Exact Solution of All Real Bessel LHODE Formula

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Abstract

The Bessel linear homogeneous ordinary differential equation (LHODE) with real parameter may be solved exactly using my "Vector Space Transformation Technique" similarly to the usage to solve the "Exact Solution of All Half-Integer Bessel LHODE Formula" - using "Exact solution of ODEs - Vector Space Transformation Technique - Part 2", Theorem I.1.

Corollary I.5: The real-parameter Bessel ODE solutions may be written:

If $Y_i'' + \frac{1}{x} Y_i' + \left[-\frac{\lambda^2}{x^2} - 1 \right] Y_i = 0$;

then:

$$\Rightarrow \left\{ \begin{array}{l} \boxed{\begin{array}{l} Y_{1k} = v_k J_{\frac{1}{2}} + u_k J_{-\frac{1}{2}} \\ Y_{2k} = u_k J_{\frac{1}{2}} - v_k J_{-\frac{1}{2}} \end{array}} \\ \\ u_k = \sum_{j=0}^{\frac{k}{2}} a_{k(2j)} x^{-2j} + \sum_{j=0}^{\frac{k}{2}} a_{k(2j+1)} x^{-(2j+1)} \\ v_k = \sum_{k=0}^{\frac{k}{2}} b_{m(2k)} x^{-2k} + \sum_{j=0}^{\frac{k}{2}} b_{k(2j+1)} x^{-(2j+1)} \\ \\ where: \\ \left\{ \begin{array}{l} b_{k0} = \frac{2}{k((k \cdot \delta) + 1)} a_{k1} \\ b_{k1} = -\frac{k((k \cdot \delta) + 1)}{2} a_{k0} \quad , \quad (0 \le n \le k) \ , \ \left(\delta = \frac{1}{k} \left(-\frac{1}{2} \pm \lambda \right) > 0 \right) \ , \ k \in \mathbb{N} \end{array} \right. \\ \left\{ \begin{array}{ll} a_{k(2h)} = \left(-\frac{1}{4} \right)^h \frac{\left(\frac{(k + (2h-1))!}{-(k+1)!} \right) \prod_{j=0}^{2h-1} ((k+j) \cdot \delta + 1)}{(2h)!} a_{k0} & , \quad (0 \le 2h \le k) \\ a_{k(1+2h)} = \left(-\frac{1}{4} \right)^h \frac{\left(\frac{(1 - k + (2h-1))!}{-k!} \right) \prod_{j=0}^{2h-1} ((k+j) \cdot \delta + 1)}{(1+2h)!} a_{k1} & , \quad (1 \le 2h \le k-1) \\ b_{k(2h)} = \left(-\frac{1}{4} \right)^h \frac{2 \left(\frac{(1 - k + (2h-1))!}{-k!} \right) \prod_{j=0}^{2h-1} ((k+j) \cdot \delta + 1)}{(2h-k)(2h+k+1)(2h)!} a_{k1} & , \quad (0 \le 2h \le k-1) \\ b_{k(2h-1)} = -\left(-\frac{1}{4} \right)^h \frac{2 \left(\frac{(k + (2h-1))!}{-(k+1)!} \right) \prod_{j=0}^{2h-1} ((k+j) \cdot \delta + 1)}{(2h-1-k)(2h+k)(2h-1)!} a_{k0} & , \quad (1 \le 2h \le k) \end{array} \right. \end{array} \right\}$$

Proof:

As with Corollary I.1:

Given the half-integer Bessel ODE, from the main theorem:

$y_1 = J_{\frac{1}{2}} = \sqrt{\frac{2}{\pi}} x^{-\frac{1}{2}} \sin x$	$y_2 = J_{-\frac{1}{2}} = \sqrt{\frac{2}{\pi}} x^{-\frac{1}{2}} \cos x$
$y_1' = J_{\frac{1}{2}}' = -\frac{1}{2} x^{-1} y_1 + y_2$	$y_2' = J_{-\frac{1}{2}}' = -y_1 - \frac{1}{2} x^{-1} y_2$

$$\Rightarrow y_i'' + \frac{1}{x} y_i' + \left(-\frac{\left(\pm \frac{1}{2} \right)^2}{x^2} - 1^2 \right) y_i = 0$$

$r_1 = -\frac{1}{2} x^{-1}$	$s_1 = 1$
$r_2 = -1$	$s_2 = -\frac{1}{2} x^{-1}$

$(r_2 + s_1) = 0$
$(r_1 - s_2) = 0$

$0 = \varphi_1 - \varphi_2$
$0 = \varphi_1 - \varphi_2$
$0 = u'' + \varphi_1 u + 2v'$
$0 = v'' + \varphi_1 v - 2u'$
$0 = 0$
$0 = 0$

So:

$$-\frac{\left(\frac{1}{2} \right)^2}{x^2} + \varphi_1 = -\frac{\lambda^2}{x^2} = -\frac{\lambda^2}{x^2} \Rightarrow \varphi_1 = \frac{1}{x^2} \left(-\lambda^2 + \frac{1}{4} \right) = -\frac{\left(\lambda - \frac{1}{2} \right) \left(\lambda + \frac{1}{2} \right)}{x^2}$$

In the half-integer case, the integers are ued to determine the half-integer $\lambda's$ rooted at $\frac{1}{2}$.

For the real case, the $\lambda's$ are used to determine the partition of λ .

Either: $\lambda : \varphi_1 = -\frac{(\lambda - \frac{1}{2})(\lambda + \frac{1}{2})}{x^2}$ or: $m = \lambda - \frac{1}{2} \Rightarrow \varphi_1 = -\frac{m(m+1)}{x^2}$

may be used (each, again rooted at $\frac{1}{2}$). The later is chosen for cinsistency with the half-integer case.

$$-2v_m' = u_m'' - \frac{m(m+1)}{x^2}u_m$$

$$\Rightarrow u_{mh}'' - \frac{m(m+1)}{x^2}u_{mh} = 0$$

Note that: $m = \lambda - \frac{1}{2} \Leftrightarrow m+1 = \lambda + \frac{1}{2} \Rightarrow m(m+1) = (\lambda - \frac{1}{2})(\lambda + \frac{1}{2}) = \lambda^2 - \frac{1}{4}$

Let: $u_m = \sum_{n=0}^{m} a_{mn}x^{-(n\cdot\delta)}$, $v_m = \sum_{n=0}^{m} b_{mn}x^{-(n\cdot\delta)}$:

where the partition of λ is given by:

$$\delta = \frac{m}{k} \ , \ k \in \mathbb{N}$$

$$\Rightarrow m(m+1) = \lambda^2 - \tfrac{1}{4} \Rightarrow m^2 + m - (\lambda^2 - \tfrac{1}{4}) = 0$$

$$\Rightarrow m = \tfrac{1}{2}\left(-1 \pm \sqrt{1 + 4(\lambda^2 - \tfrac{1}{4})}\right) = \tfrac{1}{2}\left(-1 \pm \sqrt{1 + 4\lambda^2 - 1}\right) = \tfrac{1}{2}(-1 \pm 2\lambda)$$

$$\Rightarrow m = -\tfrac{1}{2} \pm \lambda \Rightarrow \delta = \frac{1}{k}(-\tfrac{1}{2} \pm \lambda)$$

so, for: $\delta > 0$:

$$\lambda > \tfrac{1}{2} : \delta = \frac{1}{k}(-\tfrac{1}{2} + \lambda) = \frac{1}{k}(\lambda - \tfrac{1}{2})$$

$$\lambda < \tfrac{1}{2} :: \delta = \frac{1}{k}(-\tfrac{1}{2} - \lambda) = -\frac{1}{k}(\lambda + \tfrac{1}{2})$$

Note:

examples: $\lambda - \frac{1}{2} = e = 2.718281828 -.5 = 2.2182818$

$2.2182818/1 = 2.2182818 > 1$
$2.2182818/2 = 1.10914098 > 1$
$2.2182818/3 = 0.73942727 < 1 \Rightarrow k = 3$
$2.2182818/4 = 0.55457045 < 1 \Rightarrow k = 4$
$\vdots$

$\lambda - \frac{1}{2} = \pi = 3.14159265358979 -.5 = 2.64159265358979$
$2.64159265358979/1 = 2.64159265358979 > 1$
$2.64159265358979/2 = 1.32079632679489 > 1$
$2.64159265358979/3 = 0.88053088452993 < 1 \Rightarrow k = 3$
$2.64159265358979/4 = 0.66039816339744 < 1 \Rightarrow k = 4$
$\vdots$

Thus, the solutions of the real Bessel HLODE are "quantized" by whole number k for each λ .
(So, aparently, the Froebenius series solution is either the solution for the minimum k for each λ ; or the solution for $\lim_{k\to\infty} k$ for each λ .)
So, ODE’s transforming to or from the Bessel ODEs will have this same characteristic.
And, use of series solutions, given-point(s)-value problem solutions, and Green’s function solutions may produce limited results.

In order to obtain a solution of the form consistent with the half-integer solutions; for each half-integer solution choose u_m & v_m as inhomogeneous functions:

$$\left\{\begin{array}{|l|} \hline 0 = u_m'' + \varphi_1 u_m + 2v_m' \\ \hline 0 = v_m'' + \varphi_1 v_m - 2u_m' \\ \hline \end{array}\right\} \Rightarrow \left\{\begin{array}{|l|} \hline u_m'' + \varphi_1 u_m = -2v_m' \\ \hline v_m'' + \varphi_1 v_m = 2u_m' \\ \hline \end{array}\right.$$

$$\Rightarrow \left\{\begin{array}{|l|} \hline u_m'' - \frac{m(m+1)}{x^2}u_m = -2v_m' \\ \hline v_m'' - \frac{m(m+1)}{x^2}v_m = 2u_m' \\ \hline \end{array}\right.$$

Let: $u_m = \sum_{n=0}^{k} a_{mn}x^{-(n\cdot\delta)}$, $v_m = \sum_{n=0}^{k} b_{mn}x^{-(n\cdot\delta)}$:

$$\Rightarrow \begin{cases} \left(\sum_{n=0}^{k} a_{mn}x^{-(n\cdot\delta)}\right)'' - \frac{m(m+1)}{x^2}\left(\sum_{n=0}^{k} a_{mn}x^{-(n\cdot\delta)}\right) = -2\left(\sum_{n=0}^{k} b_{mn}x^{-(n\cdot\delta)}\right)' \\ \left(\sum_{n=0}^{k} b_{mn}x^{-(n\cdot\delta)}\right)'' - \frac{m(m+1)}{x^2}\left(\sum_{n=0}^{k} b_{mn}x^{-(n\cdot\delta)}\right) = 2\left(\sum_{n=0}^{k} a_{mn}x^{-(n\cdot\delta)}\right)' \end{cases}$$

$$\Rightarrow \begin{cases} \left(\sum_{n=0}^{k}(-(n\bullet\delta))a_{mn}x^{-(n\cdot\delta)-1}\right)' - m(m+1)\left(\sum_{n=0}^{k} a_{mn}x^{-(n\cdot\delta)-2}\right) = -2\left(\sum_{n=0}^{k}(-(n\bullet\delta))b_{mn}x^{-(n\cdot\delta)-1}\right) \\ \left(\sum_{n=0}^{k}(-(n\bullet\delta))b_{mn}x^{-(n\cdot\delta)-1}\right)' - m(m+1)\left(\sum_{n=0}^{k} b_{mn}x^{-(n\cdot\delta)-2}\right) = 2\left(\sum_{n=0}^{k}(-(n\bullet\delta))a_{mn}x^{-(n\cdot\delta)-1}\right) \end{cases}$$

$$\Rightarrow \begin{cases} \sum_{n=0}^{k}(-(n\bullet\delta)-1)a_{mn}x^{-(n\cdot\delta)-2} - m(m+1)\sum_{n=0}^{k} a_{mn}x^{-(n\cdot\delta)-2} = -2\sum_{n=0}^{k}(-(n\bullet\delta))b_{mn}x^{-(n\cdot\delta)-1} \\ \sum_{n=0}^{k}(-(n\bullet\delta))(-(n\bullet\delta)-1)b_{mn}x^{-(n\cdot\delta)-2} - m(m+1)\sum_{n=0}^{k} b_{mn}x^{-(n\cdot\delta)-2} = 2\sum_{n=0}^{k}(-(n\bullet\delta))a_{mn}x^{-(n\cdot\delta)-1} \end{cases}$$

$$\Rightarrow \begin{cases} \sum_{n=0}^{k}[(-(n\bullet\delta))(-(n\bullet\delta)-1) - m(m+1)]a_{mn}x^{-(n\cdot\delta)-2} = -2\sum_{n=0}^{k}(-(n\bullet\delta))b_{mn}x^{-(n\cdot\delta)-1} \\ \sum_{n=0}^{k}[(-(n\bullet\delta))(-(n\bullet\delta)-1) - m(m+1)]b_{mn}x^{-(n\cdot\delta)-2} = 2\sum_{n=0}^{k}(-(n\bullet\delta))a_{mn}x^{-(n\cdot\delta)-1} \end{cases}$$

$$\Rightarrow \begin{cases} \sum_{n=0}^{k}[(-(n\cdot\delta))(-(n\cdot\delta)-1)-m(m+1)]a_{mn}x^{-(n\cdot\delta)-2} = 2\left(\sum_{n=1}^{k}(-(n\cdot\delta))b_{mn}x^{-(n\cdot\delta)-1} + 0\cdot b_{m0}x^{-0-1}\right) \\ \sum_{n=0}^{k}[(-(n\cdot\delta))(-(n\cdot\delta)-1)-m(m+1)]b_{mn}x^{-(n\cdot\delta)-2} = -2\left(\sum_{n=1}^{k}(-(n\cdot\delta))a_{mn}x^{-(n\cdot\delta)-1} + 0\cdot a_{m0}x^{-0-1}\right) \end{cases}$$

$$\Rightarrow \begin{cases} \sum_{n=0}^{k}[(-(n\cdot\delta))(-(n\cdot\delta)-1)-m(m+1)]a_{mn}x^{-(n\cdot\delta)-2} = 2\left(\sum_{j=0}^{k-1}(-((j+1)\cdot\delta))b_{m(j+1)}x^{-(j\cdot\delta)-2} + 0\cdot b_{m0}x^{-0-1}\right) \\ \sum_{n=0}^{k}[(-(n\cdot\delta))(-(n\cdot\delta)-1)-m(m+1)]b_{mn}x^{-(n\cdot\delta)n-2} = -2\left(\sum_{j=0}^{k-1}(-((j+1)\cdot\delta))a_{m(j+1)}x^{-(j\cdot\delta)-2} + 0\cdot a_{m0}x^{-0-1}\right) \end{cases}$$

$$\Rightarrow \begin{cases} \sum_{n=0}^{k}[(-(n\cdot\delta))(-(n\cdot\delta)-1)-m(m+1)]a_{mn}x^{-(n\cdot\delta)-2} = 2\sum_{n=0}^{k}(-((n+1)\cdot\delta))b_{m(n+1)}x^{-(n\cdot\delta)-2} \\ \sum_{n=0}^{k}[(-(n\cdot\delta))(-(n\cdot\delta)-1)-m(m+1)]b_{mn}x^{-(n\cdot\delta)-2} = -2\sum_{n=0}^{k}(-((n+1)\cdot\delta))a_{m(n+1)}x^{-(n\cdot\delta)-2} \end{cases}$$

$$\Rightarrow \begin{cases} \sum_{n=0}^{k-1}[(-(n\cdot\delta))(-(n\cdot\delta)-1)-m(m+1)]a_{mn}x^{-(n\cdot\delta)-2} + [(-(k\cdot\delta))(-(k\cdot\delta)-1)-m(m+1)]a_{mm}x^{-m-2} = 2\sum_{n=0}^{k-1}(-((n+1)\cdot\delta))b_{m(n+1)}x \\ \sum_{n=0}^{k-1}[(-(n\cdot\delta))(-(n\cdot\delta)-1)-m(m+1)]b_{mn}x^{-(n\cdot\delta)-2} + [(-(k\cdot\delta))(-(k\cdot\delta)-1)-m(m+1)]b_{mm}x^{-m-2} = -2\sum_{n=0}^{m-1}(-((n+1)\cdot\delta))a_{m(n+1} \end{cases}$$

$$\Rightarrow \begin{cases} \sum_{n=0}^{k-1}[(-(n\cdot\delta))(-(n\cdot\delta)-1)-m(m+1)]a_{mn}x^{-(n\cdot\delta)-2} + [m(m+1)-m(m+1)]a_{mm}x^{-m-2} = 2\sum_{n=0}^{k-1}(-((n+1)\cdot\delta))b_{m(n+1)}x^{-(n\cdot\delta)-2} \\ \sum_{n=0}^{k-1}[(-(n\cdot\delta))(-(n\cdot\delta)-1)-m(m+1)]b_{mn}x^{-(n\cdot\delta)-2} + [m(m+1)-m(m+1)]b_{mm}x^{-m-2} = -2\sum_{n=0}^{m-1}(-((n+1)\cdot\delta))a_{m(n+1)}x^{-n-2} \end{cases}$$

$$\Rightarrow \begin{cases} \sum_{n=0}^{k-1}[(-(n\cdot\delta))(-(n\cdot\delta)-1)-m(m+1)]a_{mn}x^{-(n\cdot\delta)-2} = 2\sum_{n=0}^{k-1}(-((n+1)\cdot\delta))b_{m(n+1)}x^{-(n\cdot\delta)-2} \\ \sum_{n=0}^{k-1}[(-(n\cdot\delta))(-(n\cdot\delta)-1)-m(m+1)]b_{mn}x^{-(n\cdot\delta)-2} = -2\sum_{n=0}^{k-1}(-((n+1)\cdot\delta))a_{m(n+1)}x^{-(n\cdot\delta)-2} \end{cases}$$

$$\Rightarrow \begin{cases} \sum_{n=0}^{k-1}([(-(n\cdot\delta))(-(n\cdot\delta)-1)-m(m+1)]a_{mn} - 2(-((n+1)\cdot\delta))b_{m(n+1)})x^{-(n\cdot\delta)-2} = 0 \\ \sum_{n=0}^{k-1}([(-(n\cdot\delta))(-(n\cdot\delta)-1)-m(m+1)]b_{mn} + 2(-((n+1)\cdot\delta))a_{m(n+1)})x^{-(n\cdot\delta)-2} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} [(-(n\cdot\delta))(-(n\cdot\delta)-1)-m(m+1)]a_{mn} - 2(-((n+1)\cdot\delta))b_{m(n+1)} = 0 \\ [(-(n\cdot\delta))(-(n\cdot\delta)-1)-m(m+1)]b_{mn} + 2(-((n+1)\cdot\delta))a_{m(n+1)} = 0 \end{cases} \quad (0 \le n \le k-1)$$

$$\Rightarrow \begin{cases} a_{mn} = \dfrac{2((n+1)\cdot\delta)}{((n\cdot\delta)-m)((n\cdot\delta)+m+1)}b_{m(n+1)} \\ b_{mn} = -\dfrac{2((n+1)\cdot\delta)}{((n\cdot\delta)-m)((n\cdot\delta)+m+1)}a_{m(n+1)} \end{cases} , \left(\begin{matrix} (0 \le n\cdot\delta \le m-\delta) \;,\; m = -\tfrac{1}{2} \pm \lambda > 0 \\ \left(\delta = \tfrac{1}{k}\left(-\tfrac{1}{2} \pm \lambda\right) \;,\; k \in \mathbb{N}\right) \end{matrix} \right)$$

The first data point of the partitions of u_m & v_m is at $k = 0 \Rightarrow k\cdot\delta = 0$.
The second data point of the partition is at $k = 1 \Rightarrow k\cdot\delta = \delta$.
Each successive data point of the partition is at $k = n \Rightarrow k\cdot\delta = n\cdot\delta$.
The final data point of the partition is at $k = k \Rightarrow k\cdot\delta = m = -\frac{1}{2} \pm \lambda > 0$.
If λ is a positive half-integer, $m = -\frac{1}{2} \pm \lambda > 0 \Rightarrow \lambda \ge \frac{3}{2} \Rightarrow m \ge 1$ and the smallest $\delta > 0$ is 1;
so m begins at the minimal $\delta = 1$, and each m is separated by a minimal $\delta = 1$;
i.e.: each integral m has a corresponding positive half-integer λ.
Likewise, if λ is a positive real number, $k \in \mathbb{N} : k\cdot\delta = m = -\frac{1}{2} \pm \lambda > 0 \Rightarrow \lambda > \frac{1}{2} \Rightarrow m > 0$
so m begins at the given δ, and each m is separated by the given δ.
m is the selector for λ. The m/λ parameters partition the Bessel ODE and it's solutions.

$$\Rightarrow \begin{cases} a_{m0} = -\dfrac{2\delta}{-m(m+1)}b_{m1} = \delta\left(-\dfrac{2}{-m(m+1)}b_{m1}\right) \\ b_{m0} = \dfrac{2\delta}{m(m+1)}a_{m1} = \delta\left(\dfrac{2}{m(m+1)}a_{m1}\right) \end{cases}$$

Note the similarity to the half-integer case. In fact, a generalization of the half-integer case.

$k = 1 \Rightarrow m = k\cdot\delta = \delta = -\frac{1}{2} \pm \lambda$:

$$\Rightarrow \begin{cases} a_{\delta 0} = \dfrac{2((n+1)\cdot\delta)}{((n\cdot\delta)-m)((n\cdot\delta)+m+1)}b_{\delta(n+1)} \\ b_{\delta 0} = -\dfrac{2((n+1)\cdot\delta)}{((n\cdot\delta)-m)((n\cdot\delta)+m+1)}a_{\delta(n+1)} \end{cases} \quad (0 \le n \le 1-1) \Leftrightarrow (0 \le n \le 0)$$

$$\Rightarrow \begin{cases} a_{\delta 0} = -\dfrac{2\delta}{m(m+1)}b_{\delta 1} = \delta\left(-\dfrac{2}{\delta(\delta+1)}b_{\delta 1}\right) = -\dfrac{2}{\delta+1}b_{\delta 1} \\ b_{\delta 0} = \dfrac{2\delta}{m(m+1)}a_{\delta 1} = \delta\left(\dfrac{2}{\delta(\delta+1)}a_{\delta 1}\right) = \dfrac{2}{\delta+1}a_{\delta 1} \end{cases}$$

$$\Rightarrow \left\{ \begin{matrix} a_{\delta 0} = -\dfrac{2}{-\delta+1}b_{\delta 1} \\ b_{\delta 0} = \dfrac{2}{-\delta+1}a_{\delta 1} \\ ,\ \delta = -\tfrac{1}{2} \pm \lambda > 0 \end{matrix} \right\}$$

$$\Rightarrow \left\{ \begin{array}{l} u_1 = a_{\delta 0}x^{-0} + a_{\delta 1}x^{-\delta} \\ v_1 = b_{\delta 0}x^{-0} + b_{\delta 1}x^{-\delta} = \frac{\delta+1}{2}a_{\delta 1}x^{-0} - \frac{\delta+1}{2}a_{\delta 0}x^{-\delta} \end{array} \right\}$$

NOTE: instead of using the index δ the index k will be used:

(since $m = k \bullet \delta$ and the final constants are arbitrary, so without loss of generality)

$$\Rightarrow \left\{ \begin{array}{l} Y_{11} = v_1 J_{\frac{1}{2}} + u_1 J_{-\frac{1}{2}} = \frac{\delta+1}{2}(a_{21}x^{-0} - a_{20}x^{-\delta})J_{\frac{1}{2}} + (a_{20}x^{-0} + a_{21}x^{-\delta})J_{-\frac{1}{2}} \\ Y_{21} = u_1 J_{\frac{1}{2}} - v_1 J_{-\frac{1}{2}} = (a_{20}x^{-0} + a_{21}x^{-\delta})J_{\frac{1}{2}} - \frac{\delta+1}{2}(a_{21}x^{-0} - a_{20}x^{-\delta})J_{-\frac{1}{2}} \end{array} \right.$$

$$\Rightarrow \left\{ \begin{array}{l} Y_{11} = \frac{\frac{1}{2}\pm\lambda}{2}\left(a_{21} - a_{20}x^{-\left(-\frac{1}{2}\pm\lambda\right)}\right)J_{\frac{1}{2}} + \left(a_{20} + a_{21}x^{-\left(-\frac{1}{2}\pm\lambda\right)}\right)J_{-\frac{1}{2}} \\ Y_{21} = \left(a_{20} + a_{21}x^{-\left(-\frac{1}{2}\pm\lambda\right)}\right)J_{\frac{1}{2}} - \frac{\frac{1}{2}\pm\lambda}{2}\left(a_{21} - a_{20}x^{-\left(-\frac{1}{2}\pm\lambda\right)}\right)J_{-\frac{1}{2}} \end{array} \right.$$

$$\Rightarrow \left\{ \begin{array}{l} Y_{11} = \left(\frac{\frac{1}{2}\pm\lambda}{2}a_{21}J_{\frac{1}{2}} + a_{20}J_{-\frac{1}{2}}\right) - \left(\frac{\frac{1}{2}\pm\lambda}{2}a_{20}J_{\frac{1}{2}} - a_{21}J_{-\frac{1}{2}}\right)x^{-\left(-\frac{1}{2}\pm\lambda\right)} \\ Y_{21} = \left(a_{20}J_{\frac{1}{2}} - \frac{\frac{1}{2}\pm\lambda}{2}a_{21}J_{-\frac{1}{2}}\right) + \left(a_{21}J_{\frac{1}{2}} + \frac{\frac{1}{2}\pm\lambda}{2}a_{20}J_{-\frac{1}{2}}\right)x^{-\left(-\frac{1}{2}\pm\lambda\right)} \end{array} \right.$$

$$\Rightarrow \left\{ \begin{array}{l} Y_{11} = a_{21}\left(\frac{\frac{1}{2}\pm\lambda}{2}J_{\frac{1}{2}} + x^{-\left(-\frac{1}{2}\pm\lambda\right)}J_{-\frac{1}{2}}\right) + a_{20}\left(J_{-\frac{1}{2}} - x^{-\left(-\frac{1}{2}\pm\lambda\right)}\frac{\frac{1}{2}\pm\lambda}{2}J_{\frac{1}{2}}\right) \\ Y_{21} = a_{20}\left(J_{\frac{1}{2}} + x^{-\left(-\frac{1}{2}\pm\lambda\right)}\frac{\frac{1}{2}\pm\lambda}{2}J_{-\frac{1}{2}}\right) - a_{21}\left(\frac{\frac{1}{2}\pm\lambda}{2}J_{-\frac{1}{2}} - x^{-\left(-\frac{1}{2}\pm\lambda\right)}J_{\frac{1}{2}}\right) \end{array} \right.$$

$k = 2 \Rightarrow m = k \bullet \delta = 2\delta = 2\left(-\frac{1}{2} \pm \lambda\right) = -1 \pm 2\lambda$:

$$\Rightarrow \left\{ \begin{array}{ll} a_{k0} = \frac{2((0+1)\bullet\delta)}{((0\bullet\delta)-m)((0\bullet\delta)+m+1)}b_{k(0+1)} & b_{k0} = -\frac{2((0+1)\bullet\delta)}{((0\bullet\delta)-m)((0\bullet\delta)+m+1)}a_{k(0+1)} \\ \quad = -\frac{2\delta}{m(m+1)}b_{k1} & \quad = \frac{2\delta}{m(m+1)}a_{k1} \\ \quad = -\frac{2\delta}{2\delta(2\delta+1)}b_{k1} = -\frac{1}{2\delta+1}b_{k1} & \quad = \frac{2\delta}{2\delta(2\delta+1)}a_{m1} = \frac{1}{2\delta+1}a_{k1} \\ a_{k1} = \frac{2((1+1)\bullet\delta)}{((1\bullet\delta)-m)((1\bullet\delta)+m+1)}b_{k(1+1)} & b_{k1} = -\frac{2((1+1)\bullet\delta)}{((1\bullet\delta)-m)((1\bullet\delta)+m+1)}a_{k(1+1)} \\ \quad = \frac{4\delta}{(\delta-m)(\delta+m+1)}b_{k2} & \quad = -\frac{4\delta}{(\delta-m)(\delta+m+1)}a_{k2} \\ \quad = -\frac{4\delta}{(2\delta-\delta)(2\delta+\delta+1)}b_{k2} & \quad = \frac{4\delta}{(2\delta-\delta)(2\delta+\delta+1)}a_{k2} \\ \quad = -\frac{4}{(3\delta+1)}b_{k2} & \quad = \frac{4}{(3\delta+1)}a_{k2} \end{array} \right.$$

$$, \left(\begin{array}{l} (0 \le n\bullet\delta \le m) \ , \ m = -\frac{1}{2} \pm \lambda > 0 \\ \left(\delta = \frac{1}{2}\left(-\frac{1}{2}\pm\lambda\right)\right) \ , \ n \le k-1 = 2-1 = 1 \end{array} \right)$$

$$\Rightarrow \left\{ \begin{array}{l} a_{k0} = -\frac{1}{2\delta+1}b_{k1} \Rightarrow b_{k1} = -(2\delta+1)a_{k0} \\ b_{k0} = \frac{1}{2\delta+1}a_{k1} \\ b_{k1} = \frac{4}{3\delta+1}a_{k2} \Rightarrow a_{k2} = \frac{3\delta+1}{4}b_{k1} = -\frac{(2\delta+1)(3\delta+1)}{4}a_{k0} \\ b_{k2} = -\frac{3\delta+1}{4}a_{k1} \\ \qquad , \ \delta = \frac{1}{2}\left(-\frac{1}{2}\pm\lambda\right) > 0 \end{array} \right.$$

$$\Rightarrow \left\{ \begin{array}{l} u_2 = \sum_{n=0}^{2} a_{2n}x^{-(n\bullet\delta)} = a_{20}x^{-0} + a_{21}x^{-\delta} + a_{22}x^{-2\delta} = a_{20}x^{-0} + a_{21}x^{-\delta} - \frac{(2\delta+1)(3\delta+1)}{4}a_{20}x^{-2\delta} \\ v_2 = \sum_{n=0}^{2} b_{2n}x^{-(n\bullet\delta)} = b_{20}x^{-0} + b_{21}x^{-\delta} + b_{22}x^{-2\delta} = \frac{1}{2\delta+1}a_{21}x^{-0} - (2\delta+1)a_{20}x^{-\delta} - \frac{3\delta+1}{4}a_{21}x^{-2\delta} \end{array} \right.$$

$$\Rightarrow \left\{ \begin{array}{l} u_2 = \sum_{n=0}^{2} a_{2n}x^{-(n\bullet\delta)} = a_{20}\left(x^{-0} - \frac{(2\delta+1)(3\delta+1)}{4}x^{-2\delta}\right) + a_{21}x^{-\delta} \\ v_2 = \sum_{n=0}^{2} b_{2n}x^{-(n\bullet\delta)} = b_{20}x^{-0} + b_{21}x^{-\delta} + b_{22}x^{-2\delta} = \frac{1}{2\delta+1}a_{21}\left(x^{-0} - \frac{(2\delta+1)(3\delta+1)}{4}x^{-2\delta}\right) - (2\delta+1)a_{20}x^{-\delta} \end{array} \right.$$

$$\Rightarrow \left\{ \begin{array}{l} Y_{12} = v_2 J_{\frac{1}{2}} + u_2 J_{-\frac{1}{2}} = \left[\frac{1}{2\delta+1}a_{21}\left(x^{-0} - \frac{(2\delta+1)(3\delta+1)}{4}x^{-2\delta}\right) - (2\delta+1)a_{20}x^{-\delta}\right]J_{\frac{1}{2}} + \\ \qquad + \left[a_{20}\left(x^{-0} - \frac{(2\delta+1)(3\delta+1)}{4}x^{-2\delta}\right) + a_{21}x^{-\delta}\right]J_{-\frac{1}{2}} \\ Y_{22} = u_2 J_{\frac{1}{2}} - v_2 J_{-\frac{1}{2}} = \left[a_{20}\left(x^{-0} - \frac{(2\delta+1)(3\delta+1)}{4}x^{-2\delta}\right) + a_{21}x^{-\delta}\right]J_{\frac{1}{2}} + \\ \qquad - \left[\frac{1}{2\delta+1}a_{21}\left(x^{-0} - \frac{(2\delta+1)(3\delta+1)}{4}x^{-2\delta}\right) - (2\delta+1)a_{20}x^{-\delta}\right]J_{-\frac{1}{2}} \end{array} \right.$$

$$
\Rightarrow \begin{cases}
Y_{12} = v_2 J_{\frac{1}{2}} + u_2 J_{-\frac{1}{2}} = \left[\frac{1}{\frac{1}{2} \pm \lambda} a_{21} \left(x^{-0} - \frac{\left(\frac{1}{2} \pm \lambda\right)\left(\frac{1}{4} \pm \frac{3}{2}\lambda\right)}{4} x^{-\left(-\frac{1}{2} \pm \lambda\right)} \right) - \left(\frac{1}{2} \pm \lambda\right) a_{20} x^{-\frac{1}{2}\left(-\frac{1}{2} \pm \lambda\right)} \right] J_{\frac{1}{2}} + \\
\qquad + \left[a_{20} \left(x^{-0} - \frac{\left(\frac{1}{2} \pm \lambda\right)\left(\frac{1}{4} \pm \frac{3}{2}\lambda\right)}{4} x^{-\left(-\frac{1}{2} \pm \lambda\right)} \right) + a_{21} x^{-\frac{1}{2}\left(-\frac{1}{2} \pm \lambda\right)} \right] J_{-\frac{1}{2}} \\
Y_{22} = u_2 J_{\frac{1}{2}} - v_2 J_{-\frac{1}{2}} = \left[a_{20} \left(x^{-0} - \frac{\left(\frac{1}{2} \pm \lambda\right)\left(\frac{1}{4} \pm \frac{3}{2}\lambda\right)}{4} x^{-\left(-\frac{1}{2} \pm \lambda\right)} \right) + a_{21} x^{-\frac{1}{2}\left(-\frac{1}{2} \pm \lambda\right)} \right] J_{\frac{1}{2}} + \\
\qquad - \left[\frac{1}{2\delta + 1} a_{21} \left(x^{-0} - \frac{\left(\frac{1}{2} \pm \lambda\right)\left(\frac{1}{4} \pm \frac{3}{2}\lambda\right)}{4} x^{-\left(-\frac{1}{2} \pm \lambda\right)} \right) - (2\delta + 1) a_{20} x^{-\frac{1}{2}\left(-\frac{1}{2} \pm \lambda\right)} \right] J_{-\frac{1}{2}}
\end{cases}
$$

$k = 3 \Rightarrow m = 3 \cdot \delta = 3\delta = 3\left(-\frac{1}{2} \pm \lambda\right) = -\frac{3}{2} \pm 3\lambda$:

$$
\Rightarrow \begin{cases}
a_{3n} = \frac{2((n+1) \cdot \delta)}{((n \cdot \delta) - m)((n \cdot \delta) + m + 1)} b_{3(n+1)} \\
b_{3n} = -\frac{2((n+1) \cdot \delta)}{((n \cdot \delta) - m)((n \cdot \delta) + m + 1)} a_{3(n+1)}
\end{cases}
, \quad \begin{pmatrix} (0 \le 3 \cdot \delta = m) \\ \left(\delta = \frac{1}{3}\left(-\frac{1}{2} \pm \lambda\right)\right) \end{pmatrix}
$$

$$
\Rightarrow \begin{cases}
a_{30} = \frac{2((0+1) \cdot \delta)}{((0 \cdot \delta) - m)((0 \cdot \delta) + m + 1)} b_{3(0+1)} & b_{30} = -\frac{2((0+1) \cdot \delta)}{((0 \cdot \delta) - m)((0 \cdot \delta) + m + 1)} a_{3(0+1)} \\
\quad = -\frac{2\delta}{m(m+1)} b_{31} & \quad = \frac{2\delta}{m(m+1)} a_{31} \\
\quad = -\frac{2\delta}{3\delta(3\delta+1)} b_{31} = -\frac{2}{3(3\delta+1)} b_{31} & \quad = \frac{2\delta}{3\delta(3\delta+1)} a_{m1} = \frac{2}{3(3\delta+1)} a_{31} \\
a_{31} = \frac{2((1+1) \cdot \delta)}{((1 \cdot \delta) - m)((1 \cdot \delta) + m + 1)} b_{3(1+1)} & b_{31} = -\frac{2((1+1) \cdot \delta)}{((1 \cdot \delta) - m)((1 \cdot \delta) + m + 1)} a_{3(1+1)} \\
\quad = \frac{4\delta}{(\delta - 3\delta)(\delta + 3\delta + 1)} b_{32} & \quad = -\frac{4\delta}{(\delta - 3\delta)(\delta + 3\delta + 1)} a_{32} \\
\quad = -\frac{4\delta}{(-2\delta)(4\delta + 1)} b_{32} & \quad = \frac{4\delta}{(-2\delta)(4\delta + 1)} a_{32} \\
\quad = \frac{2}{4\delta + 1} b_{32} & \quad = -\frac{2}{4\delta + 1} a_{32} \\
a_{32} = \frac{2((2+1) \cdot \delta)}{((2 \cdot \delta) - m)((2 \cdot \delta) + m + 1)} b_{3(2+1)} & b_{32} = -\frac{2((2+1) \cdot \delta)}{((2 \cdot \delta) - m)((2 \cdot \delta) + m + 1)} a_{3(2+1)} \\
\quad = \frac{6\delta}{(2\delta - 3\delta)((2 \cdot \delta) + 3\delta + 1)} b_{33} & \quad = -\frac{6\delta}{(2\delta - 3\delta)((2 \cdot \delta) + 3\delta + 1)} b_{33} \\
\quad = -\frac{6}{(5\delta + 1)} b_{33} & \quad = \frac{6}{(5\delta + 1)} b_{33}
\end{cases}
$$

$$
\Rightarrow \begin{cases}
b_{31} = -\frac{3(3\delta+1)}{2} a_{30} & b_{30} = -\frac{2}{3(3\delta+1)} a_{31} & \\
b_{32} = \frac{4\delta+1}{2} a_{31} & b_{31} = -\frac{2}{4\delta+1} a_{32} & \Rightarrow a_{32} = -\frac{3(3\delta+1)(4\delta+1)}{4} a_{30} \\
b_{33} = -\frac{(5\delta+1)}{6} a_{32} & b_{32} = \frac{6}{(5\delta+1)} a_{33} & \Rightarrow a_{33} = \frac{(4\delta+1)(5\delta+1)}{12} a_{31} \\
b_{33} = \frac{(3\delta+1)(4\delta+1)(5\delta+1)}{8} a_{30} & & \delta = \frac{1}{3}\left(-\frac{1}{2} \pm \lambda\right) > 0
\end{cases}
$$

$$
\Rightarrow \begin{cases}
u_3 = \sum_{n=0}^{3} a_{3n} x^{-(n \cdot \delta)} = a_{30} x^{-0} + a_{31} x^{-\delta} + a_{32} x^{-2\delta} + a_{33} x^{-3\delta} = \\
\quad = a_{30} x^{-0} + a_{31} x^{-\delta} - \frac{3(3\delta+1)(4\delta+1)}{4} a_{30} x^{-2\delta} + \frac{(4\delta+1)(5\delta+1)}{12} a_{31} x^{-3\delta} \\
\quad = a_{30} \left(x^{-0} - \frac{3(3\delta+1)(4\delta+1)}{4} x^{-\frac{1}{3}(-1 \pm 2\lambda)} \right) + a_{31} \left(x^{-\frac{1}{3}\left(-\frac{1}{2} \pm \lambda\right)} + \frac{(4\delta+1)(5\delta+1)}{12} x^{-\left(-\frac{1}{2} \pm \lambda\right)} \right) \\
\quad = a_{30} \left(x^{-0} - \frac{3(3\delta+1)(4\delta+1)}{4} x^{-\frac{1}{3}(-1 \pm 2\lambda)} \right) + a_{31} \left(x^{-\frac{1}{3}\left(-\frac{1}{2} \pm \lambda\right)} + \frac{(4\delta+1)(5\delta+1)}{12} x^{-\left(-\frac{1}{2} \pm \lambda\right)} \right) \\
v_3 = \sum_{n=0}^{3} b_{3n} x^{-(n \cdot \delta)} = b_{30} x^{-0} + b_{31} x^{-\delta} + b_{32} x^{-2\delta} + b_{33} x^{-3\delta} = \\
\quad = a_{30} \left(-\frac{3(3\delta+1)}{2} x^{-\delta} + \frac{(3\delta+1)(4\delta+1)(5\delta+1)}{8} x^{-3\delta} \right) - a_{31} \left(\frac{2}{3(3\delta+1)} x^{-0} - \frac{4\delta+1}{2} x^{-2\delta} \right) \\
\quad = a_{30} \left(-\frac{3(3\delta+1)}{2} x^{-\delta} + \frac{(3\delta+1)(4\delta+1)(5\delta+1)}{8} x^{-3\delta} \right) - a_{31} \left(\frac{2}{3(3\delta+1)} x^{-0} - \frac{4\delta+1}{2} x^{-2\delta} \right) \\
\qquad , \quad \delta = \frac{1}{3}\left(-\frac{1}{2} \pm \lambda\right) > 0
\end{cases}
$$

$$
\Rightarrow \begin{cases}
Y_{13} = v_3 J_{\frac{1}{2}} + u_3 J_{-\frac{1}{2}} \\
Y_{23} = u_3 J_{\frac{1}{2}} - v_3 J_{-\frac{1}{2}}
\end{cases}
$$

Continuing:

$$
\Rightarrow \begin{cases}
a_{kn} = \frac{2((n+1) \cdot \delta)}{((n \cdot \delta) - m)((n \cdot \delta) + m + 1)} b_{k(n+1)} \\
b_{kn} = -\frac{2((n+1) \cdot \delta)}{((n \cdot \delta) - m)((n \cdot \delta) + m + 1)} a_{k(n+1)}
\end{cases}
, \quad \begin{pmatrix} (0 \le n \le k) \ , \ m = k \cdot \delta = -\frac{1}{2} \pm \lambda \\ \left(\delta = \frac{1}{k}\left(-\frac{1}{2} \pm \lambda\right) > 0 \ , \ k \in \mathbb{N}\right) \end{pmatrix}
$$

$$\Rightarrow \begin{cases} a_{kn} = \dfrac{2((n+1)\bullet\delta)}{((n\bullet\delta)-k\bullet\delta)((n\bullet\delta)+k\bullet\delta+1)} b_{k(n+1)} \\ b_{kn} = -\dfrac{2((n+1)\bullet\delta)}{((n\bullet\delta)-k\bullet\delta)((n\bullet\delta)+k\bullet\delta+1)} a_{k(n+1)} \end{cases} \quad , \left(\begin{matrix} (0 \le n \le k) \ , \ k \in \mathbb{N} \\ \left(\delta = \frac{1}{k}\left(-\frac{1}{2} \pm \lambda\right) > 0\right) \end{matrix} \right)$$

$$\Rightarrow \begin{cases} a_{kn} = \dfrac{2(n+1)}{(n-k)((n\bullet\delta)+k\bullet\delta+1)} b_{k(n+1)} \\ b_{kn} = -\dfrac{2(n+1)}{(n-k)((n+k)\bullet\delta+1)} a_{k(n+1)} \end{cases} \quad , \left(\begin{matrix} (0 \le n \le k) \ , \ k \in \mathbb{N} \\ \left(\delta = \frac{1}{k}\left(-\frac{1}{2} \pm \lambda\right) > 0\right) \end{matrix} \right)$$

$$\Rightarrow \begin{cases} a_{k0} = -\dfrac{2}{k((k\bullet\delta)+1)} b_{k1} \\ b_{k0} = \dfrac{2}{k((k\bullet\delta)+1)} a_{k1} \end{cases}$$

$$\Rightarrow \begin{cases} a_{k(n-1)} = \dfrac{2((n-1)+1)}{((n-1)-k)(((n-1)\bullet\delta)+k\bullet\delta+1)} b_{kn} \\ \qquad (0 \le n-1 \le k-2) \\ b_{kn} = -\dfrac{2(n+1)}{(n-k)((n+k)\bullet\delta+1)} a_{k(n+1)} \\ \qquad (0 \le n \le k-1) \end{cases}$$

$$\Rightarrow \begin{cases} a_{k(n-1)} = -\dfrac{4n(n+1)}{(n-k-1)(n-k)(((n+k-1)\bullet\delta)+1)((n+k)\bullet\delta+1)} a_{k(n+1)} \\ \qquad (1 \le n \le k-1) \\ b_{kn} = -\dfrac{2(n+1)}{(n-k)((n+k)\bullet\delta+1)} a_{k(n+1)} \\ \qquad (0 \le n \le k-1) \end{cases}$$

$$\Rightarrow \begin{cases} a_{k((n+1)-1)} = -\dfrac{4(n+1)((n+1)+1)}{((n+1)-k-1)((n+1)-k)((((n+1)+k-1)\bullet\delta)+1)(((n+1)+k)\bullet\delta+1)} a_{k((n+1)+1)} \\ \qquad (1 \le n \le k-1) \\ b_{kn} = -\dfrac{2(n+1)}{(n-k)((n+k)\bullet\delta+1)} a_{k(n+1)} \\ \qquad (0 \le n \le k-1) \end{cases}$$

$$\Rightarrow \begin{cases} a_{kn} = -\dfrac{4(n+1)(n+2)}{(n-k)(n-k+1)((n+k)\bullet\delta+1)((n+k+1)\bullet\delta+1)} a_{k(n+2)} \\ \qquad (1 \le n \le k-1) \\ b_{kn} = -\dfrac{2(n+1)}{(n-k)((n+k)\bullet\delta+1)} a_{k(n+1)} \\ \qquad (0 \le n \le k-1) \end{cases}$$

$$\Rightarrow \begin{cases} a_{k(n+2)} = -\dfrac{(n-k)(n-k+1)((n+k)\bullet\delta+1)((n+k+1)\bullet\delta+1)}{4(n+1)(n+2)} a_{kn} \\ \qquad (1 \le n \le k-1) \\ b_{kn} = -\dfrac{2(n+1)}{(n-k)((n+k)\bullet\delta+1)} a_{k(n+1)} \\ \qquad (0 \le n \le k-1) \end{cases}$$

$$\Rightarrow \begin{cases} a_{kn} = -\dfrac{((n-1)-k-1)((n-1)-k)((((n-1)+k-1)\bullet\delta)+1)(((n-1)+k)\bullet\delta+1)}{4(n-1)((n-1)+1)} a_{k(n-2)} \\ \qquad (1 \le n \le k-1) \\ b_{kn} = -\dfrac{2(n+1)}{(n-k)((n+k)\bullet\delta+1)} a_{k(n+1)} \\ \qquad (0 \le n \le k-1) \end{cases}$$

$$\Rightarrow \begin{cases} a_{kn} = -\dfrac{(n-k-2)(n-k-1)(((n+k-2)\bullet\delta)+1)((n+k-1)\bullet\delta+1)}{4(n-1)n} a_{k(n-2)} \\ \qquad (1 \le n \le k-1) \\ b_{kn} = -\dfrac{2(n+1)}{(n-k)((n+k)\bullet\delta+1)} a_{k(n+1)} \\ \qquad (0 \le n \le k-1) \end{cases}$$

Let:

$$W_{kn} \equiv \frac{(n-k)(n-k+1)((n+k)\bullet\delta+1)((n+k+1)\bullet\delta+1)}{4(n+1)(n+2)} \quad , \quad (0 \le n \le k-2)$$

$$\Rightarrow a_{k(n+2)} = (-1)^1 W_{kn} a_{kn}$$

$$\Rightarrow a_{k(n+4)} = (-1)^1 W_{k(n+2)} a_{k(n+2)} \quad , \quad (0 \le n \le k-4)$$

$$= (-1)^1 W_{k(n+2)} W_{kn} a_{kn} \quad , \quad (0 \le n \le k-4)$$

$$= (-1)^1 \frac{((n+2)-k)((n+2)-k+1)(((n+2)+k)\bullet\delta+1)(((n+2)+k+1)\bullet\delta+1)}{4((n+2)+1)((n+2)+2)} a_{k(n+2)} \quad , \quad (0 \le n \le k-4)$$

$$= (-1)^1 \frac{(n-k+2)(n-k+3)((n+k+2)\bullet\delta+1)((n+k+3)\bullet\delta+1)}{4(n+3)(n+4)} a_{k(n+2)} \quad , \quad (0 \le n \le k-4)$$

$$= (-1)^1 \frac{(n-k+2)(n-k+3)((n+k+2)\bullet\delta+1)((n+k+3)\bullet\delta+1)}{4(n+3)(n+4)} (-1)^1 \frac{(n-k)(n-k+1)((n+k)\bullet\delta+1)((n+k+1)\bullet\delta+1)}{4(n+1)(n+2)}$$

$$= (-1)^2 \frac{(n-k)(n-k+1)(n-k+2)(n-k+3)((n+k)\bullet\delta+1)((n+k+1)\bullet\delta+1)((n+k+2)\bullet\delta+1)((n+k+3)\bullet\delta+1)}{4^2(n+1)(n+2)(n+3)(n+4)} a_{kn}$$

$$= \left(-\frac{1}{4}\right)^2 \frac{\left(\frac{(n-k+(4-1))!}{(n-k-1)!}\right)\prod_{j=0}^{4-1}((n+k+j)\bullet\delta+1)}{\left(\frac{(n+4)!}{n!}\right)} a_{kn} \quad , \quad (0 \le n \le k-4)$$

$$\Rightarrow a_{k(n+6)} = -W_{k(n+4)}a_{k(n+4)} = (-1)^3 W_{k(n+4)}W_{k(n+2)}W_{kn}a_{kn} \quad , \quad (0 \le n \le m-6)$$

$$= \left(-\frac{1}{4}\right)^3 \frac{\left(\frac{(n-k+(6-1))!}{(n-k-1)!}\right)\prod_{j=0}^{6-1}((n+k+j)\bullet\delta+1)}{\left(\frac{(n+6)!}{n!}\right)} a_{mn} \quad , \quad (0 \le n \le k-6)$$

$$\vdots$$

$$\Rightarrow a_{k(n+2h)} = -W_{k(n+2(h-1))}a_{k(n+2(h-1))} = (-1)^k W_{k(n+2(h-1))}\cdots W_{kn}a_{kn} \quad , \quad (0 \le n \le k-2h)$$

$$= \left(-\frac{1}{4}\right)^h \frac{\left(\frac{(n-k+(2h-1))!}{(n-k-1)!}\right)\prod_{j=0}^{2h-1}((n+k+j)\bullet\delta+1)}{\left(\frac{(n+2h)!}{n!}\right)} a_{mn} \quad , \quad (0 \le n \le k-2h)$$

So:

$$a_{k(0+2h)} = \left(-\frac{1}{4}\right)^h \frac{\left(\frac{(0-k+(2h-1))!}{(0-k-1)!}\right)\prod_{j=0}^{2h-1}((0+k+j)\bullet\delta+1)}{\left(\frac{(0+2h)!}{0!}\right)} a_{k0} \quad , \quad (0 \le k-2h)$$

$$= \left(-\frac{1}{4}\right)^h \frac{\left(\frac{(k+(2h-1))!}{-(k+1)!}\right)\prod_{j=0}^{2h-1}((k+j)\bullet\delta+1)}{\left(\frac{(2h)!}{0!}\right)} a_{k0} \quad , \quad (2h \le k)$$

$$\Rightarrow a_{k(2h)} = \left(-\frac{1}{4}\right)^h \frac{\left(\frac{(k+(2h-1))!}{-(k+1)!}\right)\prod_{j=0}^{2h-1}((k+j)\bullet\delta+1)}{(2h)!} a_{k0} \quad , \quad (2h \le k)$$

$$a_{k(1+2h)} = \left(-\frac{1}{4}\right)^h \frac{\left(\frac{(1-k+(2h-1))!}{(1-k-1)!}\right)\prod_{j=0}^{2h-1}((k+j)\bullet\delta+1)}{\left(\frac{(1+2h)!}{1!}\right)} a_{k1} \quad , \quad (1+2h \le k)$$

$$\Rightarrow a_{k(1+2h)} = \left(-\frac{1}{4}\right)^h \frac{\left(\frac{(1-k+(2h-1))!}{-k!}\right)\prod_{j=0}^{2h-1}((k+j)\bullet\delta+1)}{(1+2h)!} a_{k1} \quad , \quad (1+2h \le k)$$

$$\Rightarrow b_{k(2h)} = \frac{2(2h+1)}{(2h-k)(2h+k+1)} a_{k(2h+1)} \quad , \quad (0 \le 2h \le k-1)$$

$$\Rightarrow b_{k(2h)} = \frac{2(2h+1)}{(2h-k)(2h+k+1)}\left(-\frac{1}{4}\right)^h \frac{\left(\frac{(1-k+(2h-1))!}{-k!}\right)\prod_{j=0}^{2h-1}((k+j)\bullet\delta+1)}{(1+2h)!} a_{k1} \quad , \quad (0 \le 2h \le k-1)$$

$$\Rightarrow b_{k(2h)} = \left(-\frac{1}{4}\right)^h \frac{2\left(\frac{(1-k+(2h-1))!}{-k!}\right)\prod_{j=0}^{2h-1}((k+j)\bullet\delta+1)}{(2h-k)(2h+k+1)(2h)!} a_{k1} \quad , \quad (0 \le 2h \le k-1)$$

$$\Rightarrow b_{k(2h-1)} = -\frac{4h}{(2h-1-k)(2h+k)} a_{k(2h)} \quad , \quad (1 \le 2h \le k)$$

$$\Rightarrow b_{k(2h-1)} = -\frac{4h}{(2h-1-k)(2h+k)}\left(-\frac{1}{4}\right)^h \frac{\left(\frac{(k+(2h-1))!}{-(k+1)!}\right)\prod_{j=0}^{2h-1}((k+j)\bullet\delta+1)}{(2h)!} a_{k0} \quad , \quad (2h \le k)$$

$$\Rightarrow b_{k(2h-1)} = -\left(-\frac{1}{4}\right)^h \frac{2\left(\frac{(k+(2h-1))!}{-(k+1)!}\right)\prod_{j=0}^{2h-1}((k+j)\bullet\delta+1)}{(2h-1-k)(2h+k)(2h-1)!} a_{k0} \quad , \quad (2h \le k)$$

$$
\Rightarrow \left\{ \begin{array}{l}
\left\{ \begin{array}{l}
b_{k0} = \dfrac{2}{k((k \cdot \delta) + 1)} a_{k1} \\
b_{k1} = -\dfrac{k((k \cdot \delta) + 1)}{2} a_{k0} \quad , \quad (0 \le n \le k) \;,\; \left(\delta = \dfrac{1}{k}\left(-\tfrac{1}{2} \pm \lambda\right) > 0\right) \;,\; k \in \mathbb{N}
\end{array} \right. \\
\left\{ \begin{array}{ll}
a_{k(2h)} = \left(-\dfrac{1}{4}\right)^h \dfrac{\left(\dfrac{(k+(2h-1))!}{-(k+1)!}\right) \prod_{j=0}^{2h-1} ((k+j) \cdot \delta + 1)}{(2h)!} a_{k0} & , \quad (0 \le 2h \le k) \\
a_{k(1+2h)} = \left(-\dfrac{1}{4}\right)^h \dfrac{\left(\dfrac{(1-k+(2h-1))!}{-k!}\right) \prod_{j=0}^{2h-1} ((k+j) \cdot \delta + 1)}{(1+2h)!} a_{k1} & , \quad (1 \le 2h \le k-1) \\
b_{k(2h)} = \left(-\dfrac{1}{4}\right)^h \dfrac{2\left(\dfrac{(1-k+(2h-1))!}{-k!}\right) \prod_{j=0}^{2h-1} ((k+j) \cdot \delta + 1)}{(2h-k)(2h+k+1)(2h)!} a_{k1} & , \quad (0 \le 2h \le k-1) \\
b_{k(2h-1)} = -\left(-\dfrac{1}{4}\right)^h \dfrac{2\left(\dfrac{(k+(2h-1))!}{-(k+1)!}\right) \prod_{j=0}^{2h-1} ((k+j) \cdot \delta + 1)}{(2h-1-k)(2h+k)(2h-1)!} a_{k0} & , \quad (1 \le 2h \le k)
\end{array} \right.
\end{array} \right\}
$$

So, the real-parameter Bessel ODE solutions may be written:

$$
\Rightarrow \left\{ \begin{array}{l}
\begin{array}{|l|}
\hline
Y_{1k} = v_k J_{\frac{1}{2}} + u_k J_{-\frac{1}{2}} \\
\hline
Y_{2k} = u_k J_{\frac{1}{2}} - v_k J_{-\frac{1}{2}} \\
\hline
\end{array} \\
u_k = \sum_{j=0}^{\frac{k}{2}} a_{k(2j)} x^{-2j} + \sum_{j=0}^{\frac{k}{2}} a_{k(2j+1)} x^{-(2j+1)} \\
v_k = \sum_{k=0}^{\frac{k}{2}} b_{m(2k)} x^{-2k} + \sum_{j=0}^{\frac{k}{2}} b_{k(2j+1)} x^{-(2j+1)} \\
\textit{where}: \\
\left\{ \begin{array}{l}
b_{k0} = \dfrac{2}{k((k \cdot \delta) + 1)} a_{k1} \\
b_{k1} = -\dfrac{k((k \cdot \delta) + 1)}{2} a_{k0} \quad , \quad (0 \le n \le k) \;,\; \left(\delta = \dfrac{1}{k}\left(-\tfrac{1}{2} \pm \lambda\right) > 0\right) \;,\; k \in \mathbb{N}
\end{array} \right. \\
\left\{ \begin{array}{ll}
a_{k(2h)} = \left(-\dfrac{1}{4}\right)^h \dfrac{\left(\dfrac{(k+(2h-1))!}{-(k+1)!}\right) \prod_{j=0}^{2h-1} ((k+j) \cdot \delta + 1)}{(2h)!} a_{k0} & , \quad (0 \le 2h \le k) \\
a_{k(1+2h)} = \left(-\dfrac{1}{4}\right)^h \dfrac{\left(\dfrac{(1-k+(2h-1))!}{-k!}\right) \prod_{j=0}^{2h-1} ((k+j) \cdot \delta + 1)}{(1+2h)!} a_{k1} & , \quad (1 \le 2h \le k-1) \\
b_{k(2h)} = \left(-\dfrac{1}{4}\right)^h \dfrac{2\left(\dfrac{(1-k+(2h-1))!}{-k!}\right) \prod_{j=0}^{2h-1} ((k+j) \cdot \delta + 1)}{(2h-k)(2h+k+1)(2h)!} a_{k1} & , \quad (0 \le 2h \le k-1) \\
b_{k(2h-1)} = -\left(-\dfrac{1}{4}\right)^h \dfrac{2\left(\dfrac{(k+(2h-1))!}{-(k+1)!}\right) \prod_{j=0}^{2h-1} ((k+j) \cdot \delta + 1)}{(2h-1-k)(2h+k)(2h-1)!} a_{k0} & , \quad (1 \le 2h \le k)
\end{array} \right.
\end{array} \right\}
$$

□

The simplest way to transform

$$Y_i''(z) + \frac{1}{z} Y_i'(z) + \left[-\frac{\lambda^2}{z^2} - 1\right] Y_i(z) = 0$$

into the general Bessel HLODE is as follows:

Let: $z = vx$

$$\Rightarrow Y_i''(vx) + \frac{1}{vx} Y_i'(vx) + \left[-\frac{\lambda^2}{(vx)^2} - 1\right] Y_i(vx) = 0$$

$$\Rightarrow \frac{d^2}{dz^2} Y_i(vx) + \frac{1}{vx} \frac{d}{dz} Y_i'(vx) + \left[-\frac{\lambda^2}{(vx)^2} - 1\right] Y_i(vx) = 0$$

$$\Rightarrow \frac{d}{dz}\left(\frac{dx}{dz} \frac{d}{dx} Y_i(vx)\right) + \frac{1}{vx} \frac{dx}{dz} \frac{d}{dx} Y_i'(vx) + \left[-\frac{\lambda^2}{(vx)^2} - 1\right] Y_i(vx) = 0$$

$$\Rightarrow \frac{d}{dz}\left(\frac{1}{v} \frac{d}{dx} Y_i(vx)\right) + \frac{1}{vx} \frac{1}{v} \frac{d}{dx} Y_i'(vx) + \left[-\frac{\lambda^2}{(vx)^2} - 1\right] Y_i(vx) = 0$$

$$\Rightarrow \frac{dx}{dz}\left(\frac{d}{dx} \frac{1}{v} \frac{d}{dx} Y_i(vx)\right) + \frac{1}{vx} \frac{1}{v} \frac{d}{dx} Y_i'(vx) + \left[-\frac{\lambda^2}{(vx)^2} - 1\right] Y_i(vx) = 0$$

$$\Rightarrow \frac{1}{v}\left(\frac{d}{dx} \frac{1}{v} \frac{d}{dx} Y_i(vx)\right) + \frac{1}{vx} \frac{1}{v} \frac{d}{dx} Y_i'(vx) + \left[-\frac{\lambda^2}{(vx)^2} - 1\right] Y_i(vx) = 0$$

$$\Rightarrow \frac{1}{v^2}\left(\frac{d^2}{dx^2} Y_i(vx)\right) + \frac{1}{v^2} \frac{1}{x} \frac{d}{dx} Y_i'(vx) + \left[-\frac{\lambda^2}{(vx)^2} - 1\right] Y_i(vx) = 0$$

$$\Rightarrow \frac{d^2}{dx^2} Y_i(vx) + \frac{1}{x} \frac{d}{dx} Y_i'(vx) + \left[-\frac{\lambda^2}{x^2} - v^2\right] Y_i(vx) = 0$$

Thus, the above simplified Bessel HLODE is equivalent to transforming to the general Bessel HLODE.