Very Elementary Proof of Invariance of Domain for the Real Line

Yu-Lin Chou*

Abstract

That every Euclidean subset homeomorphic to the ambient Euclidean space is open, a version of invariance of domain, is a relatively deep result whose typical proof is far from elementary. When it comes to the real line, the version of invariance of domain admits a simple proof that depends precisely on some elementary results of "common sense". It seems a pity that an elementary proof of the version of invariance of domain for the real line is not well-documented in the related literature even as an exercise, and it certainly deserves a space. Apart from the main purpose, as we develop the ideas we also make present some pedagogically enlightening remarks, which may or may not be well-documented.

Keywords: homeomorphisms; invariance of domain; manifolds; point-set topology; topology of the real line
MSC 2020: 26A03; 58C07

1 Introduction

There is the deep result: If $n \in \mathbb{N}$, and if $A \subset \mathbb{R}^n$ is homeomorphic to $\mathbb{R}^n$, then A is open with respect to the usual topology of $\mathbb{R}^n$. We refer the reader to, for instance, Munkres [2]. In what follows, let us refer to the assertion as *invariance of domain*.

For what it is worth, a *homeomorphism* between two topological spaces, by definition, is precisely a continuous bijection whose inverse is also continuous. The reader is already more than familiar with some homeomorphism between $\mathbb{R}$ and $\{x \in \mathbb{R} \mid x > 0\}$, e.g. the (real) exponential function. A topological space is said to be *homeomorphic* to a topological space if and only if there is some homeomorphism between the spaces. Thus, by considering the exponential function on $\mathbb{R}$, the real line $\mathbb{R}$ is homeomorphic to $\{x \in \mathbb{R} \mid x > 0\}$.[1]

*Yu-Lin Chou, Institute of Statistics, National Tsing Hua University, Hsinchu 30013, Taiwan, R.O.C.; Email: `y.l.chou@gapp.nthu.edu.tw`.

[1]If the reader is not yet familiar with the concept of homeomorphicness, it would be in your interest to check that the topologies of two homeomorphic spaces can actually be identified with each other; so the topological structures of the homeomorphic spaces are "the same". This observation would help justify the terminology. If the reader is familiar with the concept of isomorphicness, then you can now see that "topological isomorphism" would in principle be another reasonable choice to address a homeomorphism.

Invariance of domain, whose statement is so simple, takes however a serious proof depending on techniques of algebraic topology. This phenomenon — a simple mathematical statement deciding whose truth is far from simple — comes as no surprise in mathematics; we invite the reader to read, e.g. the statement of Poincaré conjecture, or of Goldbach conjecture, or of the (famous or infamous) Fermat's last theorem.

To have a feeling of the importance of invariance of domain, the reader may try to reason out as a typical exercise that $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}^m$ for all natural numbers $n < m$ by considering, for example, the cylinder $\mathbb{R}^n \times \{(0)_{j=1}^{m-n}\} \subset \mathbb{R}^m$; this fact contributes to defining the dimension of a manifold.

For $n = 1$, it is possible to obtain an elementary proof within the scope of elementary point-set topology. Although the present article is not intended as a technical research paper, for clarity we find it advisable to present the ideas in terms of a theorem-proof style. The elementary proof of invariance of domain for $\mathbb{R}$, together with the "common sense"[2] that every open interval of $\mathbb{R}$ is homeomorphic to $\mathbb{R}$, which will also be justified here, jointly lead to the intuitively "apparent" and elegant result that a subset of $\mathbb{R}$ homeomorphic to $\mathbb{R}$ is precisely an open interval of $\mathbb{R}$. We will also establish this result as a proposition here in the present article.

2 Proof

For our purposes, by an *interval* (of $\mathbb{R}$) we mean either a set of the form $]a, b[\coloneqq \{x \in \mathbb{R} \mid a < x < b\}$ with $-\infty \leq a < b \leq +\infty$, or of the form $[a, b] \coloneqq \{x \in \mathbb{R} \mid a \leq x \leq b\}$ with $-\infty < a < b < +\infty$, or of the form $[a, b[\coloneqq \{x \in \mathbb{R} \mid a \leq x < b\}$ with $-\infty < a < b \leq +\infty$, or of the form $]a, b] \coloneqq \{x \in \mathbb{R} \mid a < x \leq b\}$ with $-\infty \leq a < b < +\infty$. In other words, an interval, in our setting, is never degenerate (becoming a singleton or the empty set) and, except for a closed interval, can be unbounded. The standard terminology associated with intervals carries over into the present context. For example, an open interval is precisely an interval of the form $]a, b[$; and $\mathbb{R}$, accordingly, is also considered as an open interval.

We should like to prove

Theorem 1 (Invariance of Domain for $\mathbb{R}$)**.** *If $A \subset \mathbb{R}$ is homeomorphic to $\mathbb{R}$, then A is an open subset of $\mathbb{R}$.*

Proof. Since every interval is a connected set (subspace) of the topological space $\mathbb{R}$ (by, e.g. the proof of Theorem 1.2 in Chapter 5 of Dugundji [1])[3], in particular $\mathbb{R}$ is connected; the assumption then implies that A, being some continuous image of $\mathbb{R}$, is a connected set of $\mathbb{R}$. Although the empty set and a singleton are (trivially) connected sets of $\mathbb{R}$, the set A cannot be a singleton nor empty as A is by assumption in bijection with $\mathbb{R}$. Since a nonempty, non-singleton connected set of $\mathbb{R}$ is precisely an interval (again by, e.g. the proof of Theorem 1.2 in Chapter 5 of Dugundji [1]), it follows that

[2] If the reader does not yet find it a common sense, let there be no worries; you will take it into your common-sense account after reading this article.

[3] Indeed, regardless of the "age" of the reference, there is nothing more mysterious about Dugundji's proof; the main proof idea is already well-known. We select it as the reference for its refinedness and clarity from a pedagogical consideration.

A is an interval. For what it is worth, to prove that every nonempty, non-singleton connected set of $\mathbb{R}$ is an interval is much less elaborate than the converse; the definition of an interval implies that every nonempty, nonsingleton subset $\underline{A}$ of $\mathbb{R}$ that is not an interval admits some $x, y \in \underline{A}$ and some $z \in \underline{A}^c$ such that $x < z < y$, and we consider the sets $\underline{A} \cap \,]-\infty, z[$ and $\underline{A} \cap \,]z, +\infty[$.

Now, as a closed interval is a compact subset of $\mathbb{R}$, and as $\mathbb{R}$ is by assumption a continuous image of A, the set A cannot be a closed interval. Moreover, the set A cannot be a half-open interval, i.e. an interval of the form $]a, b]$ or of the form $[a, b[$. To see this, take a left-closed-right-open interval $[a, b[$. If $[a, b[$ is homeomorphic to $\mathbb{R}$ via some function f, then, by recalling that a preimage map preserves intersections, the open interval $]a, b[= [a, b[\setminus \{a\}$ is homeomorphic to $\mathbb{R} \setminus \{f(a)\}$; but $\mathbb{R} \setminus \{f(a)\} = \,]-\infty, f(a)[\cup]f(a), +\infty[$ is not connected. The argument also applies to every right-closed-left-open interval, i.e. to every interval of the form $]a, b]$.

We have shown that A "cannot but be" an open interval. Since every open interval is an open subset of $\mathbb{R}$, the proof is complete. □

The proof of Theorem 1 contains additional information that is helpful in establishing the following

Proposition 1. *Let $A \subset \mathbb{R}$. Then A is homeomorphic to $\mathbb{R}$ if and only if A is an open interval.*

Proof. The "only if" part follows directly from the proof of Theorem 1; that A is an open interval was a conclusion.

For the bounded open intervals, the "if" part is probably more than well-known and well-documented. For a quick check, let $a < b$ be real numbers and consider for intance the function $x \mapsto (e^x + 1)^{-1}(b + ae^x)$ from $\mathbb{R}$ to $]a, b[$.

For unbounded open intervals, the case that the open interval is the real line is immediate; one considers the identity automorphism $x \mapsto x$ of $\mathbb{R}$.

The unbounded open intervals of the form $]c, +\infty[$ or of the form $]-\infty, c[$ with $c \in \mathbb{R}$ constitute the remaining case. We show that such open intervals are homeomorphic to $\mathbb{R}$. If $a, b, c \in \mathbb{R}$, and if $a < b$, then the function $x \mapsto a + \frac{b-a}{e^{x-c}}$ from $]c, +\infty[$ to $]a, b[$ serves as a homeomorphism, and the function $x \mapsto a + \frac{b-a}{e^{c-x}}$ from $]-\infty, c[$ to $]a, b[$ is also a homeomorphism. Since every bounded open interval is homeomorphic to $\mathbb{R}$, it follows that every open interval of the form $]c, +\infty[$ is homeomorphic to $\mathbb{R}$ (via the composition of some homeomorphism between $]a, b[$ and $\mathbb{R}$ circ some homeomorphism between $]c, +\infty[$ and $]a, b[$), and likewise for every open interval of the form $]-\infty, c[$. We thus have completed the proof. □

References

[1] Dugundji, J. (1966). *Topology.* Allyn & Bacon.

[2] Munkres, J. R. (1984). *Elements of Algebraic Topology.* Addison-Wesley.