots v_{x_n}, v_{x_1x_2}, \ldots) = 0$ could be transformed into an ODE $F^d(f, f'_v, f''_v, \ldots) = 0$ whose solution is $\Phi(f, v) = 0$, then $\Phi(f, v) = 0$ is a solution of $F(x_1, \ldots x_n, u, u_{x_1}, \ldots u_{x_n}, u_{x_1x_2}, \ldots) = 0$.

In general, if $v = v(x_1, \ldots x_n)$ is the general solution of $H(x_1, \ldots x_n, v, v_{x_1}, \ldots v_{x_n}, v_{x_1x_2}, \ldots) = 0$, then $\Phi(f, v) = 0$ is the general solution of $F(x_1, \ldots x_n, u, u_{x_1}, \ldots u_{x_n}, u_{x_1x_2}, \ldots) = 0$; if $v = v(x_1, \ldots x_n)$ is a analytic solution of $H(x_1, \ldots x_n, v, v_{x_1}, \ldots v_{x_n}, v_{x_1x_2}, \ldots) = 0$, then $\Phi(f, v) = 0$ is a analytic solution of $F(x_1, \ldots x_n, u, u_{x_1}, \ldots u_{x_n}, u_{x_1x_2}, \ldots) = 0$.

Below we use Method 2 to propose Theorem 3-9.

Theorem 3. *In $\mathbb{R}^4$,*

$$a_1 u_{tt} + a_2 u_{xx} + a_3 u_{yy} + a_4 u_{zz} = A(u), \tag{21}$$

the general solution of Eq. (21) is

$$\Upsilon(\xi) + \Lambda(\eta) + \zeta = \pm \int \left(b_6 + \int K(u)\, du \right)^{-\frac{1}{2}} du, \tag{22}$$

where $a_1 - a_4$ are arbitrary known constants, $A(u)$ is an arbitrary known function, $\Upsilon(\xi)$ and $\Lambda(\eta)$ are arbitrary unary functions and

$$\xi = \left(-\frac{a_2 k_2^2 + a_3 k_3^2 + a_4 k_4^2}{a_1} \right)^{\frac{1}{2}} t + k_2 x + k_3 y + k_4 z + k_5, \tag{23}$$

$$\eta = -\left(-\frac{a_2 l_2^2 + a_3 l_3^2 + a_4 l_4^2}{a_1} \right)^{\frac{1}{2}} t + l_2 x + l_3 y + l_4 z + l_5, \tag{24}$$

$$\zeta = b_1 t + b_2 x + b_3 y + b_4 z + b_5, \tag{25}$$

$$K(u) = \frac{2A(u)}{a_1 b_1^2 + a_2 b_2^2 + a_3 b_3^2 + a_4 b_4^2}, \tag{26}$$

where k_5, l_5, b_5 and b_6 are arbitrary constants, $k_2 - k_4, l_2 - l_4, b_1 - b_4$ are relative arbitrary constants which satisfy

$$a_2 k_2 l_2 + a_3 k_3 l_3 + a_4 k_4 l_4 - \sqrt{\left(a_2 k_2^2 + a_3 k_3^2 + a_4 k_4^2\right)\left(a_2 l_2^2 + a_3 l_3^2 + a_4 l_4^2\right)} = 0 \tag{27}$$

$$a_2 b_2 k_2 + a_3 b_3 k_3 + a_4 b_4 k_4 + a_1 b_1 \left(-\frac{a_2 k_2^2 + a_3 k_3^2 + a_4 k_4^2}{a_1} \right)^{\frac{1}{2}} = 0 \tag{28}$$

$$a_2b_2l_2 + a_3b_3l_3 + a_4b_4l_4 - a_1b_1\left(-\frac{a_2l_2^2 + a_3l_3^2 + a_4l_4^2}{a_1}\right)^{\frac{1}{2}} = 0 \tag{29}$$

Prove. According to Method 2, we set

$$u(t, x, y, z) = f(v)$$

where f is an undetermined unary function, then

$$\begin{aligned}&a_1u_{tt} + a_2u_{xx} + a_3u_{yy} + a_4u_{zz}\\&= a_1\left(f_v^{''}v_t^2 + f_v^{'}v_{tt}\right) + a_2\left(f_v^{''}v_x^2 + f_v^{'}v_{xx}\right) + a_3\left(f_v^{''}v_y^2 + f_v^{'}v_{yy}\right) + a_4\left(f_v^{''}v_z^2 + f_v^{'}v_{zz}\right).\end{aligned}$$

Namely

$$f_v^{''}\left(a_1v_t^2 + a_2v_x^2 + a_3v_y^2 + a_4v_z^2\right) + f_v^{'}\left(a_1v_{tt} + a_2v_{xx} + a_3v_{yy} + a_4v_{zz}\right) = A\left(f\right). \tag{30}$$

Set

$$a_1v_{tt} + a_2v_{xx} + a_3v_{yy} + a_4v_{zz} = 0. \tag{31}$$

The general solution of (31) is [36]

$$v\left(x, y, z\right) = \Upsilon\left(\xi\right) + \Lambda\left(\eta\right) + \zeta, \tag{32}$$

where $\Upsilon(\xi)$ and $\Lambda(\eta)$ are arbitrary unary functions and

$$\xi = \left(-\frac{a_2k_2^2 + a_3k_3^2 + a_4k_4^2}{a_1}\right)^{\frac{1}{2}}t + k_2x + k_3y + k_4z + k_5$$

$$\eta = -\left(-\frac{a_2l_2^2 + a_3l_3^2 + a_4l_4^2}{a_1}\right)^{\frac{1}{2}}t + l_2x + l_3y + l_4z + l_5$$

$$\zeta = b_1t + b_2x + b_3y + b_4z + b_5$$

where $k_2 - k_5, l_2 - l_5$ and $b_1 - b_5$ are arbitrary constants. According to (32) and (23-25)

$$\begin{aligned}a_1v_t^2 = &-\left(a_2k_2^2 + a_3k_3^2 + a_4k_4^2\right)\left(\Upsilon_\xi^{'}\right)^2 - \left(a_2l_2^2 + a_3l_3^2 + a_4l_4^2\right)\left(\Lambda_\eta^{'}\right)^2 + a_1b_1^2\\&- 2\sqrt{\left(a_2k_2^2 + a_3k_3^2 + a_4k_4^2\right)\left(a_2l_2^2 + a_3l_3^2 + a_4l_4^2\right)}\Upsilon_\xi^{'}\Lambda_\eta^{'}\\&+ 2a_1b_1\left(-\frac{a_2k_2^2 + a_3k_3^2 + a_4k_4^2}{a_1}\right)^{\frac{1}{2}}\Upsilon_\xi^{'} - 2a_1b_1\left(-\frac{a_2l_2^2 + a_3l_3^2 + a_4l_4^2}{a_1}\right)^{\frac{1}{2}}\Lambda_\eta^{'},\end{aligned} \tag{33}$$

$$a_2v_x^2 = a_2k_2^2\left(\Upsilon_\xi^{'}\right)^2 + a_2l_2^2\left(\Lambda_\eta^{'}\right)^2 + a_2b_2^2 + 2a_2k_2l_2\Upsilon_\xi^{'}\Lambda_\eta^{'} + 2a_2b_2k_2\Upsilon_\xi^{'} + 2a_2b_2l_2\Lambda_\eta^{'}, \tag{34}$$

$$a_3v_y^2 = a_3k_3^2\left(\Upsilon_\xi^{'}\right)^2 + a_3l_3^2\left(\Lambda_\eta^{'}\right)^2 + a_3b_3^2 + 2a_3k_3l_3\Upsilon_\xi^{'}\Lambda_\eta^{'} + 2a_3b_3k_3\Upsilon_\xi^{'} + 2a_3b_3l_3\Lambda_\eta^{'}, \tag{35}$$

$$a_4v_z^2 = a_4k_4^2\left(\Upsilon_\xi^{'}\right)^2 + a_4l_4^2\left(\Lambda_\eta^{'}\right)^2 + a_4b_4^2 + 2a_4k_4l_4\Upsilon_\xi^{'}\Lambda_\eta^{'} + 2a_4b_4k_4\Upsilon_\xi^{'} + 2a_4b_4l_4\Lambda_\eta^{'}. \tag{36}$$

And

$$\begin{aligned}&a_1v_t^2 + a_2v_x^2 + a_3v_y^2 + a_4v_z^2\\&= 2\Upsilon_\xi^{'}\Lambda_\eta^{'}\left(a_2k_2l_2 + a_3k_3l_3 + a_4k_4l_4 - \sqrt{\left(a_2k_2^2 + a_3k_3^2 + a_4k_4^2\right)\left(a_2l_2^2 + a_3l_3^2 + a_4l_4^2\right)}\right)\\&+2\Upsilon_\xi^{'}\left(a_2b_2k_2 + a_3b_3k_3 + a_4b_4k_4 + a_1b_1\left(-\tfrac{a_2k_2^2+a_3k_3^2+a_4k_4^2}{a_1}\right)^{\frac{1}{2}}\right)\\&+2\Lambda_\eta^{'}\left(a_2b_2l_2 + a_3b_3l_3 + a_4b_4l_4 - a_1b_1\left(-\tfrac{a_2l_2^2+a_3l_3^2+a_4l_4^2}{a_1}\right)^{\frac{1}{2}}\right)\\&+a_1b_1^2 + a_2b_2^2 + a_3b_3^2 + a_4b_4^2.\end{aligned}$$

Set

$$a_2k_2l_2 + a_3k_3l_3 + a_4k_4l_4 - \sqrt{\left(a_2k_2^2 + a_3k_3^2 + a_4k_4^2\right)\left(a_2l_2^2 + a_3l_3^2 + a_4l_4^2\right)} = 0,$$

$$a_2b_2k_2 + a_3b_3k_3 + a_4b_4k_4 + a_1b_1\left(-\frac{a_2k_2^2 + a_3k_3^2 + a_4k_4^2}{a_1}\right)^{\frac{1}{2}} = 0,$$

$$a_2b_2l_2 + a_3b_3l_3 + a_4b_4l_4 - a_1b_1\left(-\frac{a_2l_2^2 + a_3l_3^2 + a_4l_4^2}{a_1}\right)^{\frac{1}{2}} = 0.$$

Since $k_2, k_3, k_4, l_2, l_3, l_4, b_1, b_2, b_3$ and b_4 are all arbitrary constants, there are 10 in total, the above equations can always be satisfied at the same time, so

$$a_1v_t^2 + a_2v_x^2 + a_3v_y^2 + a_4v_z^2 = a_1b_1^2 + a_2b_2^2 + a_3b_3^2 + a_4b_4^2, \tag{37}$$

and

$$f_v''\left(a_1b_1^2 + a_2b_2^2 + a_3b_3^2 + a_4b_4^2\right) = A(f). \tag{38}$$

The general solution of (38) is

$$v = b_7 \pm \int \left(b_6 + \frac{2\int A(f)\,df}{a_1b_1^2 + a_2b_2^2 + a_3b_3^2 + a_4b_4^2}\right)^{-\frac{1}{2}} df,$$

where b_6 and b_7 are arbitrary constants, according to Method 2, the general solution of (21) is

$$v(x, y, z) = \Upsilon(\xi) + \Lambda(\eta) + \zeta = b_7 \pm \int \left(b_6 + \frac{2\int A(u)\,du}{a_1b_1^2 + a_2b_2^2 + a_3b_3^2 + a_4b_4^2}\right)^{-\frac{1}{2}} du.$$

Set $b_7 = 0$, then the general solution of (21) can be written as

$$\Upsilon(\xi) + \Lambda(\eta) + \zeta = \pm \int \left(b_6 + \int^K (u)\,du\right)^{-\frac{1}{2}} du$$

where ξ, η, ζ and $K(u)$ satisfy (23-29), the theorem is proven. □

Eq. (21) is a nonlinear PDE that has been studied emphatically [9, 39], some nonlinear wave equations are special cases of it [30, 40-42]. Using analytical methods to obtain the exact solutions in various special situations and using qualitative theory to research the laws of the problems of definite solutions are currently main methods.

Theorem 4. *In* $\mathbb{R}^4$,

$$a_1u_{tt} + a_2u_{xx} + a_3u_{yy} + B(u)\,u_{zz} = A(u), \tag{39}$$

the general solution of Eq. (39) is

$$\Upsilon(\xi) + \Lambda(\eta) + \zeta = \pm \int (b_6 + \textstyle\int K(u)\,du)^{-\frac{1}{2}} du, \tag{40}$$

where $a_1 - a_3$ *are arbitrary known constants,* $A(u)$ *and* $B(u)$ *are arbitrary known functions,* $\Upsilon(\xi)$ *and* $\Lambda(\eta)$ *are arbitrary unary functions and*

$$\xi = \left(-\frac{a_2k_2^2 + a_3k_3^2}{a_1}\right)^{\frac{1}{2}} t + k_2x + k_3y + k_4, \tag{41}$$

$$\eta = -\left(-\frac{a_2l_2^2 + a_3l_3^2}{a_1}\right)^{\frac{1}{2}} t + l_2x + l_3y + l_4, \tag{42}$$

$$\zeta = b_1 t + b_2 x + b_3 y + b_4 z + b_5, \tag{43}$$

$$K(u) = \frac{2A(u)}{a_1 b_1^2 + a_2 b_2^2 + a_3 b_3^2 + b_4^2 B(u)}, \tag{44}$$

where k_4, l_4, b_4, b_5 and b_6 are arbitrary constants, k_2, k_3, l_2, l_3 and $b_1 - b_3$ are relative arbitrary constants which satisfy

$$a_2 k_2 l_2 + a_3 k_3 l_3 - \sqrt{\left(a_2 k_2^2 + a_3 k_3^2\right)\left(a_2 l_2^2 + a_3 l_3^2\right)} = 0, \tag{45}$$

$$a_2 b_2 k_2 + a_3 b_3 k_3 + a_1 b_1 \left(-\frac{a_2 k_2^2 + a_3 k_3^2}{a_1}\right)^{\frac{1}{2}} = 0, \tag{46}$$

$$a_2 b_2 l_2 + a_3 b_3 l_3 - a_1 b_1 \left(-\frac{a_2 l_2^2 + a_3 l_3^2}{a_1}\right)^{\frac{1}{2}} = 0. \tag{47}$$

Prove. According to Method 2, we set

$$u(t, x, y, z) = f(v),$$

where f is an undetermined unary function, then

$$\begin{aligned} & a_1 u_{tt} + a_2 u_{xx} + a_3 u_{yy} + B(u)\, u_{zz} \\ & = a_1 \left(f_v^{''} v_t^2 + f_v^{'} v_{tt}\right) + a_2 \left(f_v^{''} v_x^2 + f_v^{'} v_{xx}\right) + a_3 \left(f_v^{''} v_y^2 + f_v^{'} v_{yy}\right) + B(f) \left(f_v^{''} v_z^2 + f_v^{'} v_{zz}\right) \\ & = A(f). \end{aligned}$$

Namely

$$f_v^{''} \left(a_1 v_t^2 + a_2 v_x^2 + a_3 v_y^2 + B(f)\, v_z^2\right) + f_v^{'} \left(a_1 v_{tt} + a_2 v_{xx} + a_3 v_{yy} + B(f)\, v_{zz}\right) = A(f). \tag{48}$$

Set

$$v = g(t, x, y) + b_4 z + b_5. \tag{49}$$

Then

$$a_1 v_{tt} + a_2 v_{xx} + a_3 v_{yy} + B(f)\, v_{zz} = a_1 g_{tt} + a_2 g_{xx} + a_3 g_{yy}.$$

Set

$$a_1 g_{tt} + a_2 g_{xx} + a_3 g_{yy} = 0. \tag{50}$$

The general solution of (50) is [36]

$$g(t, x, y) = \Upsilon(\xi) + \Lambda(\eta) + b_1 t + b_2 x + b_3 y, \tag{51}$$

where $\Upsilon(\xi)$ and $\Lambda(\eta)$ are arbitrary unary functions and

$$\xi = \left(-\frac{a_2 k_2^2 + a_3 k_3^2}{a_1}\right)^{\frac{1}{2}} t + k_2 x + k_3 y + k_4,$$

$$\eta = -\left(-\frac{a_2 l_2^2 + a_3 l_3^2}{a_1}\right)^{\frac{1}{2}} t + l_2 x + l_3 y + l_4,$$

So

$$v = \Upsilon(\xi) + \Lambda(\eta) + b_1 t + b_2 x + b_3 y + b_4 z + b_5. \tag{52}$$

And

$$a_1 v_t^2 = -\left(a_2k_2^2 + a_3k_3^2\right)\left(\Upsilon_\xi^{'}\right)^2 - \left(a_2l_2^2 + a_3l_3^2\right)\left(\Lambda_\eta^{'}\right)^2 + a_1b_1^2$$
$$- 2\sqrt{\left(a_2k_2^2 + a_3k_3^2\right)\left(a_2l_2^2 + a_3l_3^2\right)}\Upsilon_\xi^{'}\Lambda_\eta^{'} + 2a_1b_1\left(-\frac{a_2k_2^2 + a_3k_3^2}{a_1}\right)^{\frac{1}{2}}\Upsilon_\xi^{'} \tag{53}$$
$$- 2a_1b_1\left(-\frac{a_2l_2^2 + a_3l_3^2}{a_1}\right)^{\frac{1}{2}}\Lambda_\eta^{'},$$

$$a_2v_x^2 = a_2k_2^2\left(\Upsilon_\xi^{'}\right)^2 + a_2l_2^2\left(\Lambda_\eta^{'}\right)^2 + a_2b_2^2 + 2a_2k_2l_2\Upsilon_\xi^{'}\Lambda_\eta^{'} + 2a_2b_2k_2\Upsilon_\xi^{'} + 2a_2b_2l_2\Lambda_\eta^{'}, \tag{54}$$

$$a_3v_y^2 = a_3k_3^2\left(\Upsilon_\xi^{'}\right)^2 + a_3l_3^2\left(\Lambda_\eta^{'}\right)^2 + a_3b_3^2 + 2a_3k_3l_3\Upsilon_\xi^{'}\Lambda_\eta^{'} + 2a_3b_3k_3\Upsilon_\xi^{'} + 2a_3b_3l_3\Lambda_\eta^{'}, \tag{55}$$

$$B(f)\,v_z^2 = b_4^2 B(f)\,. \tag{56}$$

Whereupon

$$\begin{aligned}
&a_1v_t^2 + a_2v_x^2 + a_3v_y^2 + B(f)\,v_z^2\\
&= 2\Upsilon_\xi^{'}\Lambda_\eta^{'}\left(a_2k_2l_2 + a_3k_3l_3 - \sqrt{\left(a_2k_2^2 + a_3k_3^2\right)\left(a_2l_2^2 + a_3l_3^2\right)}\right)\\
&+2\Upsilon_\xi^{'}\left(a_2b_2k_2 + a_3b_3k_3 + a_1b_1\left(-\tfrac{a_2k_2^2 + a_3k_3^2}{a_1}\right)^{\frac{1}{2}}\right)\\
&+2\Lambda_\eta^{'}\left(a_2b_2l_2 + a_3b_3l_3 - a_1b_1\left(-\tfrac{a_2l_2^2 + a_3l_3^2}{a_1}\right)^{\frac{1}{2}}\right) + a_1b_1^2 + a_2b_2^2 + a_3b_3^2 + b_4^2B(f)\,.
\end{aligned}$$

Set

$$a_2k_2l_2 + a_3k_3l_3 - \sqrt{\left(a_2k_2^2 + a_3k_3^2\right)\left(a_2l_2^2 + a_3l_3^2\right)} = 0,$$

$$a_2b_2k_2 + a_3b_3k_3 + a_1b_1\left(-\frac{a_2k_2^2 + a_3k_3^2}{a_1}\right)^{\frac{1}{2}} = 0,$$

$$a_2b_2l_2 + a_3b_3l_3 - a_1b_1\left(-\frac{a_2l_2^2 + a_3l_3^2}{a_1}\right)^{\frac{1}{2}} = 0.$$

Thereupon

$$a_1v_t^2 + a_2v_x^2 + a_3v_y^2 + B(f)\,v_z^2 = a_1b_1^2 + a_2b_2^2 + a_3b_3^2 + b_4^2B(f)\,, \tag{57}$$

and

$$\begin{aligned}
&f_v^{''}\left(a_1v_t^2 + a_2v_x^2 + a_3v_y^2 + B(f)\,v_z^2\right) + f_v^{'}\left(a_1v_{tt} + a_2v_{xx} + a_3v_{yy} + B(f)\,v_{zz}\right)\\
&= f_v^{''}\left(a_1b_1^2 + a_2b_2^2 + a_3b_3^2 + b_4^2B(f)\right) = A(f)\,.
\end{aligned}$$

That is

$$f_v^{''} = \frac{A(f)}{a_1b_1^2 + a_2b_2^2 + a_3b_3^2 + b_4^2K(f)}. \tag{58}$$

Then

$$v = b_7 \pm \int\left(b_6 + \frac{2\int A(f)\,df}{a_1b_1^2 + a_2b_2^2 + a_3b_3^2 + b_4^2K(f)}\right)^{-\frac{1}{2}} df.$$

Set $b_7 = 0$, then the general solution of (39) is

$$\Upsilon(\xi) + \Lambda(\eta) + \zeta = \pm\int\left(b_6 + \int K(u)\,du\right)^{-\frac{1}{2}} du,$$

where ξ, η, ζ and $K(u)$ satisfy (41-47), the theorem is proven. □

Theorem 5. *In* $\mathbb{R}^4$,

$$a_1 u_{tt} + a_2 u_{xx} + C(u) u_{yy} + B(u) u_{zz} = A(u), \tag{59}$$

the general solution of Eq. (59) is

$$\Upsilon(\xi) + \Lambda(\eta) + \zeta = \pm \int \left(b_6 + \int K(u) \, du \right)^{-\frac{1}{2}} du, \tag{60}$$

where a_1 *and* a_2 *are arbitrary known constants,* $A(u), B(u)$ *and* $C(u)$ *are arbitrary known functions,* $\Upsilon(\xi)$ *and* $\Lambda(\eta)$ *are arbitrary unary functions and*

$$\xi = \left(-\frac{a_2}{a_1} \right)^{\frac{1}{2}} k_2 t + k_2 x + k_3, \tag{61}$$

$$\eta = -\left(-\frac{a_2}{a_1} \right)^{\frac{1}{2}} l_2 t + l_2 x + l_3, \tag{62}$$

$$\zeta = b_3 y + b_4 z + b_5, \tag{63}$$

$$K(u) = \frac{2A(u)}{b_3^2 C(u) + b_4^2 B(u)}, \tag{64}$$

where k_2, k_3, l_2, l_3 *and* $b_3 - b_6$ *are arbitrary constants.*

Prove. According to Method 2, we set

$$u(t, x, y, z) = f(v),$$

where f is an undetermined unary function, then

$$\begin{aligned} & a_1 u_{tt} + a_2 u_{xx} + C(u) u_{yy} + B(u) u_{zz} \\ & = a_1 \left(f_v'' v_t^2 + f_v' v_{tt} \right) + a_2 \left(f_v'' v_x^2 + f_v' v_{xx} \right) + C(f) \left(f_v'' v_y^2 + f_v' v_{yy} \right) + B(f) \left(f_v'' v_z^2 + f_v' v_{zz} \right). \end{aligned}$$

Namely

$$f_v'' \left(a_1 v_t^2 + a_2 v_x^2 + C(f) v_y^2 + B(f) v_z^2 \right) + f_v' \left(a_1 v_{tt} + a_2 v_{xx} + C(f) v_{yy} + B(f) v_{zz} \right) = A(f). \tag{65}$$

Set

$$v = g(t, x) + b_3 y + b_4 z. \tag{66}$$

So

$$a_1 v_{tt} + a_2 v_{xx} + C(f) v_{yy} + B(f) v_{zz} = a_1 g_{tt} + a_2 g_{xx}.$$

Set

$$a_1 g_{tt} + a_2 g_{xx} = 0. \tag{67}$$

The general solution of (67) is [36]

$$g(t, x) = \Upsilon(\xi) + \Lambda(\eta) + b_1 t + b_2 x + b_5, \tag{68}$$

where $\Upsilon(\xi)$ and $\Lambda(\eta)$ are arbitrary unary functions and

$$\xi = \left(-\frac{a_2}{a_1} \right)^{\frac{1}{2}} k_2 t + k_2 x + k_3,$$

$$\eta=-\left(-\frac{a_2}{a_1}\right)^{\frac{1}{2}}l_2t+l_2x+l_3.$$

Then

$$v=\Upsilon\left(\xi\right)+\Lambda\left(\eta\right)+b_1t+b_2x+b_3y+b_4z+b_5. \tag{69}$$

And

$$\begin{aligned}a_1v_t^2=&-a_2k_2^2\left(\Upsilon_\xi^{'}\right)^2-a_2l_2^2\left(\Lambda_\eta^{'}\right)^2+a_1b_1^2-2a_1a_2k_2l_2\Upsilon_\xi^{'}\Lambda_\eta^{'}\\&+2a_1b_1\left(-\frac{a_2}{a_1}\right)^{\frac{1}{2}}k_2\Upsilon_\xi^{'}-2a_1b_1\left(-\frac{a_2}{a_1}\right)^{\frac{1}{2}}l_2\Lambda_\eta^{'},\end{aligned} \tag{70}$$

$$a_2v_x^2=a_2k_2^2\left(\Upsilon_\xi^{'}\right)^2+a_2l_2^2\left(\Lambda_\eta^{'}\right)^2+a_2b_2^2+2a_2k_2l_2\Upsilon_\xi^{'}\Lambda_\eta^{'}+2a_2b_2k_2\Upsilon_\xi^{'}+2a_2b_2l_2\Lambda_\eta^{'}, \tag{71}$$

$$C\left(f\right)v_y^2=b_3^2C\left(f\right),B\left(f\right)v_z^2=b_4^2B\left(f\right). \tag{72}$$

Thereupon

$$\begin{aligned}&a_1v_t^2+a_2v_x^2+C\left(f\right)v_y^2+B\left(f\right)v_z^2\\&=2\left(a_1b_1\left(-\tfrac{a_2}{a_1}\right)^{\frac{1}{2}}+a_2b_2\right)k_2\Upsilon_\xi^{'}+2\left(a_2b_2-a_1b_1\left(-\tfrac{a_2}{a_1}\right)^{\frac{1}{2}}\right)l_2\Lambda_\eta^{'}\\&+a_1b_1^2+a_2b_2^2+b_3^2C\left(f\right)+b_4^2B\left(f\right).\end{aligned}$$

Set

$$\begin{cases}a_1b_1\left(-\frac{a_2}{a_1}\right)^{\frac{1}{2}}+a_2b_2=0\\a_2b_2-a_1b_1\left(-\frac{a_2}{a_1}\right)^{\frac{1}{2}}=0.\end{cases}$$

That is

$$a_2b_2=0,$$

$$a_1b_1\left(-\frac{a_2}{a_1}\right)^{\frac{1}{2}}=0.$$

Since a_1 and a_2 are arbitrary known constants, and $a_1,a_2\neq0$ in general, we can get

$$b_1=b_2=0.$$

Whereupon

$$a_1v_t^2+a_2v_x^2+C\left(f\right)v_y^2+B\left(f\right)v_z^2=a_1b_1^2+a_2b_2^2+b_3^2C\left(f\right)+b_4^2B\left(f\right)=b_3^2C\left(f\right)+b_4^2B\left(f\right),$$

$$\begin{aligned}&f_v^{''}\left(a_1v_t^2+a_2v_x^2+C\left(f\right)v_y^2+B\left(f\right)v_z^2\right)+f_v^{'}\left(a_1v_{tt}+a_2v_{xx}+C\left(f\right)v_{yy}+B\left(f\right)v_{zz}\right)\\&=f_v^{''}\left(b_3^2C\left(f\right)+b_4^2B\left(f\right)\right)=A\left(f\right).\end{aligned}$$

Namely

$$f_v^{''}=\frac{A\left(f\right)}{b_3^2C\left(f\right)+b_4^2B\left(f\right)}. \tag{73}$$

Then

$$v=b_7\pm\int\left(b_6+\frac{2\int A\left(f\right)df}{b_3^2C\left(f\right)+b_4^2B\left(f\right)}\right)^{-\frac{1}{2}}df.$$

Set $b_7=0$, so the general solution of (59) is

$$\Upsilon\left(\xi\right)+\Lambda\left(\eta\right)+\zeta=\pm\int\left(b_6+\int K\left(u\right)du\right)^{-\frac{1}{2}}du,$$

where ξ, η, ζ and $K(u)$ satisfy (61-64), the theorem is proven. □

Theorem 6. *In* $\mathbb{R}^4$,

$$a_1 u_t + a_2 u_{xx} + a_3 u_{yy} + a_4 u_{zz} = A\left(u\right), \tag{74}$$

the general solution of Eq. (74) is

$$\Upsilon\left(\xi\right) + \Lambda\left(\eta\right) + \zeta = \pm \int \left(c_6 + \int K\left(u\right) du\right)^{-\frac{1}{2}} du, \tag{75}$$

where $a_1 - a_4$ are arbitrary known constants, $A(u)$ is an arbitrary known function, $\Upsilon(\xi)$ and $\Lambda(\eta)$ are arbitrary unary functions and

$$\zeta = c_2 x + c_3 y + c_4 z + c_5, \tag{76}$$

$$\xi = \sqrt{\frac{-a_3 l_3^2 - a_4 l_4^2}{a_2}} x + l_3 y + l_4 z + l_5, \tag{77}$$

$$\eta = \sqrt{\frac{-a_3 l_{13}^2 - a_4 l_{14}^2}{a_2}} x + l_{13} y + l_{14} z + l_{15}, \tag{78}$$

$$K\left(u\right) = \frac{2A\left(u\right)}{a_2 c_2^2 + a_3 c_3^2 + a_4 c_4^2}, \tag{79}$$

where l_5, l_{15}, c_5 and c_6 are arbitrary constants, $c_2 - c_4, k_2 - k_4, k_{11} - k_{14}, l_3, l_4, l_{13}$ and l_{14} are relative arbitrary constants which satisfy

$$-a_2 \sqrt{\frac{-a_3 l_3^2 - a_4 l_4^2}{a_2}} \sqrt{\frac{-a_3 l_{13}^2 - a_4 l_{14}^2}{a_2}} + a_3 l_3 l_{13} + a_4 l_4 l_{14} = 0, \tag{80}$$

$$a_2 c_2 \sqrt{\frac{-a_3 l_3^2 - a_4 l_4^2}{a_2}} + a_3 l_3 c_3 + a_4 c_4 l_4 = 0, \tag{81}$$

$$-a_2 c_2 \sqrt{\frac{-a_3 l_{13}^2 - a_4 l_{14}^2}{a_2}} + a_3 c_3 l_{13} + a_4 c_4 l_{14} = 0. \tag{82}$$

Prove. According to Method 2, we set

$$u(t, x, y, z) = f(v),$$

where f is an undetermined unary function, then

$$\begin{aligned} &a_1 u_t + a_2 u_{xx} + a_3 u_{yy} + a_4 u_{zz} \\ &= a_1 f_v^{'} v_t + a_2 \left(f_v^{''} v_x^2 + f_v^{'} v_{xx}\right) + a_3 \left(f_v^{''} v_y^2 + f_v^{'} v_{yy}\right) + a_4 \left(f_v^{''} v_z^2 + f_v^{'} v_{zz}\right). \end{aligned}$$

Namely

$$f_v^{''}\left(a_2 v_x^2 + a_3 v_y^2 + a_4 v_z^2\right) + f_v^{'}\left(a_1 v_t + a_2 v_{xx} + a_3 v_{yy} + a_4 v_{zz}\right) = A\left(f\right). \tag{83}$$

Set

$$a_1 v_t + a_2 v_{xx} + a_3 v_{yy} + a_4 v_{zz} = 0. \tag{84}$$

The general solution of (84) is [36]

$$v\left(x,y,z\right)=\lambda\Upsilon\left(\xi\right)+\zeta\Lambda\left(\eta\right)+c_2x+c_3y+c_4z+c_5, \tag{85}$$

where $\Upsilon(\xi)$ and $\Lambda(\eta)$ are arbitrary unary functions and

$$\lambda=e^{\frac{-a_1k_1(k_1t+k_2x+k_3y+k_4z)}{a_2k_2^2+a_3k_3^2+a_4k_4^2}}, \tag{86}$$

$$\zeta=e^{\frac{-a_1k_{11}(k_{11}t+k_{12}x+k_{13}y+k_{14}z)}{a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2}}, \tag{87}$$

$$\xi=\frac{2k_1\left(k_2\sqrt{-a_2\left(a_3l_3^2+a_4l_4^2\right)}+a_3l_3k_3+a_4l_4k_4\right)t}{a_2k_2^2+a_3k_3^2+a_4k_4^2}+\sqrt{\frac{-a_3l_3^2-a_4l_4^2}{a_2}}x+l_3y+l_4z+l_5, \tag{88}$$

$$\begin{aligned}\eta=&\frac{2k_{11}\left(-k_{12}\sqrt{-a_2\left(a_3l_{13}^2+a_4l_{14}^2\right)}+a_3l_{13}k_{13}+a_4l_{14}k_{14}\right)t}{a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2}\\&-\sqrt{\frac{-a_3l_{13}^2-a_4l_{14}^2}{a_2}}x+l_{13}y+l_{14}z+l_{15}\end{aligned} \tag{89}$$

So

$$\begin{aligned}a_2v_x^2=&a_2\left(\frac{a_1k_1k_2}{a_2k_2^2+a_3k_3^2+a_4k_4^2}\right)^2(\lambda\Upsilon)^2-\left(a_3l_3^2+a_4l_4^2\right)\left(\lambda\Upsilon_\xi^{'}\right)^2+\\&a_2\left(\frac{a_1k_{11}k_{12}}{a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2}\right)^2(\zeta\Lambda)^2-\left(a_3l_{13}^2+a_4l_{14}^2\right)\left(\zeta\Lambda_\eta^{'}\right)^2+a_2c_2^2\\&-\frac{2a_1a_2k_1k_2}{a_2k_2^2+a_3k_3^2+a_4k_4^2}\sqrt{\frac{-a_3l_3^2-a_4l_4^2}{a_2}}\lambda\Upsilon\lambda\Upsilon_\xi^{'}\\&+\frac{2a_1a_2k_1k_2}{a_2k_2^2+a_3k_3^2+a_4k_4^2}\frac{a_1k_{11}k_{12}\lambda\Upsilon\zeta\Lambda}{a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2}\\&+\frac{2a_1a_2k_1k_2}{a_2k_2^2+a_3k_3^2+a_4k_4^2}\sqrt{\frac{-a_3l_{13}^2-a_4l_{14}^2}{a_2}}\lambda\Upsilon\zeta\Lambda_\eta^{'}-\frac{2a_1a_2c_2k_1k_2\lambda\Upsilon}{a_2k_2^2+a_3k_3^2+a_4k_4^2}\\&-2a_2\sqrt{\frac{-a_3l_3^2-a_4l_4^2}{a_2}}\frac{a_1k_{11}k_{12}\lambda\Upsilon_\xi^{'}\zeta\Lambda}{a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2}\\&-2a_2\sqrt{\frac{-a_3l_3^2-a_4l_4^2}{a_2}}\sqrt{\frac{-a_3l_{13}^2-a_4l_{14}^2}{a_2}}\lambda\Upsilon_\xi^{'}\zeta\Lambda_\eta^{'}+2a_2c_2\sqrt{\frac{-a_3l_3^2-a_4l_4^2}{a_2}}\lambda\Upsilon_\xi^{'}\\&+\frac{2a_1a_2k_{11}k_{12}}{a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2}\sqrt{\frac{-a_3l_{13}^2-a_4l_{14}^2}{a_2}}\zeta\Lambda\zeta\Lambda_\eta^{'}\\&-\frac{2a_1a_2c_2k_{11}k_{12}\zeta\Lambda}{a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2}-2a_2c_2\sqrt{\frac{-a_3l_{13}^2-a_4l_{14}^2}{a_2}}\zeta\Lambda_\eta^{'},\end{aligned} \tag{90}$$

$$
\begin{aligned}
a_3v_y^2 =\ & a_3\left(\frac{a_1k_1k_3}{a_2k_2^2+a_3k_3^2+a_4k_4^2}\right)^2(\lambda\Upsilon)^2 + a_3l_3^2\left(\lambda\Upsilon_\xi^{'}\right)^2 \\
& + a_3\left(\frac{a_1k_{11}k_{13}}{a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2}\right)^2(\zeta\Lambda)^2 + a_3l_{13}^2\left(\zeta\Lambda_\eta^{'}\right)^2 + a_3c_3^2 \\
& - \frac{2a_1a_3k_1k_3l_3}{a_2k_2^2+a_3k_3^2+a_4k_4^2}\lambda\Upsilon_\xi^{'}\lambda\Upsilon + \frac{2a_1a_3k_1k_3}{a_2k_2^2+a_3k_3^2+a_4k_4^2}\frac{a_1k_{11}k_{13}\lambda\Upsilon\zeta\Lambda}{a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2} \\
& - \frac{2l_{13}a_1a_3k_1k_3\lambda\Upsilon\zeta\Lambda_\eta^{'}}{a_2k_2^2+a_3k_3^2+a_4k_4^2} - \frac{2a_1a_3c_3k_1k_3\lambda\Upsilon}{a_2k_2^2+a_3k_3^2+a_4k_4^2} - \frac{2a_1a_3l_3k_{11}k_{13}\lambda\Upsilon_\xi^{'}\zeta\Lambda}{a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2} \\
& + 2a_3l_3l_{13}\lambda\Upsilon_\xi^{'}\zeta\Lambda_\eta^{'} + 2a_3l_3c_3\lambda\Upsilon_\xi^{'} - \frac{2a_1a_3k_{11}k_{13}l_{13}}{a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2}\zeta\Lambda\zeta\Lambda_\eta^{'} \\
& - \frac{2a_1a_3c_3k_{11}k_{13}\zeta\Lambda}{a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2} + 2a_3c_3l_{13}\zeta\Lambda_\eta^{'},
\end{aligned} \tag{91}
$$

$$
\begin{aligned}
a_4v_z^2 =\ & a_4\left(\frac{a_1k_1k_4}{a_2k_2^2+a_3k_3^2+a_4k_4^2}\right)^2(\lambda\Upsilon)^2 + a_4l_4^2\left(\lambda\Upsilon_\xi^{'}\right)^2 + a_4\left(\frac{a_1k_{11}k_{14}}{a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2}\right)^2(\zeta\Lambda)^2 \\
& + a_4l_{14}^2\left(\zeta\Lambda_\eta^{'}\right)^2 + a_4c_4^2 - \frac{2a_1a_4k_1k_4l_4}{a_2k_2^2+a_3k_3^2+a_4k_4^2}\lambda\Upsilon_\xi^{'}\lambda\Upsilon \\
& + \frac{2a_1a_4k_1k_4}{a_2k_2^2+a_3k_3^2+a_4k_4^2}\frac{a_1k_{11}k_{14}\lambda\Upsilon\zeta\Lambda}{a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2} - \frac{2a_1a_4k_1k_4}{a_2k_2^2+a_3k_3^2+a_4k_4^2}l_{14}\lambda\Upsilon\zeta\Lambda_\eta^{'} \\
& - \frac{2a_1a_4c_4k_1k_4\lambda\Upsilon}{a_2k_2^2+a_3k_3^2+a_4k_4^2} - \frac{2a_1a_4k_{11}k_{14}l_4\lambda\Upsilon_\xi^{'}\zeta\Lambda}{a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2} + 2a_4l_4l_{14}\lambda\Upsilon_\xi^{'}\zeta\Lambda_\eta^{'} + 2a_4c_4l_4\lambda\Upsilon_\xi^{'} \\
& - \frac{2a_1a_4k_{11}k_{14}l_{14}}{a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2}\zeta\Lambda_\eta^{'}\zeta\Lambda - \frac{2a_1a_4c_4k_{11}k_{14}\zeta\Lambda}{a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2} + 2a_4c_4l_{14}\zeta\Lambda_\eta^{'}.
\end{aligned} \tag{92}
$$

And

$$
\begin{aligned}
& a_2v_x^2 + a_3v_y^2 + a_4v_z^2 \\
& = \tfrac{a_1^2k_1^2(\lambda\Upsilon)^2}{a_2k_2^2+a_3k_3^2+a_4k_4^2} + \tfrac{(a_1k_{11})^2}{a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2}(\zeta\Lambda)^2 + a_2c_2^2 + a_3c_3^2 + a_4c_4^2 \\
& - \left(a_2k_2\sqrt{\tfrac{-a_3l_3^2-a_4l_4^2}{a_2}} + a_3k_3l_3 + a_4k_4l_4\right)\tfrac{2a_1k_1\lambda\Upsilon_\xi^{'}\lambda\Upsilon}{a_2k_2^2+a_3k_3^2+a_4k_4^2} \\
& + \tfrac{2a_1^2k_1}{a_2k_2^2+a_3k_3^2+a_4k_4^2}\tfrac{k_{11}\lambda\Upsilon\zeta\Lambda}{a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2}\left(a_2k_2k_{12} + a_3k_3k_{13} + a_4k_4k_{14}\right) \\
& - \tfrac{2a_1k_1\lambda\Upsilon\zeta\Lambda_\eta^{'}}{a_2k_2^2+a_3k_3^2+a_4k_4^2}\left(-a_2k_2\sqrt{\tfrac{-a_3l_{13}^2-a_4l_{14}^2}{a_2}} + a_3k_3l_{13} + a_4k_4l_{14}\right) \\
& - \tfrac{2a_1k_1\lambda\Upsilon}{a_2k_2^2+a_3k_3^2+a_4k_4^2}\left(a_2c_2k_2 + a_3c_3k_3 + a_4c_4k_4\right) \\
& - \tfrac{2a_1k_{11}\lambda\Upsilon_\xi^{'}\zeta\Lambda}{a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2}\left(a_2k_{12}\sqrt{\tfrac{-a_3l_3^2-a_4l_4^2}{a_2}} + a_3k_{13}l_3 + a_4k_{14}l_4\right) \\
& + 2\lambda\Upsilon_\xi^{'}\zeta\Lambda_\eta^{'}\left(-a_2\sqrt{\tfrac{-a_3l_3^2-a_4l_4^2}{a_2}}\sqrt{\tfrac{-a_3l_{13}^2-a_4l_{14}^2}{a_2}} + a_3l_3l_{13} + a_4l_4l_{14}\right) \\
& + 2\lambda\Upsilon_\xi^{'}\left(a_2c_2\sqrt{\tfrac{-a_3l_3^2-a_4l_4^2}{a_2}} + a_3l_3c_3 + a_4c_4l_4\right) \\
& - \tfrac{2a_1k_{11}\zeta\Lambda\zeta\Lambda_\eta^{'}}{a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2}\left(-a_2k_{12}\sqrt{\tfrac{-a_3l_{13}^2-a_4l_{14}^2}{a_2}} + a_3k_{13}l_{13} + a_4k_{14}l_{14}\right) \\
& - \tfrac{2a_1k_{11}\zeta\Lambda}{a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2}\left(a_2c_2k_{12} + a_3c_3k_{13} + a_4c_4k_{14}\right) \\
& + 2\zeta\Lambda_\eta^{'}\left(-a_2c_2\sqrt{\tfrac{-a_3l_{13}^2-a_4l_{14}^2}{a_2}} + a_3c_4l_{13} + a_4c_4l_{14}\right).
\end{aligned}
$$

Since Υ and Λ are arbitrary functions, we set

$$
k_1 = k_{11} = 0. \tag{93}
$$

We get $\lambda=\zeta=1$, and

$$\begin{aligned}&a_2v_x^2+a_3v_y^2+a_4v_z^2\\&=a_2c_2^2+a_3c_3^2+a_4c_4^2\\&+2\lambda\Upsilon_\xi'\zeta\Lambda_\eta'\left(-a_2\sqrt{\tfrac{-a_3l_3^2-a_4l_4^2}{a_2}}\sqrt{\tfrac{-a_3l_{13}^2-a_4l_{14}^2}{a_2}}+a_3l_3l_{13}+a_4l_4l_{14}\right)\\&+2\lambda\Upsilon_\xi'\left(a_2c_2\sqrt{\tfrac{-a_3l_3^2-a_4l_4^2}{a_2}}+a_3l_3c_3+a_4c_4l_4\right)\\&+2\zeta\Lambda_\eta'\left(-a_2c_2\sqrt{\tfrac{-a_3l_{13}^2-a_4l_{14}^2}{a_2}}+a_3c_4l_{13}+a_4c_4l_{14}\right).\end{aligned}$$

Set

$$-a_2\sqrt{\frac{-a_3l_3^2-a_4l_4^2}{a_2}}\sqrt{\frac{-a_3l_{13}^2-a_4l_{14}^2}{a_2}}+a_3l_3l_{13}+a_4l_4l_{14}=0,$$

$$a_2c_2\sqrt{\frac{-a_3l_3^2-a_4l_4^2}{a_2}}+a_3l_3c_3+a_4c_4l_4=0,$$

$$-a_2c_2\sqrt{\frac{-a_3l_{13}^2-a_4l_{14}^2}{a_2}}+a_3c_4l_{13}+a_4c_4l_{14}=0.$$

Then

$$\begin{aligned}&f_v''\left(a_2v_x^2+a_3v_y^2+a_4v_z^2\right)+f_v'\left(a_1v_t+a_2v_{xx}+a_3v_{yy}+a_4v_{zz}\right)\\&=f_v''\left(a_2v_x^2+a_3v_y^2+a_4v_z^2\right)=f_v''\left(a_2c_2^2+a_3c_3^2+a_4c_4^2\right)=A\left(f\right).\end{aligned}$$

That is

$$f_v''=\frac{A\left(f\right)}{a_2c_2^2+a_3c_3^2+a_4c_4^2}. \tag{94}$$

Thereupon

$$v=c_7\pm\int\left(c_6+\frac{2\int A\left(f\right)df}{a_2c_2^2+a_3c_3^2+a_4c_4^2}\right)^{-\frac{1}{2}}df.$$

Set $c_7=0$, then the general solution of (74) is

$$\Upsilon\left(\xi\right)+\Lambda\left(\eta\right)+\zeta=\pm\int\left(c_6+\int K\left(u\right)du\right)^{-\frac{1}{2}}du$$

where ξ,η,ζ and $K(u)$ satisfy (76-82), the theorem is proven. □

Some nonlinear heat equations [43-47] and Schrödinger equation [48] are special cases of Eq. (74) which is a widely studied nonlinear PDE. The current main method is to qualitatively study the existence [49, 50], asymptotic behavior of solutions [51], use numerical methods to study definite solution problems [52] and so on.

Theorem 7. *In* $\mathbb{R}^4$,

$$a_1u_t+a_2u_{xx}+a_3u_{yy}+B\left(u\right)u_{zz}=A\left(u\right), \tag{95}$$

the general solution of Eq. (95) is

$$\Upsilon\left(\xi\right)+\Lambda\left(\eta\right)+c_2x+c_3y+c_4z+c_5=\pm\int\left(c_6+\int K\left(u\right)du\right)^{-\frac{1}{2}}du, \tag{96}$$

where $a_1 - a_3$ are arbitrary known constants, $A(u)$ and $B(u)$ are arbitrary known functions, $\Upsilon(\xi)$ and $\Lambda(\eta)$ are arbitrary unary functions and

$$\xi = \sqrt{\frac{-a_3 l_3^2}{a_2}} x + l_3 y + l_4, \tag{97}$$

$$\eta = \sqrt{\frac{-a_3 l_{13}^2}{a_2}} x + l_{13} y + l_{14}, \tag{98}$$

$$K(u) = \frac{2A(u)}{a_2 c_2^2 + a_3 c_3^2 + c_4^2 B(f)}, \tag{99}$$

where l_4, l_{14} and $c_4 - c_6$ are arbitrary constants, c_2, c_3, l_3 and l_{13} are relative arbitrary constants which satisfy

$$a_2 \left| \frac{a_3 l_3 l_{13}}{a_2} \right| + a_3 l_3 l_{13} = 0, \tag{100}$$

$$a_2 c_2 \sqrt{\frac{-a_3 l_3^2}{a_2}} + a_3 l_3 c_3 = 0, \tag{101}$$

$$a_2 c_2 \sqrt{\frac{-a_3 l_{13}^2}{a_2}} + a_3 c_3 l_{13} = 0. \tag{102}$$

Prove. According to Method 2, we set

$$u(t, x, y, z) = f(v),$$

where f is an undetermined unary function, then

$$\begin{aligned} &a_1 u_t + a_2 u_{xx} + a_3 u_{yy} + B(u) u_{zz} \\ &= a_1 f_v' v_t + a_2 \left(f_v'' v_x^2 + f_v' v_{xx} \right) + a_3 \left(f_v'' v_y^2 + f_v' v_{yy} \right) + B(f) \left(f_v'' v_z^2 + f_v' v_{zz} \right) = A(f). \end{aligned}$$

Namely

$$f_v'' \left(a_2 v_x^2 + a_3 v_y^2 + B(f) v_z^2 \right) + f_v' \left(a_1 v_t + a_2 v_{xx} + a_3 v_{yy} + B(f) v_{zz} \right) = A(f). \tag{103}$$

Set

$$v = g(t, x, y) + c_4 z. \tag{104}$$

So

$$a_2 v_x^2 + a_3 v_y^2 + B(f) v_z^2 = a_2 g_x^2 + a_3 g_y^2 + c_4^2 B(f),$$

$$a_1 v_t + a_2 v_{xx} + a_3 v_{yy} + B(f) v_{zz} = a_1 g_t + a_2 g_{xx} + a_3 g_{yy}.$$

Set

$$a_1 g_t + a_2 g_{xx} + a_3 g_{yy} = 0. \tag{105}$$

The general solution of (105) is [36]

$$g(t, x, y) = \lambda \Upsilon(\xi) + \zeta \Lambda(\eta) + c_2 x + c_3 y + c_4 z + c_5, \tag{106}$$

where $\Upsilon(\xi)$ and $\Lambda(\eta)$ are arbitrary unary functions and

$$\lambda = e^{\frac{-a_1 k_1 (k_1 t + k_2 x + k_3 y)}{a_2 k_2^2 + a_3 k_3^2}}, \tag{107}$$

$$\zeta = e^{\frac{-a_1 k_{11}(k_{11}t + k_{12}x + k_{13}y)}{a_2 k_{12}^2 + a_3 k_{13}^2}}, \tag{108}$$

$$\xi = \frac{2k_1\left(k_2\sqrt{-a_2 a_3 l_3^2} + a_3 l_3 k_3\right)t}{a_2 k_2^2 + a_3 k_3^2} + \sqrt{\frac{-a_3 l_3^2}{a_2}}x + l_3 y + l_4, \tag{109}$$

$$\eta = \frac{2k_{11}\left(k_{12}\sqrt{-a_2 a_3 l_{13}^2} + a_3 l_{13} k_{13}\right)t}{a_2 k_{12}^2 + a_3 k_{13}^2} + \sqrt{\frac{-a_3 l_{13}^2}{a_2}}x + l_{13} y + l_{14}. \tag{110}$$

So

$$\begin{aligned}
a_2 g_x^2 = {} & a_2\left(\frac{-a_1 k_1 k_2}{a_2 k_2^2 + a_3 k_3^2}\right)^2 (\lambda\Upsilon)^2 - a_3 l_3^2\left(\lambda\Upsilon_\xi^{'}\right)^2 + a_2\left(\frac{a_1 k_{11} k_{12}}{a_2 k_{12}^2 + a_3 k_{13}^2}\right)^2 (\zeta\Lambda)^2 - a_3 l_{13}^2\left(\zeta\Lambda_\eta^{'}\right)^2 \\
& + a_2 c_2^2 - \frac{2a_1 a_2 k_1 k_2}{a_2 k_2^2 + a_3 k_3^2}\sqrt{\frac{-a_3 l_3^2}{a_2}}\lambda\Upsilon\lambda\Upsilon_\xi^{'} + \frac{2a_1 a_2 k_1 k_2}{a_2 k_2^2 + a_3 k_3^2}\frac{a_1 k_{11} k_{12}\lambda\Upsilon\zeta\Lambda}{a_2 k_{12}^2 + a_3 k_{13}^2} \\
& - \frac{2a_1 a_2 k_1 k_2}{a_2 k_2^2 + a_3 k_3^2}\sqrt{\frac{-a_3 l_{13}^2}{a_2}}\lambda\Upsilon\zeta\Lambda_\eta^{'} - \frac{2a_1 a_2 c_2 k_1 k_2 \lambda\Upsilon}{a_2 k_2^2 + a_3 k_3^2} - 2a_2\sqrt{\frac{-a_3 l_3^2}{a_2}}\frac{a_1 k_{11} k_{12}\lambda\Upsilon_\xi^{'}\zeta\Lambda}{a_2 k_{12}^2 + a_3 k_{13}^2} \\
& + 2a_2\sqrt{\frac{-a_3 l_3^2}{a_2}}\sqrt{\frac{-a_3 l_{13}^2}{a_2}}\lambda\Upsilon_\xi^{'}\zeta\Lambda_\eta^{'} + 2a_2 c_2\sqrt{\frac{-a_3 l_3^2}{a_2}}\lambda\Upsilon_\xi^{'} - \frac{2a_1 a_2 k_{11} k_{12}}{a_2 k_{12}^2 + a_3 k_{13}^2}\sqrt{\frac{-a_3 l_{13}^2}{a_2}}\zeta\Lambda\zeta\Lambda_\eta^{'} \\
& - \frac{2a_1 a_2 c_2 k_{11} k_{12}\zeta\Lambda}{a_2 k_{12}^2 + a_3 k_{13}^2} + 2a_2 c_2\sqrt{\frac{-a_3 l_{13}^2}{a_2}}\zeta\Lambda_\eta^{'},
\end{aligned} \tag{111}$$

$$\begin{aligned}
a_3 g_y^2 = {} & a_3\left(\frac{-a_1 k_1 k_3}{a_2 k_2^2 + a_3 k_3^2}\right)^2 (\lambda\Upsilon)^2 + a_3 l_3^2\left(\lambda\Upsilon_\xi^{'}\right)^2 + a_3\left(\frac{a_1 k_{11} k_{13}}{a_2 k_{12}^2 + a_3 k_{13}^2}\right)^2 (\zeta\Lambda)^2 + a_3 l_{13}^2\left(\zeta\Lambda_\eta^{'}\right)^2 \\
& + a_3 c_3^2 - \frac{2a_1 a_3 k_1 k_3 l_3}{a_2 k_2^2 + a_3 k_3^2}\lambda\Upsilon_\xi^{'}\lambda\Upsilon + \frac{2a_1 a_3 k_1 k_3}{a_2 k_2^2 + a_3 k_3^2}\frac{a_1 k_{11} k_{13}\lambda\Upsilon\zeta\Lambda}{a_2 k_{12}^2 + a_3 k_{13}^2} - \frac{2a_1 a_3 k_1 k_3 l_{13}\lambda\Upsilon\zeta\Lambda_\eta^{'}}{a_2 k_2^2 + a_3 k_3^2} \\
& - \frac{2a_1 a_3 c_3 k_1 k_3 \lambda\Upsilon}{a_2 k_2^2 + a_3 k_3^2} - \frac{2a_1 a_3 l_3 k_{11} k_{13}\lambda\Upsilon_\xi^{'}\zeta\Lambda}{a_2 k_{12}^2 + a_3 k_{13}^2} + 2a_3 l_3 l_{13}\lambda\Upsilon_\xi^{'}\zeta\Lambda_\eta^{'} + 2a_3 l_3 c_3\lambda\Upsilon_\xi^{'} \\
& - \frac{2a_1 a_3 k_{11} k_{13} l_{13}}{a_2 k_{12}^2 + a_3 k_{13}^2}\zeta\Lambda\zeta\Lambda_\eta^{'} - \frac{2a_1 a_3 c_3 k_{11} k_{13}\zeta\Lambda}{a_2 k_{12}^2 + a_3 k_{13}^2} + 2a_3 c_3 l_{13}\zeta\Lambda_\eta^{'}.
\end{aligned} \tag{112}$$

And

$$\begin{aligned}
& a_2 v_x^2 + a_3 v_y^2 + B(f)\, v_z^2 \\
& = a_2 g_x^2 + a_3 g_y^2 + c_4^2 B(f) \\
& = \tfrac{a_1^2 k_1^2}{a_2 k_2^2 + a_3 k_3^2}(\lambda\Upsilon)^2 + \tfrac{a_1^2 k_{11}^2}{a_2 k_{12}^2 + a_3 k_{13}^2}(\zeta\Lambda)^2 + \tfrac{2a_1 k_1}{a_2 k_2^2 + a_3 k_3^2}\tfrac{a_1 k_{11}\lambda\Upsilon\zeta\Lambda}{a_2 k_{12}^2 + a_3 k_{13}^2}(a_2 k_2 k_{12} + a_3 k_3 k_{13}) \\
& -\tfrac{2a_1 k_1 \lambda\Upsilon_\xi^{'}\lambda\Upsilon}{a_2 k_2^2 + a_3 k_3^2}\left(a_2 k_2\sqrt{\tfrac{-a_3 l_3^2}{a_2}} + a_3 k_3 l_3\right) - \tfrac{2a_1 k_1}{a_2 k_2^2 + a_3 k_3^2}\lambda\Upsilon\zeta\Lambda_\eta^{'}\left(a_2 k_2\sqrt{\tfrac{-a_3 l_{13}^2}{a_2}} + a_3 k_3 l_{13}\right) \\
& -\tfrac{2a_1 k_1 \lambda\Upsilon}{a_2 k_2^2 + a_3 k_3^2}(a_2 c_2 k_2 + a_3 c_3 k_3) - \tfrac{2a_1 k_{11}\lambda\Upsilon_\xi^{'}\zeta\Lambda}{a_2 k_{12}^2 + a_3 k_{13}^2}\left(a_2 k_{12}\sqrt{\tfrac{-a_3 l_3^2}{a_2}} + a_3 l_3 k_{13}\right) \\
& +2\lambda\Upsilon_\xi^{'}\zeta\Lambda_\eta^{'}\left(a_2\sqrt{\tfrac{-a_3 l_3^2}{a_2}}\sqrt{\tfrac{-a_3 l_{13}^2}{a_2}} + a_3 l_3 l_{13}\right) + 2\lambda\Upsilon_\xi^{'}\left(a_2 c_2\sqrt{\tfrac{-a_3 l_3^2}{a_2}} + a_3 l_3 c_3\right) \\
& -\tfrac{2a_1 k_{11}\zeta\Lambda\zeta\Lambda_\eta^{'}}{a_2 k_{12}^2 + a_3 k_{13}^2}\left(a_2 k_{12}\sqrt{\tfrac{-a_3 l_{13}^2}{a_2}} + a_3 k_{13} l_{13}\right) - \tfrac{2a_1 k_{11}\zeta\Lambda}{a_2 k_{12}^2 + a_3 k_{13}^2}(a_2 c_2 k_{12} + a_3 c_3 k_{13}) \\
& +2\zeta\Lambda_\eta^{'}\left(a_2 c_2\sqrt{\tfrac{-a_3 l_{13}^2}{a_2}} + a_3 c_3 l_{13}\right) + a_2 c_2^2 + a_3 c_3^2 + c_4^2 B(f).
\end{aligned}$$

Since Υ and Λ are arbitrary functions, we set

$$k_1 = k_{11} = 0 \tag{113}$$

So

$$\begin{array}{l} a_2v_x^2+a_3v_y^2+B\left(f\right)v_z^2 \\ =2\lambda\Upsilon_\xi^{'}\zeta\Lambda_\eta^{'}\left(a_2\sqrt{\frac{-a_3l_3^2}{a_2}}\sqrt{\frac{-a_3l_{13}^2}{a_2}}+a_3l_3l_{13}\right)+2\lambda\Upsilon_\xi^{'}\left(a_2c_2\sqrt{\frac{-a_3l_3^2}{a_2}}+a_3l_3c_3\right) \\ +2\zeta\Lambda_\eta^{'}\left(a_2c_2\sqrt{\frac{-a_3l_{13}^2}{a_2}}+a_3c_3l_{13}\right)+a_2c_2^2+a_3c_3^2+c_4^2B\left(f\right). \end{array}$$

Set

$$a_2\sqrt{\frac{-a_3l_3^2}{a_2}}\sqrt{\frac{-a_3l_{13}^2}{a_2}}+a_3l_3l_{13}=a_2\left|\frac{a_3l_3l_{13}}{a_2}\right|+a_3l_3l_{13}=0,$$

$$a_2c_2\sqrt{\frac{-a_3l_3^2}{a_2}}+a_3l_3c_3=0,$$

$$a_2c_2\sqrt{\frac{-a_3l_{13}^2}{a_2}}+a_3c_3l_{13}=0.$$

Whereupon

$$v=\Upsilon\left(\xi\right)+\Lambda\left(\eta\right)+c_2x+c_3y+c_4z+c_5, \tag{114}$$

where $\Upsilon(\xi)$ and $\Lambda(\eta)$ are arbitrary unary functions and

$$\xi=\sqrt{\frac{-a_3l_3^2}{a_2}}x+l_3y+l_4,$$

$$\eta=\sqrt{\frac{-a_3l_{13}^2}{a_2}}x+l_{13}y+l_{14}.$$

So

$$\begin{array}{l} f_v^{''}\left(a_2v_x^2+a_3v_y^2+B\left(f\right)v_z^2\right)+f_v^{'}\left(a_1v_t+a_2v_{xx}+a_3v_{yy}+B\left(f\right)v_{zz}\right) \\ =f_v^{''}\left(a_2v_x^2+a_3v_y^2+B\left(f\right)v_z^2\right)=f_v^{''}\left(a_2g_x^2+a_3g_y^2+c_4^2B\left(f\right)\right) \\ =f_v^{''}\left(a_2c_2^2+a_3c_3^2+c_4^2B\left(f\right)\right)=A\left(f\right). \end{array}$$

That is

$$f_v^{''}=\frac{A\left(f\right)}{a_2c_2^2+a_3c_3^2+c_4^2B\left(f\right)}. \tag{115}$$

Thereupon

$$v=c_7\pm\int\left(c_6+2\int\frac{A\left(f\right)df}{a_2c_2^2+a_3c_3^2+c_4^2B\left(f\right)}\right)^{-\frac{1}{2}}df.$$

Set $c_7=0$, then the general solution of (95) is

$$\Upsilon\left(\xi\right)+\Lambda\left(\eta\right)+c_2x+c_3y+c_4z+c_5=\pm\int\left(c_6+\int K\left(u\right)du\right)^{-\frac{1}{2}}du,$$

where ξ,η and $K(u)$ satisfy (97-102), the theorem is proven. □

Theorem 8. *In* $\mathbb{R}^4$,

$$a_0u_t+a_1u_{tt}+a_2u_{xx}+a_3u_{yy}+a_4u_{zz}=0, \tag{116}$$

the general solution of Eq. (116) is

$$u = e^{\frac{-a_0k_1(k_1t+k_2x+k_3y+k_4z)}{a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2}} h_1\left(\sqrt{\frac{-a_2l_2^2-a_3l_3^2-a_4l_4^2}{a_1}}t+l_2x+l_3y+l_4z+l_5\right)$$
$$+e^{\frac{-a_0k_{11}(k_{11}t+k_{12}x+k_{13}y+k_{14}z)}{a_1k_{11}^2+a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2}} h_2\left(-\sqrt{\frac{-a_2l_{12}^2-a_3l_{13}^2-a_4l_{14}^2}{a_1}}t+l_{12}x+l_{13}y+l_{14}z+l_{15}\right), \tag{117}$$

where a_0-a_4 are arbitrary known constants, h_1 and h_2 are arbitrary second differentiable functions, $k_1-k_4, k_{11}-k_{14}, l_2-l_4$ and $l_{12}-l_{14}$ are relative arbitrary constants which satisfy

$$\frac{2k_1\left(a_2k_2l_2+a_3k_3l_3+a_4k_4l_4\right)}{-a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2}=\sqrt{\frac{-a_2l_2^2-a_3l_3^2-a_4l_4^2}{a_1}} \tag{118}$$

$$\frac{2k_{11}\left(a_2k_{12}l_{12}+a_3k_{13}l_{13}+a_4k_{14}l_{14}\right)}{-a_1k_{11}^2+a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2}=-\sqrt{\frac{-a_2l_{12}^2-a_3l_{13}^2-a_4l_{14}^2}{a_1}} \tag{119}$$

Proof. According to Z_3 Transformation [36], we set

$$u(t,x,y,z)=f(v)=f(k_1t+k_2x+k_3y+k_4z+k_5), \tag{120}$$

where f is an undetermined unary function, k_1-k_5 are undetermined constants, then

$$a_0u_t+a_1u_{tt}+a_2u_{xx}+a_3u_{yy}+a_4u_{zz}=a_0k_1f_v^{'}+\left(a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2\right)f_v^{''}=0.$$

Set $w=f_v^{'}$, then

$$\begin{aligned}&a_0k_1f_v^{'}+\left(a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2\right)f_v^{''}=0\\ &\Longrightarrow \left(a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2\right)w_v^{'}=-a_0k_1w\\ &\Longrightarrow w=k_8e^{\frac{-a_0k_1v}{a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2}}\\ &\Longrightarrow f(v)=-k_8\frac{a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2}{a_0k_1}e^{\frac{-a_0k_1v}{a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2}}+k_7.\end{aligned}$$

So the particular solution of Eq. (116) is

$$u(t,x,y,z)=k_6e^{\frac{-a_0k_1(k_1t+k_2x+k_3y+k_4z+k_5)}{a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2}}+k_7, \tag{121}$$

where k_0-k_8 are arbitrary constants.

Next we use Z_3 Transformation to obtain the general solution of Eq. (116), set

$$u(t,x,y,z)=gh(w)=g(t,x,y,z)h(l_1t+l_2x+l_3y+l_4z+l_5), \tag{122}$$

where $w(x,y,z)=l_1t+l_2x+l_3y+l_4z+l_5$, l_1-l_5 are undetermined constants, h and g are undetermined second differentiable functions, so

$$\begin{aligned}&a_0u_t+a_1u_{tt}+a_2u_{xx}+a_3u_{yy}+a_4u_{zz}\\ &=a_0l_1gh_w^{'}+a_0hg_t+a_1l_1^2gh_w^{''}+2a_1l_1g_th_w^{'}+a_1hg_{tt}+a_2l_2^2gh_w^{''}+2a_2l_2g_xh_w^{'}+a_2hg_{xx}\\ &+a_3l_3^2gh_w^{''}+2a_3l_3g_yh_w^{'}+a_3hg_{yy}+a_4l_4^2gh_w^{''}+2a_4l_4g_zh_w^{'}+a_4hg_{zz}.\end{aligned}$$

Namely

$$\left(a_1l_1^2+a_2l_2^2+a_3l_3^2+a_4l_4^2\right)gh_w^{''}+\left(a_0l_1g+2a_1l_1g_t+2a_2l_2g_x+2a_3l_3g_y+2a_4l_4g_z\right)h_w^{'} + \left(a_0g_t+a_1g_{tt}+a_2g_{xx}+a_3g_{yy}+a_4g_{zz}\right)h=0. \tag{123}$$

Set $h(w)$ an arbitrary second differentiable function, according to (123) we obtain

$$a_1l_1^2+a_2l_2^2+a_3l_3^2+a_4l_4^2=0 \Longrightarrow l_1=\pm\sqrt{\frac{-a_2l_2^2-a_3l_3^2-a_4l_4^2}{a_1}}, \tag{124}$$

$$a_0l_1g+2a_1l_1g_t+2a_2l_2g_x+2a_3l_3g_y+2a_4l_4g_z=0, \tag{125}$$

$$a_0g_t+a_1g_{tt}+a_2g_{xx}+a_3g_{yy}+a_4g_{zz}=0. \tag{126}$$

By (121) the particular solution of Eq. (126) is

$$g\left(t,x,y,z\right)=k_6e^{\frac{-a_0k_1(k_1t+k_2x+k_3y+k_4z+k_5)}{a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2}}+k_7. \tag{127}$$

Set $k_7=0$, and substitute from (127) into (125)

$$\begin{aligned}
&a_0l_1g+2a_1l_1g_t+2a_2l_2g_x+2a_3l_3g_y+2a_4l_4g_z\\
&=a_0l_1k_6e^{\frac{-a_0k_1(k_1t+k_2x+k_3y+k_4z+k_5)}{a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2}}+2a_1l_1k_6\frac{-a_0k_1^2}{a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2}e^{\frac{-a_0k_1(k_1t+k_2x+k_3y+k_4z+k_5)}{a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2}}\\
&+2a_2l_2k_6\frac{-a_0k_1k_2}{a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2}e^{\frac{-a_0k_1(k_1t+k_2x+k_3y+k_4z+k_5)}{a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2}}\\
&+2a_3l_3k_6\frac{-a_0k_1k_3}{a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2}e^{\frac{-a_0k_1(k_1t+k_2x+k_3y+k_4z+k_5)}{a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2}}\\
&+2a_4l_4k_6\frac{-a_0k_1k_4}{a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2}e^{\frac{-a_0k_1(k_1t+k_2x+k_3y+k_4z+k_5)}{a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2}}=0.
\end{aligned}$$

We get

$$l_1=\frac{2k_1\left(a_2k_2l_2+a_3k_3l_3+a_4k_4l_4\right)}{-a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2}. \tag{128}$$

Then

$$u\left(x,y,z,t\right)=g\left(x,y,z,t\right)h\left(w\right)=k_6e^{\frac{-a_0k_1(k_1t+k_2x+k_3y+k_4z+k_5)}{a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2}}h\left(l_1t+l_2x+l_3y+l_4z+l_5\right)$$

So the general solution of Eq. (116) is

$$\begin{aligned}
u&=g_1h\left(w_1\right)+g_2h\left(w_2\right)\\
&=e^{\frac{-a_0k_1(k_1t+k_2x+k_3y+k_4z)}{a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2}}h_1\left(\sqrt{\frac{-a_2l_2^2-a_3l_3^2-a_4l_4^2}{a_1}}t+l_2x+l_3y+l_4z+l_5\right)\\
&+e^{\frac{-a_0k_{11}(k_{11}t+k_{12}x+k_{13}y+k_{14}z)}{a_1k_{11}^2+a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2}}h_2\left(-\sqrt{\frac{-a_2l_{12}^2-a_3l_{13}^2-a_4l_{14}^2}{a_1}}t+l_{12}x+l_{13}y+l_{14}z+l_{15}\right),
\end{aligned} \tag{129}$$

where h_1 and h_2 are arbitrary second differentiable functions, g_i and w_i are independent of each other $(i=1,2)$, $k_1-k_4, k_{11}-k_{14}, l_2-l_4$ and $l_{12}-l_{14}$ are relative arbitrary constants which satisfy

$$\frac{2k_1\left(a_2k_2l_2+a_3k_3l_3+a_4k_4l_4\right)}{-a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2}=\sqrt{\frac{-a_2l_2^2-a_3l_3^2-a_4l_4^2}{a_1}}, \tag{118}$$

$$\frac{2k_{11}\left(a_2k_{12}l_{12}+a_3k_{13}l_{13}+a_4k_{14}l_{14}\right)}{-a_1k_{11}^2+a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2}=-\sqrt{\frac{-a_2l_{12}^2-a_3l_{13}^2-a_4l_{14}^2}{a_1}}. \tag{119}$$

□

Theorem 9. *In* $\mathbb{R}^4$,

$$a_0u_t+a_1u_{tt}+a_2u_{xx}+a_3u_{yy}+a_4u_{zz}=A\left(u\right), \tag{130}$$

the general solution of Eq. (130) is

$$\Upsilon\left(\xi\right)+\Lambda\left(\eta\right)+\varsigma=\pm\int\left(c_9+\int K\left(u\right)du\right)^{-\frac{1}{2}}du, \tag{131}$$

where a_0-a_4 *are arbitrary known constants,* $A(u)$ *is an arbitrary known function,* $\Upsilon(\xi)$ *and* $\Lambda(\eta)$ *are arbitrary unary functions and*

$$\varsigma=c_2x+c_3y+c_4z+c_8, \tag{132}$$

$$\xi=\sqrt{\frac{-a_3l_3^2-a_4l_4^2}{a_2}}x+l_3y+l_4z+l_5, \tag{133}$$

$$\eta=-\sqrt{\frac{-a_3l_{13}^2-a_4l_{14}^2}{a_2}}x+l_{13}y+l_{14}z+l_{15}, \tag{134}$$

$$K\left(u\right)=\frac{2A\left(u\right)}{a_2c_2^2+a_3c_3^2+a_4c_4^2}, \tag{135}$$

where l_5,l_{15},c_8 *and* c_9 *are arbitrary constants,* c_2-c_4,l_3,l_4,l_{13} *and* l_{14} *are relative arbitrary constants which satisfy*

$$-a_2\sqrt{\frac{-a_3l_3^2-a_4l_4^2}{a_2}}\sqrt{\frac{-a_3l_{13}^2-a_4l_{14}^2}{a_2}}+a_3l_3l_{13}+a_4l_4l_{14}=0, \tag{136}$$

$$a_2c_2\sqrt{\frac{-a_3l_3^2-a_4l_4^2}{a_2}}+a_3l_3c_3+a_4c_4l_4=0, \tag{137}$$

$$-a_2c_2\sqrt{\frac{-a_3l_{13}^2-a_4l_{14}^2}{a_2}}+a_3c_3l_{13}+a_4c_4l_{14}=0. \tag{138}$$

Prove. Some nonlinear wave equations [53, 54] and Schrödinger equation [55] are special cases of (130). According to Method 2, we set

$$u(t,x,y,z)=f(v),$$

where f is an undetermined unary function, then

$$\begin{aligned}&a_0u_t+a_1u_{tt}+a_2u_{xx}+a_3u_{yy}+a_4u_{zz}\\&=a_0f_v^{'}v_t+a_1\left(f_v^{''}v_t^2+f_v^{'}v_{tt}\right)+a_2\left(f_v^{''}v_x^2+f_v^{'}v_{xx}\right)+a_3\left(f_v^{''}v_y^2+f_v^{'}v_{yy}\right)\\&+a_4\left(f_v^{''}v_z^2+f_v^{'}v_{zz}\right).\end{aligned}$$

Namely

$$f_v^{''}\left(a_1v_t^2+a_2v_x^2+a_3v_y^2+a_4v_z^2\right)+f_v^{'}\left(a_0v_t+a_1v_{tt}+a_2v_{xx}+a_3v_{yy}+a_4v_{zz}\right)=A\left(f\right). \quad (139)$$

Set

$$a_0v_t+a_1v_{tt}+a_2v_{xx}+a_3v_{yy}+a_4v_{zz}=0. \quad (140)$$

The general solution of (140) is

$$v\left(x,y,z\right)=\lambda\Upsilon\left(\xi\right)+\zeta\Lambda\left(\eta\right)+\varsigma. \quad (141)$$

where $\Upsilon(\xi)$ and $\Lambda(\eta)$ are arbitrary unary functions and

$$\lambda=e^{\frac{-a_0k_1(k_1t+k_2x+k_3y+k_4z)}{a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2}}, \quad (142)$$

$$\zeta=e^{\frac{-a_0k_{11}(k_{11}t+k_{12}x+k_{13}y+k_{14}z)}{a_1k_{11}^2+a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2}}, \quad (143)$$

$$\xi=\sqrt{\frac{-a_2l_2^2-a_3l_3^2-a_4l_4^2}{a_1}}t+l_2x+l_3y+l_4z+l_5, \quad (144)$$

$$\eta=-\sqrt{\frac{-a_2l_{12}^2-a_3l_{13}^2-a_4l_{14}^2}{a_1}}t+l_{12}x+l_{13}y+l_{14}z+l_{15}, \quad (145)$$

$$\frac{2k_1\left(a_2k_2l_2+a_3k_3l_3+a_4k_4l_4\right)}{-a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2}=\sqrt{\frac{-a_2l_2^2-a_3l_3^2-a_4l_4^2}{a_1}}, \quad (118)$$

$$\frac{2k_{11}\left(a_2k_{12}l_{12}+a_3k_{13}l_{13}+a_4k_{14}l_{14}\right)}{-a_1k_{11}^2+a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2}=-\sqrt{\frac{-a_2l_{12}^2-a_3l_{13}^2-a_4l_{14}^2}{a_1}}, \quad (119)$$

$$\varsigma=c_2x+c_3y+c_4z+c_5xy+c_6xz+c_7yz+c_8.$$

Set

$$c_5=c_6=c_7=0.$$

Then

$$\begin{aligned}v_t=&-\tfrac{a_0k_1^2\lambda\Upsilon}{a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2}+\sqrt{\tfrac{-a_2l_2^2-a_3l_3^2-a_4l_4^2}{a_1}}\lambda\Upsilon_\xi^{'}\\&-\tfrac{a_0k_{11}^2\zeta\Lambda}{a_1k_{11}^2+a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2}-\sqrt{\tfrac{-a_2l_{12}^2-a_3l_{13}^2-a_4l_{14}^2}{a_1}}\zeta\Lambda_\eta^{'},\end{aligned}$$

$$v_x=-\frac{a_0k_1k_2\lambda\Upsilon}{a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2}+l_2\lambda\Upsilon_\xi^{'}-\frac{a_0k_{11}k_{12}\zeta\Lambda}{a_1k_{11}^2+a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2}+l_{12}\zeta\Lambda_\eta^{'}+c_2,$$

$$v_y=-\frac{a_0k_1k_3\lambda\Upsilon}{a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2}+l_3\lambda\Upsilon_\xi^{'}-\frac{a_0k_{11}k_{13}\zeta\Lambda}{a_1k_{11}^2+a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2}+l_{13}\zeta\Lambda_\eta^{'}+c_3,$$

$$v_z=-\frac{a_0k_1k_4\lambda\Upsilon}{a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2}+l_4\lambda\Upsilon_\xi^{'}-\frac{a_0k_{11}k_{14}\zeta\Lambda}{a_1k_{11}^2+a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2}+l_{14}\zeta\Lambda_\eta^{'}+c_4.$$

Whereupon

$$
\begin{aligned}
a_1 v_t^2 &= a_1\left(\frac{a_0 k_1^2}{a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2}\right)^2(\lambda\Upsilon)^2-\left(a_2l_2^2+a_3l_3^2+a_4l_4^2\right)\left(\lambda\Upsilon_\xi^{'}\right)^2\\
&+a_1\left(\frac{a_0 k_{11}^2}{a_1k_{11}^2+a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2}\right)^2(\zeta\Lambda)^2-\left(a_2l_{12}^2+a_3l_{13}^2+a_4l_{14}^2\right)\left(\zeta\Lambda_\eta^{'}\right)^2\\
&-\frac{2a_0a_1k_1^2}{a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2}\sqrt{\frac{-a_2l_2^2-a_3l_3^2-a_4l_4^2}{a_1}}\lambda\Upsilon\lambda\Upsilon_\xi^{'}\\
&+\frac{2a_0^2a_1k_1^2k_{11}^2\lambda\Upsilon\zeta\Lambda}{\left(a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2\right)\left(a_1k_{11}^2+a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2\right)}\\
&+\frac{2a_0a_1k_1^2}{a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2}\sqrt{\frac{-a_2l_{12}^2-a_3l_{13}^2-a_4l_{14}^2}{a_1}}\lambda\Upsilon\zeta\Lambda_\eta^{'}\\
&-\sqrt{\frac{-a_2l_2^2-a_3l_3^2-a_4l_4^2}{a_1}}\frac{2a_0a_1k_{11}^2\lambda\Upsilon_\xi^{'}\zeta\Lambda}{a_1k_{11}^2+a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2}\\
&-2a_1\sqrt{\frac{-a_2l_2^2-a_3l_3^2-a_4l_4^2}{a_1}}\sqrt{\frac{-a_2l_{12}^2-a_3l_{13}^2-a_4l_{14}^2}{a_1}}\lambda\Upsilon_\xi^{'}\zeta\Lambda_\eta^{'}\\
&+\sqrt{\frac{-a_2l_{12}^2-a_3l_{13}^2-a_4l_{14}^2}{a_1}}\frac{2a_0a_1k_{11}^2\zeta\Lambda\zeta\Lambda_\eta^{'}}{a_1k_{11}^2+a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2},
\end{aligned}
\tag{146}
$$

$$
\begin{aligned}
a_2 v_x^2 &= a_2\left(\frac{a_0 k_1k_2}{a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2}\right)^2(\lambda\Upsilon)^2+a_2l_2^2\left(\lambda\Upsilon_\xi^{'}\right)^2\\
&+a_2\left(\frac{a_0 k_{11}k_{12}}{a_1k_{11}^2+a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2}\right)^2(\zeta\Lambda)^2+a_2l_{12}^2\left(\zeta\Lambda_\eta^{'}\right)^2\\
&-\frac{2a_0a_2k_1k_2l_2\lambda\Upsilon\lambda\Upsilon_\xi^{'}}{a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2}\\
&+\frac{2a_0^2a_2k_1k_2k_{11}k_{12}\lambda\Upsilon\zeta\Lambda}{\left(a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2\right)\left(a_1k_{11}^2+a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2\right)}\\
&-\frac{2a_0a_2k_1k_2l_{12}\lambda\Upsilon\zeta\Lambda_\eta^{'}}{a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2}-\frac{2a_0a_2k_{11}k_{12}l_2\lambda\Upsilon_\xi^{'}\zeta\Lambda}{a_1k_{11}^2+a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2}+2a_2l_2l_{12}\lambda\Upsilon_\xi^{'}\zeta\Lambda_\eta^{'}\\
&-\frac{2a_0a_2k_{11}k_{12}l_{12}\zeta\Lambda\zeta\Lambda_\eta^{'}}{a_1k_{11}^2+a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2}+a_2c_2^2-\frac{2a_2c_2a_0k_1k_2\lambda\Upsilon}{a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2}+2a_2c_2l_2\lambda\Upsilon_\xi^{'}\\
&-\frac{2a_2c_2a_0k_{11}k_{12}\zeta\Lambda}{a_1k_{11}^2+a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2}+2a_2c_2l_{12}\zeta\Lambda_\eta^{'},
\end{aligned}
\tag{147}
$$

$$
\begin{aligned}
a_3 v_y^2 = {} & a_3 \left(\frac{a_0 k_1 k_3}{a_1 k_1^2 + a_2 k_2^2 + a_3 k_3^2 + a_4 k_4^2} \right)^2 (\lambda \Upsilon)^2 + a_3 l_3^2 \left(\lambda \Upsilon_\xi^{'} \right)^2 \\
& + a_3 \left(\frac{a_0 k_{11} k_{13}}{a_1 k_{11}^2 + a_2 k_{12}^2 + a_3 k_{13}^2 + a_4 k_{14}^2} \right)^2 (\zeta \Lambda)^2 + a_3 l_{13}^2 \left(\zeta \Lambda_\eta^{'} \right)^2 \\
& - \frac{2 a_0 a_3 k_1 k_3 l_3 \lambda \Upsilon \lambda \Upsilon_\xi^{'}}{a_1 k_1^2 + a_2 k_2^2 + a_3 k_3^2 + a_4 k_4^2} \\
& + \frac{2 a_0^2 a_3 k_1 k_3 k_{11} k_{13} \lambda \Upsilon \zeta \Lambda}{\left(a_1 k_1^2 + a_2 k_2^2 + a_3 k_3^2 + a_4 k_4^2 \right) \left(a_1 k_{11}^2 + a_2 k_{12}^2 + a_3 k_{13}^2 + a_4 k_{14}^2 \right)} \\
& - \frac{2 a_0 a_3 k_1 k_3 l_{13} \lambda \Upsilon \zeta \Lambda_\eta^{'}}{a_1 k_1^2 + a_2 k_2^2 + a_3 k_3^2 + a_4 k_4^2} - \frac{2 a_0 a_3 k_{11} k_{13} l_3 \lambda \Upsilon_\xi^{'} \zeta \Lambda}{a_1 k_{11}^2 + a_2 k_{12}^2 + a_3 k_{13}^2 + a_4 k_{14}^2} + 2 a_3 l_3 l_{13} \lambda \Upsilon_\xi^{'} \zeta \Lambda_\eta^{'} \\
& - \frac{2 a_0 a_3 k_{11} k_{13} l_{13} \zeta \Lambda \zeta \Lambda_\eta^{'}}{a_1 k_{11}^2 + a_2 k_{12}^2 + a_3 k_{13}^2 + a_4 k_{14}^2} + a_3 c_3^2 - \frac{2 a_3 c_3 a_0 k_1 k_3 \lambda \Upsilon}{a_1 k_1^2 + a_2 k_2^2 + a_3 k_3^2 + a_4 k_4^2} + 2 a_3 c_3 l_3 \lambda \Upsilon_\xi^{'} \\
& - \frac{2 a_3 c_3 a_0 k_{11} k_{13} \zeta \Lambda}{a_1 k_{11}^2 + a_2 k_{12}^2 + a_3 k_{13}^2 + a_4 k_{14}^2} + 2 a_3 c_3 l_{13} \zeta \Lambda_\eta^{'},
\end{aligned}
\tag{148}
$$

$$
\begin{aligned}
a_4 v_z^2 = {} & a_4 \left(\frac{a_0 k_1 k_4}{a_1 k_1^2 + a_2 k_2^2 + a_3 k_3^2 + a_4 k_4^2} \right)^2 (\lambda \Upsilon)^2 + a_4 l_4^2 \left(\lambda \Upsilon_\xi^{'} \right)^2 \\
& + a_4 \left(\frac{a_0 k_{11} k_{14}}{a_1 k_{11}^2 + a_2 k_{12}^2 + a_3 k_{13}^2 + a_4 k_{14}^2} \right)^2 (\zeta \Lambda)^2 + a_4 l_{14}^2 \left(\zeta \Lambda_\eta^{'} \right)^2 \\
& - \frac{2 a_0 a_4 k_1 k_4 l_4 \lambda \Upsilon \lambda \Upsilon_\xi^{'}}{a_1 k_1^2 + a_2 k_2^2 + a_3 k_3^2 + a_4 k_4^2} \\
& + \frac{2 a_0^2 a_4 k_1 k_4 k_{11} k_{14} \lambda \Upsilon \zeta \Lambda}{\left(a_1 k_1^2 + a_2 k_2^2 + a_3 k_3^2 + a_4 k_4^2 \right) \left(a_1 k_{11}^2 + a_2 k_{12}^2 + a_3 k_{13}^2 + a_4 k_{14}^2 \right)} \\
& - \frac{2 a_0 a_4 k_1 k_4 l_{14} \lambda \Upsilon \zeta \Lambda_\eta^{'}}{a_1 k_1^2 + a_2 k_2^2 + a_3 k_3^2 + a_4 k_4^2} - \frac{2 a_0 a_4 k_{11} k_{14} l_4 \lambda \Upsilon_\xi^{'} \zeta \Lambda}{a_1 k_{11}^2 + a_2 k_{12}^2 + a_3 k_{13}^2 + a_4 k_{14}^2} + 2 a_4 l_4 l_{14} \lambda \Upsilon_\xi^{'} \zeta \Lambda_\eta^{'} \\
& - \frac{2 a_0 a_4 k_{11} k_{14} l_{14} \zeta \Lambda \zeta \Lambda_\eta^{'}}{a_1 k_{11}^2 + a_2 k_{12}^2 + a_3 k_{13}^2 + a_4 k_{14}^2} + a_4 c_4^2 - \frac{2 a_4 c_4 a_0 k_1 k_4 \lambda \Upsilon}{a_1 k_1^2 + a_2 k_2^2 + a_3 k_3^2 + a_4 k_4^2} + 2 a_4 c_4 l_4 \lambda \Upsilon_\xi^{'} \\
& - \frac{2 a_4 c_4 a_0 k_{11} k_{14} \zeta \Lambda}{a_1 k_{11}^2 + a_2 k_{12}^2 + a_3 k_{13}^2 + a_4 k_{14}^2} + 2 a_4 c_4 l_{14} \zeta \Lambda_\eta^{'}.
\end{aligned}
\tag{149}
$$

Thereupon

$$
\begin{aligned}
&a_1v_t^2+a_2v_x^2+a_3v_y^2+a_4v_z^2\\
&=a_1\left(\frac{a_0k_1^2}{a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2}\right)^2(\lambda\Upsilon)^2-\left(a_2l_2^2+a_3l_3^2+a_4l_4^2\right)\left(\lambda\Upsilon_\xi'\right)^2\\
&+a_1\left(\frac{a_0k_{11}^2}{a_1k_{11}^2+a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2}\right)^2(\zeta\Lambda)^2-\left(a_2l_{12}^2+a_3l_{13}^2+a_4l_{14}^2\right)\left(\zeta\Lambda_\eta'\right)^2\\
&-\frac{2a_0a_1k_1^2}{a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2}\sqrt{\frac{-a_2l_2^2-a_3l_3^2-a_4l_4^2}{a_1}}\lambda\Upsilon\lambda\Upsilon_\xi'\\
&+\frac{2a_0^2a_1k_1^2k_{11}^2\lambda\Upsilon\zeta\Lambda}{\left(a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2\right)\left(a_1k_{11}^2+a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2\right)}\\
&+\frac{2a_0a_1k_1^2}{a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2}\sqrt{\frac{-a_2l_{12}^2-a_3l_{13}^2-a_4l_{14}^2}{a_1}}\lambda\Upsilon\zeta\Lambda_\eta'\\
&-\sqrt{\frac{-a_2l_2^2-a_3l_3^2-a_4l_4^2}{a_1}}\frac{2a_0a_1k_{11}^2\lambda\Upsilon_\xi'\zeta\Lambda}{a_1k_{11}^2+a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2}\\
&-2a_1\sqrt{\frac{-a_2l_2^2-a_3l_3^2-a_4l_4^2}{a_1}}\sqrt{\frac{-a_2l_{12}^2-a_3l_{13}^2-a_4l_{14}^2}{a_1}}\lambda\Upsilon_\xi'\zeta\Lambda_\eta'\\
&+\sqrt{\frac{-a_2l_{12}^2-a_3l_{13}^2-a_4l_{14}^2}{a_1}}\frac{2a_0a_1k_{11}^2\zeta\Lambda\zeta\Lambda_\eta'}{a_1k_{11}^2+a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2}\\
&+a_2\left(\frac{a_0k_1k_2}{a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2}\right)^2(\lambda\Upsilon)^2+a_2l_2^2\left(\lambda\Upsilon_\xi'\right)^2\\
&+a_2\left(\frac{a_0k_{11}k_{12}}{a_1k_{11}^2+a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2}\right)^2(\zeta\Lambda)^2+a_2l_{12}^2\left(\zeta\Lambda_\eta'\right)^2-\frac{2a_0a_2k_1k_2l_2\lambda\Upsilon\lambda\Upsilon_\xi'}{a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2}\\
&+\frac{2a_0^2a_2k_1k_2k_{11}k_{12}\lambda\Upsilon\zeta\Lambda}{\left(a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2\right)\left(a_1k_{11}^2+a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2\right)}-\frac{2a_0a_2k_1k_2l_{12}\lambda\Upsilon\zeta\Lambda_\eta'}{a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2}\\
&-\frac{2a_0a_2k_{11}k_{12}l_2\lambda\Upsilon_\xi'\zeta\Lambda}{a_1k_{11}^2+a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2}+2a_2l_2l_{12}\lambda\Upsilon_\xi'\zeta\Lambda_\eta'-\frac{2a_0a_2k_{11}k_{12}l_{12}\zeta\Lambda\zeta\Lambda_\eta'}{a_1k_{11}^2+a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2}\\
&+a_3\left(\frac{a_0k_1k_3}{a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2}\right)^2(\lambda\Upsilon)^2+a_3l_3^2\left(\lambda\Upsilon_\xi'\right)^2+a_3\left(\frac{a_0k_{11}k_{13}}{a_1k_{11}^2+a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2}\right)^2(\zeta\Lambda)^2\\
&+a_3l_{13}^2\left(\zeta\Lambda_\eta'\right)^2-\frac{2a_0a_3k_1k_3l_3\lambda\Upsilon\lambda\Upsilon_\xi'}{a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2}+\frac{2a_0^2a_3k_1k_3k_{11}k_{13}\lambda\Upsilon\zeta\Lambda}{\left(a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2\right)\left(a_1k_{11}^2+a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2\right)}\\
&-\frac{2a_0a_3k_1k_3l_{13}\lambda\Upsilon\zeta\Lambda_\eta'}{a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2}-\frac{2a_0a_3k_{11}k_{13}l_3\lambda\Upsilon_\xi'\zeta\Lambda}{a_1k_{11}^2+a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2}+2a_3l_3l_{13}\lambda\Upsilon_\xi'\zeta\Lambda_\eta'\\
&-\frac{2a_0a_3k_{11}k_{13}l_{13}\zeta\Lambda\zeta\Lambda_\eta'}{a_1k_{11}^2+a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2}+a_4\left(\frac{a_0k_1k_4}{a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2}\right)^2(\lambda\Upsilon)^2+a_4l_4^2\left(\lambda\Upsilon_\xi'\right)^2\\
&+a_4\left(\frac{a_0k_{11}k_{14}}{a_1k_{11}^2+a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2}\right)^2(\zeta\Lambda)^2+a_4l_{14}^2\left(\zeta\Lambda_\eta'\right)^2-\frac{2a_0a_4k_1k_4l_4\lambda\Upsilon\lambda\Upsilon_\xi'}{a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2}\\
&+\frac{2a_0^2a_4k_1k_4k_{11}k_{14}\lambda\Upsilon\zeta\Lambda}{\left(a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2\right)\left(a_1k_{11}^2+a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2\right)}-\frac{2a_0a_4k_1k_4l_{14}\lambda\Upsilon\zeta\Lambda_\eta'}{a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2}\\
&-\frac{2a_0a_4k_{11}k_{14}l_4\lambda\Upsilon_\xi'\zeta\Lambda}{a_1k_{11}^2+a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2}+2a_4l_4l_{14}\lambda\Upsilon_\xi'\zeta\Lambda_\eta'-\frac{2a_0a_4k_{11}k_{14}l_{14}\zeta\Lambda\zeta\Lambda_\eta'}{a_1k_{11}^2+a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2}\\
&+a_2c_2^2-\frac{2a_2c_2a_0k_1k_2\lambda\Upsilon}{a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2}+2a_2c_2l_2\lambda\Upsilon_\xi'-\frac{2a_2c_2a_0k_{11}k_{12}\zeta\Lambda}{a_1k_{11}^2+a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2}\\
&+2a_2c_2l_{12}\zeta\Lambda_\eta'+a_3c_3^2-\frac{2a_3c_3a_0k_1k_3\lambda\Upsilon}{a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2}+2a_3c_3l_3\lambda\Upsilon_\xi'-\frac{2a_3c_3a_0k_{11}k_{13}\zeta\Lambda}{a_1k_{11}^2+a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2}\\
&+2a_3c_3l_{13}\zeta\Lambda_\eta'+a_4c_4^2-\frac{2a_4c_4a_0k_1k_4\lambda\Upsilon}{a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2}+2a_4c_4l_4\lambda\Upsilon_\xi'-\frac{2a_4c_4a_0k_{11}k_{14}\zeta\Lambda}{a_1k_{11}^2+a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2}\\
&+2a_4c_4l_{14}\zeta\Lambda_\eta'\\
&=a_0^2k_1^2\frac{a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2}{\left(a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2\right)^2}(\lambda\Upsilon)^2+a_0^2k_{11}^2\frac{a_1k_{11}^2+a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2}{\left(a_1k_{11}^2+a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2\right)^2}(\zeta\Lambda)^2\\
&-\frac{2a_0k_1\lambda\Upsilon\lambda\Upsilon_\xi'}{a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2}\left(a_1k_1\sqrt{\frac{-a_2l_2^2-a_3l_3^2-a_4l_4^2}{a_1}}+a_2k_2l_2+a_3k_3l_3+a_4k_4l_4\right)\\
&+\frac{2a_0^2k_1k_{11}\lambda\Upsilon\zeta\Lambda(a_1k_1k_{11}+a_2k_2k_{12}+a_3k_3k_{13}+a_4k_4k_{14})}{\left(a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2\right)\left(a_1k_{11}^2+a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2\right)}\\
&+\frac{2a_0k_1\lambda\Upsilon\zeta\Lambda_\eta'}{a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2}\left(a_1k_1\sqrt{\frac{-a_2l_{12}^2-a_3l_{13}^2-a_4l_{14}^2}{a_1}}-a_2k_2l_{12}-a_3k_3l_{13}-a_4k_4l_{14}\right)\\
&-\frac{2a_0k_{11}\lambda\Upsilon_\xi'\zeta\Lambda}{a_1k_{11}^2+a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2}\left(a_1k_{11}\sqrt{\frac{-a_2l_2^2-a_3l_3^2-a_4l_4^2}{a_1}}+a_2k_{12}l_2+a_3k_{13}l_3+a_4k_{14}l_4\right)\\
&-2\lambda\Upsilon_\xi'\zeta\Lambda_\eta'\left(a_1\sqrt{\frac{-a_2l_2^2-a_3l_3^2-a_4l_4^2}{a_1}}\sqrt{\frac{-a_2l_{12}^2-a_3l_{13}^2-a_4l_{14}^2}{a_1}}-a_2l_2l_{12}-a_3l_3l_{13}-a_4l_4l_{14}\right)\\
&+\frac{2a_0k_{11}\zeta\Lambda\zeta\Lambda_\eta'}{a_1k_{11}^2+a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2}\left(a_1k_{11}\sqrt{\frac{-a_2l_{12}^2-a_3l_{13}^2-a_4l_{14}^2}{a_1}}-a_2k_{12}l_{12}-a_3k_{13}l_{13}-a_4k_{14}l_{14}\right)\\
&+a_2c_2^2+a_3c_3^2+a_4c_4^2-\frac{2a_0k_1\lambda\Upsilon(a_2c_2k_2+a_3c_3k_3+a_4c_4k_4)}{a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2}+2\lambda\Upsilon_\xi'\left(a_2c_2l_2+a_3c_3l_3+a_4c_4l_4\right)\\
&-\frac{2a_0k_{11}\zeta\Lambda(a_2c_2k_{12}+a_3c_3k_{13}+a_4c_4k_{14})}{a_1k_{11}^2+a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2}+2\zeta\Lambda_\eta'\left(a_2c_2l_{12}+a_3c_3l_{13}+a_4c_4l_{14}\right).
\end{aligned}
$$

Since Υ and Λ are arbitrary functions, we set

$$k_1 = k_{11} = 0. \tag{150}$$

So

$$\begin{aligned}&\tfrac{2k_1(a_2k_2l_2+a_3k_3l_3+a_4k_4l_4)}{-a_1k_1^2+a_2k_2^2+a_3k_3^2+a_4k_4^2} = \sqrt{\tfrac{-a_2l_2^2-a_3l_3^2-a_4l_4^2}{a_1}} = 0\\ &\Longrightarrow l_2 = \pm\sqrt{\tfrac{-a_3l_3^2-a_4l_4^2}{a_2}},\end{aligned}$$

$$\begin{aligned}&\tfrac{2k_{11}(a_2k_{12}l_{12}+a_3k_{13}l_{13}+a_4k_{14}l_{14})}{-a_1k_{11}^2+a_2k_{12}^2+a_3k_{13}^2+a_4k_{14}^2} = -\sqrt{\tfrac{-a_2l_{12}^2-a_3l_{13}^2-a_4l_{14}^2}{a_1}} = 0\\ &\Longrightarrow l_{12} = \pm\sqrt{\tfrac{-a_3l_{13}^2-a_4l_{14}^2}{a_2}}.\end{aligned}$$

Set

$$l_2 = \sqrt{\frac{-a_3l_3^2 - a_4l_4^2}{a_2}}, l_{12} = -\sqrt{\frac{-a_3l_{13}^2 - a_4l_{14}^2}{a_2}}.$$

Then

$$\begin{aligned}&a_1v_t^2 + a_2v_x^2 + a_3v_y^2 + a_4v_z^2\\ &= -2\lambda\Upsilon_\xi'\zeta\Lambda_\eta'\left(-a_2l_2l_{12} - a_3l_3l_{13} - a_4l_4l_{14}\right) + a_2c_2^2 + a_3c_3^2 + a_4c_4^2\\ &+2\lambda\Upsilon_\xi'\left(a_2c_2l_2 + a_3c_3l_3 + a_4c_4l_4\right) + 2\zeta\Lambda_\eta'\left(a_2c_2l_{12} + a_3c_3l_{13} + a_4c_4l_{14}\right)\\ &= 2\lambda\Upsilon_\xi'\zeta\Lambda_\eta'\left(-a_2\sqrt{\tfrac{-a_3l_3^2-a_4l_4^2}{a_2}}\sqrt{\tfrac{-a_3l_{13}^2-a_4l_{14}^2}{a_2}} + a_3l_3l_{13} + a_4l_4l_{14}\right) + a_2c_2^2 + a_3c_3^2\\ &+a_4c_4^2 + 2\lambda\Upsilon_\xi'\left(a_2c_2\sqrt{\tfrac{-a_3l_3^2-a_4l_4^2}{a_2}} + a_3l_3c_3 + a_4c_4l_4\right)\\ &+2\zeta\Lambda_\eta'\left(-a_2c_2\sqrt{\tfrac{-a_3l_{13}^2-a_4l_{14}^2}{a_2}} + a_3c_3l_{13} + a_4c_4l_{14}\right).\end{aligned}$$

Set

$$-a_2\sqrt{\frac{-a_3l_3^2 - a_4l_4^2}{a_2}}\sqrt{\frac{-a_3l_{13}^2 - a_4l_{14}^2}{a_2}} + a_3l_3l_{13} + a_4l_4l_{14} = 0, \tag{136}$$

$$a_2c_2\sqrt{\frac{-a_3l_3^2 - a_4l_4^2}{a_2}} + a_3l_3c_3 + a_4c_4l_4 = 0, \tag{137}$$

$$-a_2c_2\sqrt{\frac{-a_3l_{13}^2 - a_4l_{14}^2}{a_2}} + a_3c_3l_{13} + a_4c_4l_{14} = 0. \tag{138}$$

Whereupon

$$\lambda = \zeta = 1, \tag{151}$$

$$\xi = \sqrt{\frac{-a_3l_3^2 - a_4l_4^2}{a_2}}x + l_3y + l_4z + l_5, \tag{133}$$

$$\eta = -\sqrt{\frac{-a_3l_{13}^2 - a_4l_{14}^2}{a_2}}x + l_{13}y + l_{14}z + l_{15}, \tag{134}$$

So

$$\begin{aligned}&f_v''\left(a_1v_t^2 + a_2v_x^2 + a_3v_y^2 + a_4v_z^2\right) + f_v'\left(a_0v_t + a_1v_{tt} + a_2v_{xx} + a_3v_{yy} + a_4v_{zz}\right)\\ &= f_v''\left(a_1v_t^2 + a_2v_x^2 + a_3v_y^2 + a_4v_z^2\right) = A\left(f\right).\end{aligned}$$

That is

$$f_v'' = \frac{A(f)}{a_2c_2^2 + a_3c_3^2 + a_4c_4^2}. \tag{152}$$

Thereupon

$$v = c_{10} \pm \int \left(c_9 + \frac{2\int A(f)\,df}{a_2c_2^2 + a_3c_3^2 + a_4c_4^2} \right)^{-\frac{1}{2}} df.$$

Set $c_{10} = 0$, then the general solution of (130) is

$$\Upsilon(\xi) + \Lambda(\eta) + \varsigma = \pm \int \left(c_9 + \int K(u)\,du \right)^{-\frac{1}{2}} du, \tag{131}$$

where ξ, η, ς and $K(u)$ satisfy (132-138), the theorem is proven. □

3. Discussion and summary

According to Theorem 3-9, it can be intuitively found that the structures of the general solutions of these equations are all the same. The general solutions of the two equations (74,130) are identical, and neither contains time, which shows that we have obtained the general solutions for equilibrium. Whether there are general solutions with time for these two equations is a question that needs to be studied further.

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