Notation of zeta function using tangent number and research irrational number of special value of Riemann zeta function(n) and L-function(n)

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Representation of $\zeta(2n)$ using the tangent number and consideration of $\zeta(n)$ and $L(n)$.

1 Introduction

First, this sentence is created by machine translation.[1] There may be some strange sentences.

The value does not change even if you use a function that uses the tangent number, but I think it is easier to understand intuitively than the Bernoulli number.

We investigated the zeta function (2n+1) and the L function L (2n). I didn't know the exact value, but I investigated the relationship with irrational numbers.

2 Riemann zeta-function and L-function

2.1 Riemann zeta-function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (s > 1)$$

2.2 L-function

(Dirichlet L-function)

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \qquad \chi(n) = \begin{cases} 0 & (n \equiv 0, 2 \bmod 4) \\ 1 & (n \equiv 1 \bmod 4) \\ -1 & (n \equiv 3 \bmod 4) \end{cases}$$

(Euler L-function)

$$L(s) = \sum_{n \geqq 1:(Odd)}^{\infty} (-1)^{\frac{n-1}{2}} n^{-s} = 1 - \frac{1}{3^s} + \frac{1}{5^s} - \frac{1}{7^s} + \frac{1}{9^s} - \cdots \quad [2]$$

$$L(s) = \prod_{p;(odd\ \ prime)}^{\infty} (1 - (-1)^{\frac{p-1}{2}} p^{-s})^{-1} = \frac{1}{1+3^{-s}} \times \frac{1}{1-5^{-s}} \times \frac{1}{1+7^{-s}} \times \cdots \quad [2]$$

$$L(2n+1) = \frac{E_{2n}}{(2n)!\ 2^{2n+2}} \pi^{2n+1} \qquad (n = 0, 1, 2 \cdots) \qquad E_{2n} : (Euler\ number) \quad [3]$$

3 Notation of zeta function using tangent number

3.1 Tangent number

$\tan z = \sum_{k=0}^{\infty} \frac{T_k}{k!} z^k$

T_n	1	2	3	4	5	6	7	8	9	10	$\cdots$
tangent number	1	-	2	-	16	-	272	-	7936	-	$\cdots$

3.2 $\zeta(2n)$

$$(n \geqq 2) \quad n \equiv 0 \pmod 2 \qquad \zeta(n) = \frac{T_{(n-1)}\ \pi^n}{\Gamma(n)\ 2^{n+1}} \times \frac{2^n}{2^n - 1}$$

3.3 $\zeta(-n)$

$$(n \geqq 1) \quad n \equiv 1 \pmod 2 \qquad B_n = Bernoulli\ number$$

$$\zeta(-n) = (-1)^n \frac{B_{(n+1)}}{n+1}$$

$$T_n = \frac{2^{n+1}(2^{n+1}-1)\ |B_{n+1}|}{n+1}$$

$$\zeta(-n) = (-1)^{\frac{n+1}{2}} \times T_n \times \frac{n+1}{2^{n+1}(2^{n+1}-1)} \times \frac{1}{n+1}$$

$$= (-1)^{\frac{n+1}{2}} \times T_n \times \frac{1}{2^{n+1}(2^{n+1}-1)}$$

4 Abbreviation

4.1 Display method

$$\prod_{\substack{p;odd \\ prime \\ (p\equiv x \bmod a)}}^{\infty} \frac{p^n}{p^n - 1} = \prod_{\substack{p;odd \\ prime \\ (\bmod a)}}^{\infty} \left[\frac{p \overset{f(x)}{n}}{p^n \underset{f(x)}{-} 1} \right]$$

$$(a = 8) \qquad f(x) = \begin{cases} (p \equiv 1) & (p \equiv 1 \bmod 8) \\ (p \equiv 5,7) & (p \equiv 5,7 \bmod 8) \\ (prime - all = p) & (p \equiv 1,3,5,7 \bmod 8) \end{cases}$$

$$L(n) = \prod_{\substack{p;odd \\ prime \\ (p\equiv 1 \bmod 4)}}^{\infty} \frac{p^n}{p^n - 1} \times \prod_{\substack{p;odd \\ prime \\ (p\equiv 3 \bmod 4)}}^{\infty} \frac{p^n}{p^n + 1} = \prod_{\substack{p;odd \\ prime \\ (\bmod 4)}}^{\infty} \left[\frac{p \overset{(p\equiv1)}{n}}{p^n \underset{(p\equiv1)}{-} 1} \times \frac{p \overset{(p\equiv3)}{n}}{p^n \underset{(p\equiv3)}{+} 1} \right]$$

$$\zeta(n) = \prod_{p;prime}^{\infty} \frac{p^n}{p^n - 1} = \prod_{p;prime}^{\infty} \left[\frac{p^n}{p^n - 1} \right]$$

4.2 Calculation

$$\prod_{\substack{p;odd\\prime\\(\text{mod } 4)}}^{\infty} \left[\frac{p_{(p\equiv1)}^{n}}{p^n_{(p\equiv1)} - 1} \times \frac{p_{(p\equiv3)}^{n}}{p^n_{(p\equiv3)} - 1} = \frac{p_{(p)}^{n}}{p^n_{(p)} - 1} \right]$$

$$\prod_{\substack{p;odd\\prime\\(\text{mod } 4)}}^{\infty} \left[\frac{p_{(p\equiv1)}^{n}}{p^n_{(p\equiv1)} - 1} \times \frac{p_{(p\equiv3)}^{n}}{p^n_{(p\equiv3)} + 1} = \frac{p_{(p)}^{n}}{(p^n_{(p\equiv1)} - 1)(p^n_{(p\equiv3)} + 1)} \right]$$

$$\prod_{\substack{p;odd\\prime}}^{\infty} \left[\frac{p^n}{p^n - 1} \times \frac{p^n}{p^n + 1} = \frac{p^{2n}}{p^{2n} - 1} \right] \qquad \prod_{\substack{p;odd\\prime}}^{\infty} \left[\frac{p^n}{p^n - 1} \times \frac{p^n}{p^n - 1} = \frac{p^{2n}}{(p^n - 1)^2} \right]$$

$$\prod_{\substack{p;odd\\prime\\(\text{mod } 4)}}^{\infty} \left[\frac{p_{(p)}^{n}}{p^n_{(p)} - 1} \times \frac{p^n_{(p\equiv3)} - 1}{p^n_{(p\equiv3)}} = \frac{p_{(p\equiv1)}^{n}}{p^n_{(p\equiv1)} - 1} \right]$$

$$\prod_{\substack{p;odd\\prime\\(\text{mod } 4)}}^{\infty} \left[\frac{p_{(p\equiv1)}^{n}}{p^n_{(p\equiv1)} - 1} \times \frac{p^n_{(p\equiv3)} - 1}{p^n_{(p\equiv3)}} = \frac{p_{(p\equiv1)}^{n} \times (p^n_{(p\equiv3)} - 1)}{p^n_{(p\equiv3)} \times (p^n_{(p\equiv1)} - 1)} \right]$$

5 Factorization

$(n = 1, 2, 3, 4 \cdots)$

$$\begin{aligned}
(x^n - 1) &= (x - 1)(x^{n-1} + x^{n-2} + \cdots + x + 1) \\
(x^n - 1) &= (x + 1)(x^{n-1} - x^{n-2} + x^{n-3} \cdots - 1) \qquad n \equiv 0 \pmod 2 \\
(x^n + 1) &= (x + 1)(x^{n-1} - x^{n-2} + x^{n-3} \cdots - 1) \qquad n \equiv 1 \pmod 2 \\
(x^{2n} - 1) &= (x^n - 1)(x^n + 1) \\
(x^{3n} - 1) &= (x^n - 1)(x^{2n} + x^n + 1) = (x^{2n} - 1)(x^n + \frac{1}{x^n + 1}) \\
(x^{3n} + 1) &= (x^n + 1)(x^{2n} - x^n + 1) = (x^{2n} - 1)(x^n + \frac{1}{x^n - 1}) \\
(x^{4n} + x^{2n} + 1) &= (x^{2n} + x^n + 1)(x^{2n} - x^n + 1) = (x^{2n} + 1)(x^{2n} + \frac{1}{x^{2n} + 1}) \\
(x^{4n} + x^{2n} + 1) &= (x^{3n} - 1)(x^n + \frac{1}{x^n - 1}) = (x^{3n} + 1)(x^n + \frac{1}{x^n + 1})
\end{aligned}$$

$$
\begin{aligned}
(x^{4n}-x^{2n}+1) &= (x^{2n}-1)(x^{2n}+\frac{1}{x^{2n}-1}) \\
(x^n+\frac{1}{x^n-1}) &= \frac{(x^{2n}-x^n+1)}{(x^n-1)} \\
(x^n+\frac{1}{x^n+1}) &= \frac{(x^{2n}+x^n+1)}{(x^n+1)}
\end{aligned}
$$

6 Euler product and L-function

$(n \geqq 2) \qquad (n = 2,3,4,\cdots)$

$$\zeta(n) = \prod_{p;prime}^{\infty} \frac{p^n}{p^n-1}$$

$$L(n) = \prod_{\substack{p;odd\\ prime\\ (p\equiv 1 \bmod 4)}}^{\infty} \frac{p^n}{p^n-1} \times \prod_{\substack{p;odd\\ prime\\ (p\equiv 3 \bmod 4)}}^{\infty} \frac{p^n}{p^n+1}$$

6.1 Special value zeta-function

$(n \geqq 2) \quad n \equiv 1 \pmod 2$

$$\zeta(n) = \prod_{\substack{p;odd\\ prime\\ (\bmod 4)}}^{\infty} \left[\frac{p^{\overset{(p)}{n}}}{(p^n \underset{(p\equiv 1)}{-} 1)(p^n \underset{(p\equiv 3)}{+} 1)} \times \frac{p^n \overset{(p\equiv 3)}{+} 1}{p^n \underset{(p\equiv 3)}{-} 1} \right] \times \frac{2^n}{2^n-1}$$

$(n \geqq 1) \qquad (n = 1,2,3,\cdots)$

$$\zeta(3n) = \prod_{p;prime}^{\infty} \left[\frac{p^{2n}}{(p^{2n}-1)} \times \frac{p^n}{(p^n+\frac{1}{p^n+1})} \right]$$

6.2 Special value L-function

$(n \geqq 1) \qquad (n = 1,2,3,\cdots)$

$$L(2n) = \prod_{\substack{p;odd\\ prime\\ (\bmod 4)}}^{\infty} \left[\frac{p^{\overset{(p)}{2\,n}}}{(p^{2n} \underset{(p)}{-} 1)} \times \frac{p^{2n} \overset{(p\equiv 3)}{-} 1}{p^{2n} \underset{(p\equiv 3)}{+} 1} \right]$$

6.3 Other

$(n \geqq 2) \quad n \equiv 1 \pmod 2$

$$\prod_{\substack{p;odd\\ prime\\ (\bmod 4)}}^{\infty} \left[\frac{p^{\overset{(p)}{2\,n}}}{(p^n \underset{(p)}{-} 1)(p^n \underset{(p)}{+} 1)} \times \frac{(p^n \overset{(p\equiv 1)}{-} 1)(p^n \overset{(p\equiv 3)}{+} 1)}{p^n_{(p)}} = \frac{p^{\overset{(p)}{n}}}{(p^n \underset{(p\equiv 3)}{-} 1)(p^n \underset{(p\equiv 1)}{+} 1)} \right]$$

$$\prod_{\substack{p;odd\\ prime\\ (\bmod 4)}}^{\infty}\left[\frac{p^{\overset{(p)}{n}}}{(p^n \underset{(p\equiv1)}{-} 1)(p^n \underset{(p\equiv3)}{+} 1)} = \frac{p^{\overset{(p)}{n}}}{(p^n \underset{(p)}{-} 1)} \times \frac{p^n \overset{(p\equiv3)}{-} 1}{p^n \underset{(p\equiv3)}{+} 1}\right]$$

$$\prod_{\substack{p;odd\\ prime\\ (\bmod 4)}}^{\infty}\left[\frac{(p^n \overset{(p\equiv1)}{-} 1)(p^n \overset{(p\equiv3)}{+} 1)}{p^n_{(p)}} \times \frac{p^{\overset{(p)}{n}}}{(p^n \underset{(p\equiv3)}{-} 1)(p^n \underset{(p\equiv3)}{+} 1)} = \frac{(p^n \overset{(p\equiv1)}{-} 1)(p^n \overset{(p\equiv3)}{+} 1)}{(p^n \underset{(p\equiv1)}{+} 1)(p^n \underset{(p\equiv3)}{-} 1)}\right]$$

$$\prod_{\substack{p;odd\\ prime\\ (\bmod 4)}}^{\infty}\left[\frac{p^{\overset{(p)}{n}}}{(p^n \underset{(p\equiv1)}{-} 1)(p^n \underset{(p\equiv3)}{+} 1)} + \frac{p^{\overset{(p)}{n}}}{(p^n \underset{(p\equiv3)}{-} 1)(p^n \underset{(p\equiv1)}{+} 1)} = \right]$$

$$\prod_{\substack{p;odd\\ prime\\ (\bmod 4)}}^{\infty}\left[= \frac{p^{\overset{(p)}{n}}}{(p^n \underset{(p)}{-} 1)} \times \left(\frac{p^n \overset{(p\equiv1)}{-} 1}{p^n \underset{(p\equiv1)}{+} 1} + \frac{p^n \overset{(p\equiv3)}{-} 1}{p^n \underset{(p\equiv3)}{+} 1}\right) \right]$$

$(n \geqq 1) \qquad (n = 1, 2, 3, \cdots)$

$$\prod_{\substack{p;odd\\ prime}}^{\infty}\left[\frac{p^{6n}}{(p^{6n}-1)} = \frac{p^{6n}}{(p^{3n}-1)(p^{3n}+1)} = \frac{p^{6n}}{(p^{2n}-1)(p^{4n}+p^{2n}+1)}\right]$$

$$\prod_{\substack{p;odd\\ prime}}^{\infty}\left[\frac{p^{6n}}{(p^{2n}-1)(p^{4n}+p^{2n}+1)} \times \frac{(p^{2n}-1)}{p^{2n}} = \frac{p^{4n}}{(p^{4n}+p^{2n}+1)}\right]$$

$$\prod_{\substack{p;odd\\ prime}}^{\infty}\left[\frac{p^{4n}}{(p^{4n}+p^{2n}+1)} \times \frac{(p^{2n}-1)}{p^{2n}} = \frac{p^{2n}(p^{2n}-1)}{(p^{4n}+p^{2n}+1)} = \frac{p^{2n}}{(p^n+\frac{1}{p^n+1})(p^n+\frac{1}{p^n-1})}\right]$$

$$\prod_{\substack{p;odd\\ prime}}^{\infty}\left[\frac{p^{6n}}{(p^{6n}+1)} = \frac{p^{6n}}{(p^{2n}+1)(p^{4n}-p^{2n}+1)}\right]$$

7 special value of Riemann zeta function(n) and L-function(n)

$$\prod_{\substack{p;odd\\ prime}}^{\infty}\left[\frac{p^4}{(p^2-1)(p^2+1)} \times \frac{(p^2-1)}{p^2} = \frac{p^2}{(p^2+1)}\right] = \frac{\pi^4}{96} \times \frac{8}{\pi^2} = \frac{\pi^2}{12}$$

$$\prod_{\substack{p;odd\\ prime}}^{\infty}\left[\frac{p^2}{(p^2-1)} \times \frac{(p^2+1)}{p^2} = \frac{(p^2+1)}{(p^2-1)}\right] = \frac{\pi^2}{8} \times \frac{12}{\pi^2} = \frac{3}{2}$$

$$\prod_{\substack{p;odd\\prime\\(\bmod 4)}}^{\infty}\left[\frac{p^{\overset{(p)}{3}}}{(p^3\underset{(p\equiv1)}{-}1)(p^3\underset{(p\equiv3)}{+}1)}+\frac{p^{\overset{(p)}{3}}}{(p^3\underset{(p\equiv3)}{-}1)(p^3\underset{(p\equiv1)}{+}1)}=\right]$$

$$\prod_{\substack{p;odd\\prime\\(\bmod 4)}}^{\infty}\left[=\frac{p^{\overset{(p)}{3}}}{(p^3\underset{(p)}{-}1)}\times\left(\frac{p^3\overset{(p\equiv1)}{-}1}{p^3\underset{(p\equiv1)}{+}1}+\frac{p^3\overset{(p\equiv3)}{-}1}{p^3\underset{(p\equiv3)}{+}1}\right)\right]=\frac{\pi^3}{32}+\frac{\pi^3}{30}=\frac{31\pi^3}{480}$$

$$\prod_{\substack{p;odd\\prime\\(\bmod 4)}}^{\infty}\left[\frac{(p^3\overset{(p)}{-}1)(p^3\overset{(p)}{+}1)}{p^6_{(p)}}\times\frac{p^{\overset{(p)}{3}}}{(p^3\underset{(p)}{-}1)}\times\left(\frac{p^3\overset{(p\equiv1)}{-}1}{p^3\underset{(p\equiv1)}{+}1}+\frac{p^3\overset{(p\equiv3)}{-}1}{p^3\underset{(p\equiv3)}{+}1}\right)=\right]$$

$$\prod_{\substack{p;odd\\prime\\(\bmod 4)}}^{\infty}\left[=\frac{(p^3\overset{(p)}{+}1)}{p^3_{(p)}}\times\left(\frac{p^3\overset{(p\equiv1)}{-}1}{p^3\underset{(p\equiv1)}{+}1}+\frac{p^3\overset{(p\equiv3)}{-}1}{p^3\underset{(p\equiv3)}{+}1}\right)\right]=\frac{960}{\pi^6}\times\frac{31\pi^3}{480}=\frac{62}{\pi^3}$$

$$\prod_{\substack{p;odd\\prime\\(\bmod 4)}}^{\infty}\left[\frac{(p^3\overset{(p)}{+}1)}{(p^3\underset{(p)}{-}1)}\times\left(\frac{p^3\overset{(p\equiv1)}{-}1}{p^3\underset{(p\equiv1)}{+}1}+\frac{p^3\overset{(p\equiv3)}{-}1}{p^3\underset{(p\equiv3)}{+}1}\right)^2\right]=\frac{31\pi^3}{480}\times\frac{62}{\pi^3}=\frac{961}{240}$$

$$\prod_{\substack{p;odd\\prime}}^{\infty}\left[\frac{p^6}{(p^2-1)(p^4+p^2+1)}\times\frac{(p^2-1)}{p^2}=\frac{p^4}{(p^4+p^2+1)}\right]=\frac{\pi^6}{960}\times\frac{8}{\pi^2}=\frac{\pi^4}{120}$$

$$\prod_{\substack{p;odd\\prime}}^{\infty}\left[\frac{p^4}{(p^4+p^2+1)}=\frac{p^4}{(p^3-1)(p+\frac{1}{p-1})}=\frac{p^4}{(p^3+1)(p+\frac{1}{p+1})}\right]$$

$$\prod_{\substack{p;odd\\prime\\(\bmod 4)}}^{\infty}\left[\frac{(p\overset{(p)}{+}\frac{1}{p-1})}{(p\underset{(p)}{+}\frac{1}{p+1})}\times\left(\frac{p^3\overset{(p\equiv1)}{-}1}{p^3\underset{(p\equiv1)}{+}1}+\frac{p^3\overset{(p\equiv3)}{-}1}{p^3\underset{(p\equiv3)}{+}1}\right)^2\right]=\frac{31}{4\pi}\times\frac{31\pi}{60}=\frac{31^2}{240}$$

None of $\left(\frac{3}{2},\frac{961}{240},\frac{31^2}{240}\right)$ contains π.

From this, I think that $\prod_{\substack{p;odd\\prime\\(\bmod 4)}}^{\infty}\left[\frac{p^n\overset{(p\equiv1)}{-}1}{p^n\underset{(p\equiv1)}{+}1}\right]$ and $\prod_{\substack{p;odd\\prime\\(\bmod 4)}}^{\infty}\left[\frac{p^n\overset{(p\equiv3)}{-}1}{p^n\underset{(p\equiv3)}{+}1}\right]$ do not contain π.

$(n \geqq 3) \quad \{n = 3, 4, 5, \ldots\} \qquad n \equiv 1 \pmod 2$

$$\zeta(n) = \prod_{\substack{p;odd \\ prime \\ (\bmod 4)}}^{\infty} \left[\frac{p^{\overset{(p)}{n}}}{(p^n \underset{(p\equiv1)}{-} 1)(p^n \underset{(p\equiv3)}{+} 1)} \times \frac{p^n \overset{(p\equiv3)}{+} 1}{p^n \underset{(p\equiv3)}{-} 1} \right] \times \frac{2^n}{2^n - 1}$$

$(n \geqq 1) \qquad (n = 1, 2, 3, \cdots)$

$$L(2n) = \prod_{\substack{p;odd \\ prime \\ (\bmod 4)}}^{\infty} \left[\frac{p^{\overset{(p)}{2\,n}}}{(p^{2n} \underset{(p)}{-} 1)} \times \frac{p^{2n} \overset{(p\equiv3)}{-} 1}{p^{2n} \underset{(p\equiv3)}{+} 1} \right]$$

From this, I think that special value of zeta function(n) and L-function(n) contain π.

References

[1] https://translate.google.com google translation

[2] N.Kurokawa 『The quest for Riemann-from ABC to Z-』
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[3] N.Kurokawa 『Let's solve the Riemann conjecture
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