Another Topological Proof for Equivalent Characterizations of Continuity

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Abstract

To prove the equivalence between the ε-δ characterization and the topological characterization of the continuity of maps acting between metric spaces, there are two typical approaches in, respectively, analysis and topology. We provide another proof that would be pedagogically informative, resembling the typical proof method — principle of appropriate sets — associated with sigma-algebras.

Keywords: characterizations of continuity; continuity of maps acting between metric spaces; principle of appropriate sets

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1 Introduction

Let $(\Omega, d), (\underline{\Omega}, \underline{d})$ be metric spaces; let $\mathscr{T}$ and $\underline{\mathscr{T}}$ be the metric topologies of Ω and $\underline{\Omega}$, respectively. A continuous map in analysis is by definition precisely a map $f : \Omega \to \underline{\Omega}$ with the property that for every $x \in \Omega$ and every $\varepsilon > 0$ there is some $\delta > 0$ such that $d(x, y) < \delta$ implies $\underline{d}(f(x), f(y)) < \varepsilon$, and in topology is by definition precisely a map $f : \Omega \to \underline{\Omega}$ with the property that $f^{-1)}(\underline{G}) \in \mathscr{T}$ for all $\underline{G} \in \underline{\mathscr{T}}$. Here the map $f^{-1)} : 2^{\underline{\Omega}} \to 2^{\Omega}$ denotes the preimage map induced by f.

That the topological characterization implies the $\varepsilon - \delta$ characterization is immediate. In analysis, the typical proof (e.g. Proposition 0.23 in Folland [2]) that the two characterizations are equivalent directly "expands" the $\varepsilon - \delta$ characterization; in topology, the typical proof follows from a general result (e.g. Theorem 8.3 in Chapter III of Dugundji [1]) that depends on some topology preliminaries.

With topology preliminaries almost no more than the definition of a topology, we can actually obtain a proof of the equivalence in the style of the principle of appropriate sets that is a usual proof method in real analysis when it comes to sigma-algebras.

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2 The Proof

Proof. With the same notation in mind, let $f : \Omega \to \underline{\Omega}$ be continuous in the $\varepsilon - \delta$ sense. Let $\mathscr{E}$ be the collection of all open balls in $\underline{\Omega}$, i.e. let $\mathscr{E}$ be the usual basis generating $\underline{\mathscr{T}}$. If $\mathscr{A} := \{A \subset \underline{\Omega} \mid f^{-1)}(A) \in \mathscr{T}\}$, then $\mathscr{E} \subset \mathscr{A}$ by the assumed $\varepsilon - \delta$ property of f; the f-preimage of every given element of $\mathscr{E}$ that does not meet the range of f is simply the empty set.

On the other hand, the union-intersection preserving property of $f^{-1)}$ implies that $\mathscr{A}$ is a topology of $\underline{\Omega}$. But since $\underline{\mathscr{T}}$ is the smallest topology including $\mathscr{E}$, which follows directly from the definition of a topology, we have $\underline{\mathscr{T}} \subset \mathscr{A}$. □

Remark. Our proof via "principle of appropriate sets" stresses the fact that $\mathscr{A}$ itself is a topology, an analogous observation as in the context of sigma-algebras. Indeed, the topology $\mathscr{A}$ is by definition the identification topology of $\underline{\Omega}$ determined by f given $\mathscr{T}$.

In showing that $\mathscr{A}$ is a topology, we exploit the union-intersection preserving property of a preimage map, which indeed may be used to give a shorter proof, although this shorter proof is not as informative for pedagogical purposes: Since every $G \in \underline{\mathscr{T}}$ is some union of open balls in $\underline{\Omega}$, the preserving property of a preimage map completes the proof. □

References

[1] Dugundji, J. (1966). *Topology.* Allyn & Bacon.

[2] Folland, G. B. (2007). *Real Analysis: Modern Techniques and Their Applications*, second edition. Wiley.