Obtaining a Homeomorphism from an Arbitrary Bijection

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Abstract

We wish to demystify the concept of a homeomorphism for anyone who finds the idea "intangible", by showing that one can construct a homeomorphism out of any given bijection in a natural way. Some interdisciplinary examples are discussed for concreteness.

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1 Introduction

A homeomorphism acting between topological spaces is by definition precisely a continuous bijection whose inverse is also continuous, i.e. a bicontinuous bijection. Since speaking of continuity of a map presumes topologies of the domain and the codomain of the map, speaking of a homeomorphism also presumes topological structures of the domain and the codomain of the homeomorphism.

The exponential function from $\mathbb{R}$ to $\mathbb{R}_{++} := \{x \in \mathbb{R} \mid x > 0\}$ is an elementary, familiar example of a homeomorphism, where the underlying topologies are, respectively, the standard topology of $\mathbb{R}$ and the relative (subspace) topology of $\mathbb{R}_{++}$.

In an abstract context without a specific rule of assignment, the concept of a homeomorphism seems to suddenly become more "intangible". But this feeling is unnecessary, and we wish to clear this mythology for anyone who feels that way. Our constructions are also hoped to serve as a healthy companion helping the beginning reader to appreciate certain subtleties of the mathematics of topology.

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2 Constructions

Let $\Omega, \underline{\Omega}$ be arbitrary nonempty sets; let $f : \Omega \to \underline{\Omega}$ be a bijection. As the power set of any given set is automatically a topology (stable with respect to arbitrary unions and finite intersections, the empty set and the ambient set being taken care of by considering the empty union and the empty intersection), we can always consider Ω or $\underline{\Omega}$ as a topological space, although this consideration is not necessarily "useful" with respect to various purposes. But being useful is *a priori* not an intended purpose of our constructions to be demonstrated.

2.1 When Ω is Given an Arbitrary Topology

If $\mathscr{T}$ is a topology of Ω, we wish to make f into a homeomorphism. Indeed, let

$$\tau(f, \mathscr{T}) \coloneqq \{A \subset \underline{\Omega} \mid f^{-1)}(A) \in \mathscr{T}\},$$

i.e. let $\tau(f, \mathscr{T})$ be the collection of all subsets of $\underline{\Omega}$ whose f-preimage is open-$\mathscr{T}$. The elements of $\tau(f, \mathscr{T})$ are not so trivial as f is a bijection. Since every preimage map induced by a function preserves unions and intersections, the collection $\tau(f, \mathscr{T})$ is indeed a topology.

Now f is evidently continuous-$(\mathscr{T}, \tau(f, \mathscr{T}))$. Since f is in particular an injection, the composition $f^{-1)} \circ f^{1)}$ of the preimage map $f^{-1)}$ circ the image map $f^{1)}$ induced by f is the identity map of 2^{Ω}, and so

$$f^{-1)} \circ f^{1)}(V) = V$$

for all $V \in \mathscr{T}$. This implies that $f^{1)}(V) \in \tau(f, \mathscr{T})$ for all $V \in \mathscr{T}$, and hence f is an open map, i.e. $f^{1)}$ carries every element of $\mathscr{T}$ to an element of $\tau(f, \mathscr{T})$. But, since f is a bijection, we have $(f^{-1})^{-1)} = f^{1)}$; so f^{-1} is continuous-$(\tau(f, \mathscr{T}), \mathscr{T})$. We thus have made f into a homeomorphism with respect to the topologies $\mathscr{T}$ and $\tau(f, \mathscr{T})$.

The topology $\tau(f, \mathscr{T})$ is simply the *identification topology* determined by f (given $\mathscr{T}$), which can be shown to be the largest topology such that f is continuous with respect to $\mathscr{T}$. The reader may recall (or discover) that the topology introduced to a quotient space is exactly the identification topology determined by the canonical quotient map.

2.2 When $\underline{\Omega}$ is Given an Arbitrary Topology

Beginning with a given topology of $\underline{\Omega}$, we can also construct a homeomorphism from f.

If $\underline{\mathscr{T}}$ is a topology of $\underline{\Omega}$, let

$$\tau(f, \underline{\mathscr{T}}) \coloneqq \{f^{-1)}(G) \mid G \in \underline{\mathscr{T}}\}.$$

The union-intersection preserving property of a preimage map again ensures that $\tau(f, \underline{\mathscr{T}})$ is a topology.

The map f is trivially continuous-$(\tau(f, \underline{\mathscr{T}}), \underline{\mathscr{T}})$. To see that f^{-1} is also continuous, let $V \in \tau(f, \underline{\mathscr{T}})$, so that there is some $G \in \underline{\mathscr{T}}$ such that $V = f^{-1)}(G)$. Since f is in particular a surjection, it follows that $f^{1)} \circ f^{-1)}$ is the identity of $2^{\underline{\Omega}}$; so

$$f^{1)}(V) = f^{1)}(f^{-1)}(G)) = G.$$

Using the relation $(f^{-1})^{-1)} = f^{1)}$ once more, we see that f^{-1} is continuous-$(\underline{\mathscr{T}}, \tau(f, \underline{\mathscr{T}}))$. Thus we have shown that f is a homeomorphism with respect to the topologies $\tau(f, \underline{\mathscr{T}})$ and $\underline{\mathscr{T}}$.

The topology $\tau(f, \underline{\mathscr{T}})$ is simply the *weak topology* generated by f (given $\underline{\mathscr{T}}$), which is evidently the smallest topology such that f is continuous with respect to $\underline{\mathscr{T}}$. The reader may recall (or discover) that the weak topology generated by a collection of natural projections defined on (or of, depending on one's definition of a Cartesian product) a given Cartesian product is the product topology of the Cartesian product.

3 Discussions

The previous constructions are purely "theoretical" in the sense that they are silent on the case where both the topologies of the domain and the codomain of a given bijection are given. Thus there is no claim made here regarding their usefulness in the customary sense. However, the constructions may be useful in a conceptual sense.

Some natural interdisciplinary examples may be considered to make concrete the ideas of the constructions.

3.1 Cayley's Regular Representation

Cayley's representation theorem for groups asserts that every group is isomorphic to some subgroup of the symmetric group over the given group. If $(G, \ddot{\circ})$ is a group, the Cayley's regular representation is the isomorphism $a \mapsto (a \,\ddot{\circ}\, x)_{x \in G}$ defined on G. Here the isomorphism is considered with respect to the group operation $\ddot{\circ}$ and the usual function composition $\circ$.

If G is equipped with an arbitrary topology, which not necessarily makes G into a topological group, there is some immediate way to topologize $\varphi^{1)}(G)$ — i.e. the subgroup of the symmetric group over G that is isomorphic to G via φ — such that G is homeomorphic to $\varphi^{1)}(G)$ via φ; the construction in Subsection 2.1 just serves the purpose. We might add again that whether or not this homeomorphism is interesting is not of concern here.

3.2 Bimeasurable Bijections

To a certain extent, the concept of a sigma-algebra resembles that of a topology. It is then conceivable that a measurable bijection whose inverse is also measurable — a *bimeasurable bijection* — and a homeomorphism share some properties.

Indeed, our constructions may be immediately adapted to obtain a bimeasurable bijection from any given bijection acting between measurable spaces. Working out this assertion would be a nice exercise for those who find it helpful in self studies.