ON THE PROXIMITY OF MULTIPLICATIVE FUNCTIONS TO THE FUNCTION $\Omega(n)$

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ABSTRACT. In this paper we examine how closely a multiplicative function resembles an additive function. We show that in fact, given any small $\epsilon > 0$,

$$E(\Omega, g; x) \gg \frac{x}{(\log\log x)^{\frac{1}{2}+\epsilon}}$$

for some choice of multiplicative function, where $\Omega(n) = \sum\limits_{p||n} 1$. This is therefore an extension of an earlier result of De Koninck, Doyon and Letendre [1].

1. Introduction

Let $f : \mathbb{N} \longrightarrow \mathbb{C}$. Then f is said to be additive if $f(mn) := f(m) + f(n)$, whenever $(m,n) = 1$. It is said to be strongly additive if $f(p^\alpha) := f(p)$ for all integers $\alpha > 0$ and for all primes p. Also let $g : \mathbb{N} \longrightarrow \mathbb{C}$. Then g is said to be multiplicative if $g(mn) = g(m)g(n)$ whenever $(m,n) = 1$. It is said to be strongly multiplicative if $g(p^\alpha) := g(p)$ for all integers $\alpha > 0$ and for all primes p. Let us set $E(f, g; x) := \#\{n \leq x : f(n) = g(n)\}$, where f and g are arbitrary multiplicative and additive functions, respectively. One of the basic and natural questions one can ever ask is how small and how large can this quantity be. This quantity has been studied extensively by De koninck, Doyon and Letendre (See [1]). In 2014 they showed that, given any $\epsilon > 0$, there exist a strongly multiplicative function g and some sequence (x_n) of positive integers such that

$$E(\omega, g, x_n) \gg \frac{x_n}{(\log\log x_n)^{\frac{1}{2}+\epsilon}}.$$

Above all, they showed that no additive function can agree with a multiplicative function on a set of positive density. That is, they showed that for f an integer-valued additive function such that

$$\varphi(x) = \varphi_f(x) = \frac{B(x)}{A(x)} \longrightarrow 0$$

as $x \longrightarrow \infty$, where

$$A(x) := \sum_{p^\alpha \leq x} f(p^\alpha)\left(1 - \frac{1}{p}\right) \quad \text{and} \quad B(x) := \sum_{p^\alpha \leq x} \frac{|f(p^\alpha)|^2}{p^\alpha},$$

and that

$$\max_{z \in \mathbb{R}} \#\{n \leq x : f(n) = z\} = O\left(\frac{x}{K(x)}\right),$$

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where $K(x) \longrightarrow \infty$ as $x \longrightarrow \infty$. Then, for any multiplicative function g

$$E(f,g;x)=o(x)$$

as $x \longrightarrow \infty$. In this paper we obtain the following uniform version of the result of De Koninck, Doyon and Letendre (see [1]) at the price of working on a thin subset of primes:

Theorem 1.1. *For any small $\epsilon>0$, there exists a strongly multiplicative function g such that*

$$E(\Omega,g,x)\gg \frac{x}{(\log\log x)^{\frac{1}{2}+\epsilon}}.$$

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2. Notations

Through out this paper a prime number will either be denoted by p or the subscripts of s. Any other letter will be clarified. The functions $\omega(n):=\sum\limits_{p|n}1$ and $\Omega(n):=\sum\limits_{p||n}1$ counts the number distinct and prime factors of n with multiplicity. The function $f:\mathbb{N}\longrightarrow\mathbb{C}$ and $g:\mathbb{N}\longrightarrow\mathbb{C}$ will also denote additive and multiplicative function, respectively. The inequality $|k(n)|\leq Mp(n)$ for sufficiently large values of n will be compactly written as $k(n)\ll p(n)$ or $k(n)=O(p(n))$. Similarly the inequality $|k(n)|\geq Mp(n)$ for sufficiently large values of n will be represented by $k(n)\gg p(n)$. The limit $\lim\limits_{n\longrightarrow\infty}\frac{k(n)}{p(n)}=0$ will be represented in a compact form as $k(n)=o(p(n))$ as $n\longrightarrow\infty$. The quantities ϵ and δ are positive numbers that can be taken arbitrarily small. Also in this paper we will be interested in the regime where x is a sufficiently large integer. That is to say, $x\geq N_0$ for some $N_0>0$.

3. Preliminary results

Lemma 3.1. *Let $\pi_k(x):=\#\{n\leq x:\omega(n)=k\}$ for each positive integer k. Then the maximum value of $\pi_k(x)$ is $(1+o(1))\frac{x}{\sqrt{\log\log x}}$ and the value of k for which it occurs is $k=\log\log x+O(1)$.*

Proof. This follows from a result of Balazard [2]. □

Lemma 3.2. *For any $\delta>0$,*

$$\#\{n\leq x:|\omega(n)-\log\log n|>(\log\log x)^{1+\delta}\}=o(x)$$

as $x\longrightarrow\infty$.

Proof. This follows from Theorem 8.12 in the book of Nathanson [3]. □

Theorem 3.3. *Let $\epsilon>0$. Then there exist a strongly multiplicative function g and a sequence (x_n) of positive integers such that*

$$E(\omega,g,x_n)\gg\frac{x_n}{(\log\log x_n)^{\frac{1}{2}+\epsilon}}.$$

Proof. This follows from a result of De koninck, Doyon and Letendre [1]. □

Remark 3.4. The above result also holds with $\omega(n)$ replaced by $\Omega(n)$. But it depends greatly on some choice of sequence of positive integers. Now, by modifying the techniques devised by De koninck, Doyon and Letendre [1] we obtain a result that holds uniformly on the set of positive integers.

4. **Main result**

Theorem 4.1. *For any small $\epsilon > 0$, there exists a strongly multiplicative function g such that*

$$E(\Omega, g, x) \gg \frac{x}{(\log\log x)^{\frac{1}{2}+\epsilon}}.$$

Remark 4.2. The above result is telling us that, we can find a multiplicative function g such that the points of coincidence with the additive function Ω is somewhat uniformly large. Nevertheless, this improvement comes with the compromise of working on a thin subset of primes. The proof below follows closely the techniques of De Koninck, Doyon and Letendre, with some slight modification.

Proof. Let $\mathcal{S} = \{s_1, s_2, \ldots\}$ be an infinite set of primes such that

$$\sum_{j=1}^{\infty} \frac{1}{s_j} < \infty.$$

We leverage the regime where x is a sufficiently large integer, that is for all $x \geq N_0$ for some $N_0 > 0$. Let (z_j) be sequence of positive integers maximizing the quantity

$$\#\left\{r \leq \frac{x}{s_j} : s_i \nmid r \quad \text{for each } s_i \in \mathcal{S},\ \Omega(r) = z_j - 2\right\},$$

for $s_j \equiv 1 \pmod 4$ for each $j \geq 1$, and

$$\#\left\{r \leq \frac{x}{s_j} : s_i \nmid r \quad \text{for each } s_i \in \mathcal{S},\ \Omega(r) = z_j\right\},$$

for $s_j \equiv 3 \pmod 4$ for each $j \geq 1$, which is well defined in light of Lemma 3.1. Define g, a strongly multiplicative function, on the primes as

$$g(p) = \begin{cases} z_j - 1 & \text{if } p = s_j \equiv 1 \pmod 4,\ s_j \in \mathcal{S} \\ z_j + 1 & \text{if } p = s_j \equiv 3 \pmod 4,\ s_j \in \mathcal{S} \\ 1 \quad \text{if} & p \notin \mathcal{S}. \end{cases}$$

To obtain a lower bound for $E(\Omega, g, x)$, it suffices to consider only integers of the form $n = r \cdot s_j^{\alpha}$ for $\alpha \geq 1$, $s_i \nmid r$ for all $i \geq 1$. Clearly

$$\begin{aligned} E(\Omega, g, x) &= \#\left\{n \leq x : \Omega(n) = g(n)\right\} \\ &\geq \sum_{\alpha \geq 1} \#\{n \leq x : \ s_j^{\alpha} | n, \quad s_i \nmid n \quad \text{for} \quad i \neq j, \quad \Omega(n) = g(n)\} \\ &\geq \sum_{\alpha \geq 1} \#\left\{r \leq \frac{x}{s_j^{\alpha}} : \Omega(r) = g(s_j) - \alpha,\ s_i \nmid r \text{ for each } s_i \in \mathcal{S}, \quad s_j \equiv 1 \pmod 4\right\} \\ &\geq \#\left\{r \leq \frac{x}{s_j} : \Omega(r) = g(s_j) - 1,\ s_i \nmid r \text{ for each } s_i \in \mathcal{S}, \quad s_j \equiv 1 \pmod 4\right\} \\ &\geq \#\left\{r \leq \frac{x}{s_j} : \Omega(r) = z_j - 2,\ s_i \nmid r \text{ for each } s_i \in \mathcal{S}\right\}. \end{aligned}$$

We only need to consider the interval

$$\mathcal{I} = [\log\log x - (\log\log x)^{\frac{1}{2}+\epsilon}, \log\log x + (\log\log x)^{\frac{1}{2}+\epsilon}],$$

since by Lemma 3.2, most values of $\omega(n)$ and $\Omega(n)$ fall within such interval, for any $\epsilon > 0$. Let us consider the quantity

$$\#\left\{r \leq \frac{x}{s_j} : s_i \nmid r \text{ for each } s_i \in \mathcal{S},\ s_j \equiv 1 \pmod{4}\right\}.$$

We observe, in relation to Lemma 3.2

$$\#\left\{r \leq \frac{x}{s_j} : s_i \nmid r \text{ for each } s_i \in \mathcal{S},\ s_j \equiv 1 \pmod{4},\ \Omega(r) \notin \mathcal{I}\right\} = o\bigg(\frac{x}{s_j}\bigg)$$

as $j \longrightarrow \infty$. On the other hand by letting

$$\mathcal{U} = \left\{r \leq \frac{x}{s_j} : s_i \nmid r \quad \text{for each} \quad s_i \in \mathcal{S},\ s_j \equiv 1 \pmod{4},\ \Omega(r) \in \mathcal{I}\right\},$$

then it follows that

$$\#\mathcal{U} = \sum_{l \in \mathcal{I}} \#\left\{r \leq \frac{x}{s_j} : s_i \nmid r \quad \text{for each} \quad s_i \in \mathcal{S},\ \Omega(r) = l, \quad s_j \equiv 1 \pmod{4}\right\}$$

(4.1)

$$\leq 2(\log\log x)^{\frac{1}{2}+\epsilon} \#\left\{r \leq \frac{x}{s_j} : s_i \nmid r \text{ for each } s_i \in \mathcal{S}, \quad \Omega(r) = z_j - 2\right\}.$$

With the size of the quantity

$$\#\left\{r \leq \frac{x}{s_j} : s_i \nmid r \quad \text{for each} \quad s_i \in \mathcal{S},\ s_j \equiv 1 \pmod{4},\ \Omega(r) \in \mathcal{I}\right\}$$

being relatively small, It follows from (4.1), by letting

$$\mathcal{N} := \left\{r \leq \frac{x}{s_j} : s_i \nmid r \text{ for each } s_i \in \mathcal{S},\ \Omega(r) = z_j - 2\right\}$$

that by inversion

$$\begin{aligned}
\#\mathcal{N} &\geq \frac{1}{2(\log\log x)^{\frac{1}{2}+\epsilon}} \#\left\{r \leq \frac{x}{s_j} : s_i \nmid r \text{ for each } s_i \in \mathcal{S},\ s_j \equiv 1 \pmod{4}\right\} \\
&\geq \frac{1}{2(\log\log x)^{\frac{1}{2}+\epsilon}} \sum_{s_j \equiv 1 \pmod{4}} \#\left\{r \leq \frac{x}{s_j} : s_i \nmid r \text{ for each } s_i \in \mathcal{S}\right\} \\
&\geq \frac{1}{2(\log\log x)^{\frac{1}{2}+\epsilon}} \sum_{s_j \equiv 1 \pmod{4}} \frac{x}{s_j}(1+o(1))C(\mathcal{S}) \\
&\geq \frac{1}{2(\log\log x)^{\frac{1}{2}+\epsilon}} x(1+o(1))C(\mathcal{S}) \sum_{s_j \equiv 1 \pmod{4}} \frac{1}{s_j} \\
&\geq \frac{1}{2(\log\log x)^{\frac{1}{2}+\epsilon}} x(1+o(1))C(\mathcal{S})\mathrm{K},
\end{aligned}$$

for some positive real number K, where

$$C(\mathcal{S}) = \prod_{j=1}^{\infty}\left(1 - \frac{1}{s_j}\right) \quad \text{and} \quad \sum_{s_j \equiv 1 \pmod{4}} \frac{1}{s_j} < \sum_{j=1}^{\infty} \frac{1}{s_j} < \infty.$$

Carrying out the same process for the other residue class $s_j \equiv 3 \pmod 4$ and combining the result, we will obtain

$$E(\Omega, g, x) \gg \frac{x}{(\log\log x)^{\frac{1}{2}+\epsilon}}.$$

□

5. Conclusion

The lower bound obtained in the original work of De koninck, Doyon and Letendre [1] can be made uniform by using a similar choice of multiplicative function in the main result; that is, if we let

$$g(p) = \begin{cases} z_j - 1 & \text{if } p = s_j \equiv 1 \pmod 4,\ s_j \in \mathcal{S} \\ z_j + 1 & \text{if } p = s_j \equiv 3 \pmod 4,\ s_j \in \mathcal{S} \\ 1 \quad \text{if} & p \notin \mathcal{S}. \end{cases}$$

Then

$$E(\omega, g, x) \gg \frac{x}{(\log\log x)^{\frac{1}{2}+\epsilon}}$$

holds uniformly. In a sequel to their first paper (see [4]), De Koninck, Doyon and Letendre studied the distribution of multiplicative and additive functions on a global scale. In fact they conjectured that the distribution of multiplicative functions cannot be as narrow as additive functions, in the sense that if for any given multiplicative function satisfying

$$\frac{\sigma}{\lambda} < \epsilon$$

where σ and λ denotes the standard deviation and the mean value, respectively, of the multiplicative function in question, then that could not be said about additive functions. There abounds as much open problems concerning the study of distributions of additive and multiplicative functions. For instance, It is a notorious open problem to determine if given an additive function with a specified limiting distribution, one can or cannot construct a multiplicative function with the same limiting distribution. What is known in this regard pertains to multiplicative functions with finite support [4].

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