DERIVATION OF A GENERALIZED TRANSPORT FLUX FROM THE FOKKER-PLANCK EQUATION

Peter Bissonnet

An independent and unaffiliated researcher
Mailing address:
Calle 9 37-90
Apartado Postal 365837
Fusagasuga, Cundinamarca
Colombia, South America 252211

Abstract

The derivation of a generalized transport flux is attempted from the well-known Fokker-Planck equation using a covariant 4-dimensional approach
Four-dimensional tensor analysis is used in the derivation of this generalized transport flux.

Keywords: Generalized Transport Flux, Fokker-Planck equation, Covariant Formulation of Transport Equation

2020 Mathematics Subject Classification: 60J60, 80A19, 82C31, 82C70

1 Main Argument

The derivation result used will be that of C. Kittel[2] (and his source), in addition to other sources[3]. The derivation derives from the Smoluchowski equation using a conditional probability P(z|y, t) that a particle at z at t = 0 will be at y during the time interval Δt. The result derived is the Fokker-Planck equation without sources:

$$\frac{\partial P}{\partial t}+\frac{\partial\left[A(y)P\right]}{\partial y}-\frac{1}{2}\frac{\partial^2\left[B(y)P\right]}{\partial y^2}=0 \qquad (1)$$

If A(y) = 0, then there are equal probabilities of moving either left or right. If B(y) is independent of position or in other words there is an isotropic environment, then this reduces to the usual diffusion equation. Now let's construct the covariant formulation utilizing the symbolism[1] that $_{/\mu}$ is a partial derivative with respect to coordinate μ (or ∂_μ) and $^{/\mu}$ represents the contravariant partial derivative with respect to coordinate μ (which runs from 1 to 4).

$$\left(A^{\mu}P\right)_{/\mu} = \nabla \bullet \left(\mathbf{A}P\right) + \frac{\partial\left(A^4P\right)}{\partial\left(ct\right)} = \left(A^1P\right)_{/1} + \left(A^2P\right)_{/2} + \left(A^3P\right)_{/3} + \left(A^4P\right)_{/4} \qquad (2)$$

The above equation then incorporates the first two terms on the left of the Fokker-Planck equation.

$$\nabla^2\left(BP\right) = \nabla \bullet \nabla\left(BP\right) \qquad (3)$$

which can be represented covariantly as $\left(BP\right)^{/\mu}{}_{/\mu}$

$$\left(A^{\mu}P\right)_{/\mu} + K\left(BP\right)^{/\mu}{}_{/\mu} = S = \text{source/sink.} \qquad (4)$$

where K is a constant and is left as such to be as general as possible.

This is still not in completely covariant form since we have not used the covariant derivative // which involves the Christoffel symbols in curved coordinate systems.

The author digresses here momentarily to introduce the covariant derivative and the Christoffel symbol, actually the Christoffel symbol of the second kind.
The covariant derivative of a contravariant vector is

$\xi^{i}{}_{//l} = \xi^{i}{}_{/l} + \left\{ {i \atop lr} \right\} \xi^{r}$ and the covariant derivative of a covariant vector is

$\eta_{m//l} = \eta_{m/l} - \left\{ {r \atop ml} \right\} \eta_{r}$

The Christoffel symbol of the second kind is defined as

$\left\{ {j \atop ik} \right\} = \frac{1}{2} g^{jl}\left(g_{il/k} + g_{kl/i} - g_{ik/l}\right)$ where g_{ik} is the 4-dimensional metric with a signature of -1 -1 -1 +1 in a flat Lorentzian spacetime.

Equation 4 in covariant form now becomes:

$$\left(A^{\mu}P\right)_{//\mu} + K\left(BP\right)^{/\mu}_{//\mu} = S \qquad (5)$$

which is now a covariant equation. We can now factor out the covariant derivative and obtain:

$$\left[\left(A^{\mu}P\right) + K\left(BP\right)^{/\mu}\right]_{//\mu} = S = J^{\mu}_{//\mu} \qquad (6)$$

$$J^{\mu} = \left(A^{\mu}P\right) + K\left(BP\right)^{/\mu} = \text{ transport flux or current density} \qquad (7)$$

This is slightly more general than other transport fluxes usually found and represented in irreversible thermodynamics, such as (again from Kittel):
$\mathbf{J}_x$ K_x = - $\mathbf{\nabla}$ W
Ohms law: x = e and K_e = electrical conductivity, W = electric potential, $\mathbf{J}_e$ = current density

Fourier's law: x = q and K_q = thermal conductivity, W = Temperature, $\mathbf{J}_q$ = heat current density
Fick's law: x = m and K_m = diffusivity, W = particle or mass concentration, $\mathbf{J}_m$ = mass or particle current density.

2 Conclusion

$$J_\mu = \left(A_\mu P\right) + K\left(BP\right)_{/\mu} \quad (8)$$

The above transport flux shows that there is a flow in three dimensions determined by the vector "stress" components A_μ and the gradient in the non-isotropic factor BP for $\mu = 1, 2, 3$. Actually, the three dimensional vector components of A_μ point in the negative $\vec{A}$, due to the fact that $A^\mu = \left(\vec{A}, A^4\right)$ and $A_\mu = \left(-\vec{A}, A_4\right)$, while $\partial^\mu = \left(-\nabla, \frac{\partial}{\partial x^4}\right)$ and $\partial_\mu = \left(\nabla, \frac{\partial}{\partial x^4}\right)$.

The whole point in deriving this new generalized transport flux is to hopefully simplify investigations such as electrical conductivity, thermal conductivity, and diffusivity, etc. in environments in which the "stress" and non-isotropic factors are more complicated and by reducing the second order Fokker-Planck equation to a first order equation.
There is also an added bonus in that there is now a 4-dimensional component.
$J_4 = A_4 P + K\frac{\partial}{\partial(ct)}\left(BP\right)$ which is derived from equation 8 and which hopefully sheds light upon the variation in time of the components of the transport flux.

It would be interesting to know in how many other venues this structural form for a transport flux arises, and what are, if any, the common characteristics they all share.

Acknowledgements. This is a text of acknowledgements.

References

[1] R. Adler, M. Bazin, M. Schiffer, *Introduction to General Relativity*, McGraw-Hill, (1965).

[2] C. Kittel, *Elementary Statistical Physics*, John Wiley and Sons, Inc., New York, p. 157 (Third Printing, March 1964)
Kittel's reference is:
Ming Chen Wang and G.E. Uhlenbeck, Revs. Mod. Phys. **17**, 331, (1945).

[3] G.G. Koerber, *Properties of Solids*, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, p. 13 (1962).