A General Definition of Means and Corresponding Inequalities

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Abstract

The definitions of the Quadratic, Arithmetic, Geometric and Harmonic means all follow a certain generalisable pattern. The aim of this paper is to explicitly state that pattern, hence generalising the definition of a 'mean', and to prove inequalities for comparing different means (of which those between the four means stated previously is a special case).

1. Introduction

The aim of this paper is to introduce a general definition of means (of which the Quadratic, Arithmetic, Geometric and Harmonic Means (QM, AM, GM and HM respectively) are special cases), and to prove inequalities for comparing such means.

The most general result proven in this paper states that given a pair of invertible and twice differentiable functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ (with $\frac{f''(t)}{f'(t)} > \frac{g''(t)}{g'(t)}\ \forall\, t \in \mathbb{R}$), n numbers $x_1, x_2, \ldots, x_n \in \mathbb{R}$ and n positive real numbers $a_1, a_2, \ldots, a_n$ s.t. $a_1 + a_2 + \ldots + a_n = 1$, we have

$$f^{-1}\left(\sum_{i=1}^{n} a_i\, f(x_i)\right) \geq g^{-1}\left(\sum_{i=1}^{n} a_i\, g(x_i)\right)$$

With equality holding iff $x_1 = x_2 = \ldots = x_n$.

Note that $\mathbb{R}$ may be replaced by any open subset of the real line in all the occurrences it has in the above result, and the result will hold just as well. The same applies to all results proven in this paper.

In order to make the arguments easy to follow, I have provided the results in ascending order of generality. However, if one wishes to simply see the most general and final result I obtained (of which all other results are special cases), one may go through *Section 2* and then skip to *Result 4* and *Result 5* (Pages 7 to 10).

2 General Definitions and a Basic Property

Definition 2.1

We will define the 'f-mean' (where $f : \mathbb{R} \to \mathbb{R}$ is an invertible function) of n numbers $x_1\,, x_2\,, \ldots\,, x_n \in \mathbb{R}$ (denoted by $F(x_1\,, x_2\,, \ldots\,, x_n)$) as follows

$$F(x_1\,, x_2\,, \ldots\,, x_n) = f^{-1}\left(\frac{f(x_1) + f(x_2) + \ldots + f(x_n)}{n}\right)$$

One may note that the QM is the special case $f(t) = t^2$, the AM is the special case $f(t) = t$, the GM is the special case $f(t) = \ln(t)$ and the HM is the special case $f(t) = \frac{1}{t}$.

Definition 2.2

Similar to the f-mean, we will define the 'weighted f-mean' (where $f : \mathbb{R} \to \mathbb{R}$ is an invertible function) of n numbers $x_1\,, x_2\,, \ldots\,, x_n \in \mathbb{R}$, given n positive real numbers $a_1\,, a_2\,, \ldots\,, a_n$ s.t $a_1 + a_2 + \ldots + a_n = 1$ (denoted by $F_0(x_1\,, x_2\,, \ldots\,, x_n)$) as follows

$$F_0(x_1\,, x_2\,, \ldots\,, x_n) = f^{-1}\left[a_1\, f(x_1) + a_2\, f(x_2) + \ldots + a_n\, f(x_n)\right]$$

Property 2.3

The $(pf + q)$-mean is the same as the f-mean, where p and q are some real numbers ($p \neq 0$) and f is an invertible real function.

Hence, we may say that under the transformation T, defined as $T\{f\} = pf + q$, the f-mean is an invariant.

Furthermore, one may also notice that $\frac{f''}{f'}$ is an important entity in the final result, and just like the f-mean, it too is an invariant under T. This is no coincidence.

Result 1 : f-mean Versus AM

In this section, we prove the following result

$$\frac{f''(t)}{f'(t)} > 0 \; \forall \, t \in \mathbb{R} \implies F(x,y) > \frac{x+y}{2} \;\; \forall \; y > x \in \mathbb{R}$$

for $f : \mathbb{R} \to \mathbb{R}$ being a monotonic and twice differentiable function.

Proof
For simplicity, we will write $F(x,y)$ as F, henceforth. Also, we may assume that $f'(t) > 0 \; \forall \, t \in \mathbb{R}$ WLG using *Property 2.3*. Note that this also means that $f''(t) > 0 \; \forall \, t \in \mathbb{R}$.

Now, we shall manipulate $f(F)$ in such a way that it becomes straightforward to compare it to $f\left(\frac{x+y}{2}\right)$.

$$f(F) = \frac{f(x)+f(y)}{2} = f(y) - \frac{1}{2}\,[\,f(y) - f(x)]$$

$$= f(y) - \frac{1}{2}\int_x^y f'(t)\,dt$$

$$\implies f(F) = f(y) - \frac{1}{2}\int_0^{\frac{y-x}{2}} f'(x+t)\,dt - \frac{1}{2}\int_{\frac{x+y}{2}}^y f'(t)\,dt \qquad (1)$$

Likewise, we may also manipulate $f\left(\frac{x+y}{2}\right)$ in a similar fashion.

$$f\left(\frac{x+y}{2}\right) = f(y) - \left[f(y) - f\left(\frac{x+y}{2}\right)\right]$$

$$= f(y) - \int_{\frac{x+y}{2}}^y f'(t)\,dt$$

$$\implies f\left(\frac{x+y}{2}\right) = f(y) - \frac{1}{2}\int_0^{\frac{y-x}{2}} f'\left(\frac{x+y}{2}+t\right)\,dt - \frac{1}{2}\int_{\frac{x+y}{2}}^y f'(t)\,dt \qquad (2)$$

Hence, subtracting *(2)* from *(1)* yields

$$f(F) - f\left(\frac{x+y}{2}\right) = \frac{1}{2}\int_0^{\frac{y-x}{2}} \left[f'\left(\frac{x+y}{2} + t\right) - f'(x+t)\right] dt$$

Since $f''(t) > 0 \; \forall \, t \in \mathbb{R}$, this means that

$$f(F) - f\left(\frac{x+y}{2}\right) > 0$$

Now, since $f'(t) > 0 \; \forall \, t \in \mathbb{R}$, we have

$$F > \frac{x+y}{2} \quad \square$$

Result 2 : f-mean Versus g-mean

In this section, we prove the following result

$$\frac{f''(t)}{f'(t)} > \frac{g''(t)}{g'(t)} \; \forall \, t \in \mathbb{R} \implies F(x,y) > G(x,y) \;\; \forall \; y > x \in \mathbb{R}$$

for $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ being monotonic and twice differentiable functions.

Proof
As before, we will write $F(x,y)$ and $G(x,y)$ as F and G respectively, for simplicity. Also, we may assume that $f'(t)\,,\, g'(t) > 0 \; \forall \, t \in \mathbb{R}$, as before.

Now, we shall manipulate $f(F)$ to make it easier to compare to $f(G)$ as follows

$$f(F) = \frac{f(x) + f(y)}{2} = f(y) - \frac{1}{2}\,[\,f(y) - f(x)]$$

$$= f(y) - \frac{1}{2}\int_x^y f'(t)\,dt$$

$$\implies f(F) = f(y) - \frac{1}{2}\int_x^G f'(t)\,dt - \frac{1}{2}\int_G^y f'(t)\,dt$$

Substituting $g(t) - g(x) = u$ into the left integral, we have

$$f(F) = f(y) - \frac{1}{2}\int_0^{\frac{1}{2}\left(g(y)-g(x)\right)} h\left[\, g^{-1}(u + g(x))\,\right] du - \frac{1}{2}\int_G^y f'(t)\,dt \qquad (3)$$

Where $h(t) = \frac{f'(t)}{g'(t)}$.

In a similar fashion, we shall manipulate $f(G)$ as follows

$$f(G) = f(y) - [f(y) - f(G)]$$

$$= f(y) - \int_G^y f'(t)\,dt$$

$$\implies f(G) = f(y) - \frac{1}{2}\int_G^y f'(t)\,dt - \frac{1}{2}\int_G^y f'(t)\,dt$$

Substituting $g(t) - g(G) = u$ into the left integral, we have

$$f(G) = f(y) - \frac{1}{2}\int_0^{\frac{1}{2}\left(g(y)-g(x)\right)} h\left[g^{-1}\left(u + \frac{g(x)+g(y)}{2}\right)\right] du - \frac{1}{2}\int_G^y f'(t)\,dt \qquad (4)$$

Hence, subtracting *4* from *3* yields

$$f(F) - f(G) = \frac{1}{2}\int_0^{\frac{1}{2}\left(g(y)-g(x)\right)} \left\{h\left[g^{-1}\left(u + \frac{g(x)+g(y)}{2}\right)\right] - h\left[\, g^{-1}(u + g(x))\,\right]\right\} du$$

Given that $\frac{f''(t)}{f'(t)} > \frac{g''(t)}{g'(t)}$ and $f'(t)\,, g'(t) > 0\ \forall\, t \in \mathbb{R}$, it's easy to see that $h'(t) > 0\ \forall\, t \in \mathbb{R}$. Hence, we have

$$f(F) - f(G) > 0$$

Now, since $f'(t) > 0\ \forall\, t \in \mathbb{R}$, we have

$$F > G \quad \square$$

Result 3 : Weighted f-mean Versus Weighted AM

In this section, we prove the following result

$$\frac{f''(t)}{f'(t)} > 0 \; \forall \, t \in \mathbb{R} \implies F_0(x, y) > ax + by \;\; \forall \; y > x \in \mathbb{R}$$

for $f : \mathbb{R} \to \mathbb{R}$ being a monotonic and twice differentiable function, $F_0(x, y) = f^{-1}[af(x) + bf(y)]$ and a and b being positive real numbers with $a + b = 1$.

Proof
As before, we will write $F_0(x, y)$ as F_0 for simplicity, and assume that $f'(t) > 0 \; \forall \, t \in \mathbb{R}$ WLG.

Now, we shall manipulate $f(F_0)$ in a similar vein as in the proof of *Result 1*.

$$f(F_0) = af(x) + bf(y) = f(y) - a[f(y) - f(x)]$$

$$= f(y) - a \int_x^y f'(t) \, dt$$

$$= f(y) - a \int_0^{b(y-x)} f'(x + t) \, dt - a \int_{ax+by}^y f'(t) \, dt$$

$$\implies f(F_0) = f(y) - \int_0^{ab(y-x)} f' \left(x + \frac{t}{a} \right) \, dt - a \int_{ax+by}^y f'(t) \, dt \qquad (5)$$

Likewise, we may also manipulate $f\,(ax + by)$ in a similar fashion.

$$f\,(ax + by) = f(y) - [f(y) - f\,(ax + by)]$$

$$= f(y) - \int_{ax+by}^y f'(t) \, dt$$

$$= f(y) - b \int_0^{a(y-x)} f'\,(ax + by + t) \, dt - a \int_{ax+by}^y f'(t) \, dt$$

$$\implies f(ax+by) = f(y) - \int_0^{ab(y-x)} f'\left(ax+by+\frac{t}{b}\right)dt \; - \; a\int_{ax+by}^{y} f'(t)\,dt \tag{6}$$

Hence, subtracting *(6)* from *(5)* yields

$$f(F_0) - f\,(ax+by) = \int_0^{ab(y-x)} \left[f'\left(ax+by+\frac{t}{b}\right) - f'\left(x+\frac{t}{a}\right)\right] dt$$

Since $f''(t) > 0 \; \forall \, t \in \mathbb{R}$, this means that

$$f(F_0) - f\,(ax+by) > 0$$

Now, since $f'(t) > 0 \; \forall \, t \in \mathbb{R}$, we have

$$F_0 > ax+by \quad \square$$

Result 4 : Weighted f-mean Versus Weighted g-mean

In this section, we prove the following result

$$\frac{f''(t)}{f'(t)} > \frac{g''(t)}{g'(t)} \; \forall \, t \in \mathbb{R} \implies F_0(x,y) > G_0(x,y) \; \forall \; y > x \in \mathbb{R}$$

for $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ being monotonic and twice differentiable functions, $F_0(x,y) = f^{-1}[af(x) + bf(y)]$, $G_0(x,y) = g^{-1}[ag(x) + bg(y)]$ and a and b being positive real numbers with $a + b = 1$.

Proof
As before, we will write $F_0(x,y)$ and $G_0(x,y)$ as F_0 and G_0 respectively, for simplicity. Also, we may assume that $f'(t)\,,\, g'(t) > 0 \; \forall \, t \in \mathbb{R}$ WLG (without loss of generality), using *Property 2.3*.

Now, we shall manipulate $f(F_0)$ to make it easier to compare to $f(G_0)$ as follows

$$f(F_0) = af(x) + bf(y) = f(y) - a\ [\,f(y) - f(x)]$$

$$= f(y) - a\int_x^y f'(t)\,dt$$

$$\implies f(F_0) = f(y) - a\int_x^{G_0} f'(t)\,dt - a\int_{G_0}^y f'(t)\,dt$$

Substituting $a[g(t) - g(x)] = u$ into the left integral, we have

$$f(F_0) = f(y) - \int_0^{ab(g(y)-g(x))} h\left[\,g^{-1}\left(\frac{u}{a} + g(x)\right)\right]\,du - a\int_{G_0}^y f'(t)\,dt \qquad (7)$$

Where $h(t) = \frac{f'(t)}{g'(t)}$.

In a similar fashion, we shall manipulate $f(G_0)$ as follows

$$f(G_0) = f(y) - [f(y) - f(G_0)]$$

$$= f(y) - \int_{G_0}^y f'(t)\,dt$$

$$\implies f(G_0) = f(y) - b\int_{G_0}^y f'(t)\,dt - a\int_{G_0}^y f'(t)\,dt$$

Substituting $b[g(t) - g(G_0)] = u$ into the left integral, we have

$$f(G_0) = f(y)\ -\ \int_0^{ab(g(y)-g(x))} h\left[\,g^{-1}\left(\frac{u}{b} + ag(x) + bg(y)\right)\right]\,du\ -\ a\int_{G_0}^y f'(t)\,dt \qquad (8)$$

Hence, subtracting *(8)* from *(7)* yields

$$f(F_0) - f(G_0) = \int_0^{\,ab(g(y)-g(x))} \left\{h\left[\,g^{-1}\left(\frac{u}{b} + ag(x) + bg(y)\right)\right]\ - h\left[\,g^{-1}\left(\frac{u}{a} + g(x)\right)\right]\right\}\,du$$

Given that $\frac{f''(t)}{f'(t)} > \frac{g''(t)}{g'(t)}$ and $f'(t)\,, g'(t) > 0\ \forall\, t \in \mathbb{R}$, it's easy to see that $h'(t) > 0\ \forall\, t \in \mathbb{R}$. Hence, we have

$$f(F_0) - f(G_0) > 0$$

Now, since $f'(t) > 0\ \forall\, t \in \mathbb{R}$, we have

$$F_0 > G_0 \quad \square$$

Result 5 : The Final Result

Up until this point, we have proved the final result for the case when the data set has only two entries (x and y). Now, we will show that *Result 4* implies the final result, which is that given n numbers $x_1\,, x_2\,, \ldots\,, x_n \in \mathbb{R}$ and n positive real numbers $a_1\,, a_2\,, \ldots\,, a_n$ (with $a_1 + a_2 + \ldots + a_n = 1$), we have the following

$$F_0(x_1\,, x_2\,, \ldots\,, x_n) = f^{-1}\left[a_1\, f(x_1) + a_2\, f(x_2) + \ldots + a_n\, f(x_n)\right] \;>$$
$$G_0(x_1\,, x_2\,, \ldots\,, x_n) = g^{-1}\left[a_1\, g(x_1) + a_2\, g(x_2) + \ldots + a_n\, g(x_n)\right]$$

Where $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are invertible and twice differentiable functions, with the condition that

$$\frac{f''(t)}{f'(t)} > \frac{g''(t)}{g'(t)}$$

Proof by induction on n

We will be assuming that $f'(t), g'(t) > 0\ \forall\, t \in \mathbb{R}$ WLG, as before.

The claim has been proven for $n = 2$ as *Result 4*. Hence, we will now assume that it is true for some $n \geq 2$, and prove that it's true for $n + 1$.

Let $\mathbb{F} = f^{-1}\left[\frac{a_1}{1-a_{n+1}}\, f(x_1) + \frac{a_2}{1-a_{n+1}}\, f(x_2) + \ldots + \frac{a_n}{1-a_{n+1}}\, f(x_n)\right]$, and define $\mathbb{G}$ analogously. Now, we have

$$F_0(x_1\,, x_2\,, \ldots\,, x_{n+1}) = f^{-1}\left[\, a_1\, f(x_1) + a_2\, f(x_2) + \ldots + a_{n+1}\, f(x_{n+1})\,\right]$$

$$\implies F_0(x_1\,, x_2\,, \ldots\,, x_{n+1}) = f^{-1}\left[\,(1 - a_{n+1})\ f(\mathbb{F}) + a_{n+1}\ f(x_{n+1})\,\right] \qquad (9)$$

Likewise we have a similar result for the weighted g-mean, which is as follows

$$G_0(x_1\,,\,x_2\,,\,\ldots\,,\,x_{n+1}) = g^{-1}\,[\,(1-a_{n+1})\;g(\mathbb{G}) + a_{n+1}\;g(x_{n+1})\,] \tag{10}$$

We have $\mathbb{F} > \mathbb{G}$ using the induction hypothesis. Hence, we have

$$f^{-1}\,[\,(1-a_{n+1})\;f(\mathbb{F})+a_{n+1}\;f(x_{n+1})\,] > f^{-1}\,[\,(1-a_{n+1})\;f(\mathbb{G})+a_{n+1}\;f(x_{n+1})\,]$$

Result 4 yields

$$f^{-1}\,[\,(1-a_{n+1})\;f(\mathbb{G})+a_{n+1}\;f(x_{n+1})\,] > g^{-1}\,[\,(1-a_{n+1})\;g(\mathbb{G})+a_{n+1}\;g(x_{n+1})\,]$$

Hence, combining these two inequalities yields the desired result. □