COMMENTS ON "ANALYTICAL FEATURES OF THE SIR MODEL AND THEIR APPLICATIONS TO COVID-19"

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Abstract. In their article, Kudryashov et al. (2021) try to establish the analytical solution of the SIR epidemiological model. One of the equations given there is wrong, which invalidates the presented solution, derived from this result. The objective of the present letter is to indicate this error and present the correct analytical solution to the SIR epidemiological model.

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1. Introduction

In their article "Analytical features of the SIR model and their applications to COVID-19", Kudryashov et al. [3] try to establish the analytical solution of the SIR (Susceptible-Infected-Removed) epidemiological model of Kermack and McKendrick [2]. While the article itself is well-written, one of the equations given there is wrong (eq. 14), which invalidates further the putative solution (eq. 15). Notably, the wrong equation reads[1]

$$Y = exp\left(\frac{\alpha I - C}{\gamma} - \frac{1}{\gamma} W\left(-\frac{e^{\frac{\alpha I - C}{\gamma}}}{\gamma}\right)\right) - \gamma$$

where $Y := \frac{\dot{I}}{I}$ and W denotes the Lambert W function. The above equation is claimed to be the solution of the first integral of the system given below:

$$\gamma + \frac{\dot{I}}{I} - \gamma \log\left(\gamma + \frac{\dot{I}}{I}\right) + \alpha I = C \tag{1}$$

Unfortunately, this is not true. The objective of the present letter is to indicate this error and present the correct solution.

2. Resolution of the first integral 1

Before presenting the correct solution to eq. 1 and the correct analytical solution, few remarks about the Lambert W function are in order.

[1]The present letter uses slightly different labelling of the model – instead of β used in [3] – γ is used and $\alpha = \beta/N$.

2.1. **The Lambert W function.** The Lambert W function can be defined implicitly by the equation

$$W(z)e^{W(z)} = z, \quad z \in \mathbb{C} \tag{2}$$

Furthermore, the Lambert function obeys the differential equation for $x \neq -1$

$$W(x)' = \frac{e^{-W(x)}}{1 + W(x)}$$

The Lambert W is a multivalued function and in particular it has 2 real-valued branches denoted by W_+ and W_-, respectively. Properties of the W function are surveyed in [1]. Useful identities

$$e^{-W(z)} = \frac{W(z)}{z} \tag{3}$$

$$e^{nW(z)} = \left(\frac{z}{W(z)}\right)^n \tag{4}$$

$$\log W(z) = \log z - W(z) \tag{5}$$

2.2. **Solution of the transcendental equation.** Starting from the first integral we apply series of transformations as follows:

$$\gamma + \frac{\dot{I}}{I} - \gamma \log\left(\gamma + \frac{\dot{I}}{I}\right) + \alpha I = C \Longleftrightarrow Y - \gamma \log Y = C - \alpha I \Longleftrightarrow$$
$$-\frac{Y}{\gamma} + \log Y = \frac{\alpha I - C}{\gamma} \Longrightarrow$$
$$e^{-\frac{Y}{\gamma} + \log Y} = e^{\frac{\alpha I - C}{\gamma}} \Longrightarrow$$
$$Y e^{-\frac{Y}{\gamma}} = e^{\frac{\alpha I - C}{\gamma}} \Longleftrightarrow -Y e^{-\frac{Y}{\gamma}} = -e^{\frac{\alpha I - C}{\gamma}}$$

Further, we apply the W function on both sides of the last equation

$$W\left(-Y e^{-\frac{Y}{\gamma}}\right) = W\left(-e^{\frac{\alpha I - C}{\gamma}}\right) \Longrightarrow$$
$$-\frac{Y}{\gamma} = W\left(-e^{\frac{\alpha I - C}{\gamma}}\right)$$

where we have used the defining identity eq. 2. So finally

$$Y = -\gamma W\left(-e^{\frac{\alpha I - C}{\gamma}}\right) \tag{6}$$

Therefore,

$$\frac{\dot{I}}{I} = -\gamma W\left(-e^{\frac{\alpha I - C}{\gamma}}\right)$$

using the original notation for the incidence. This is an autonomous system which can be solved for t by separation of variables. The question remains, however, whether this is the correct differential equation for the incidence. The authors do not present the derivation procedure of the first integral so it needs to be verified. It will be answered in affirmative as will be demonstrated below. At this point some remarks about the SIR model are in order.

3. The SIR model

The SIR model is formulated in terms of 3 populations of individuals [4]. The S population consists of all individuals susceptible to the infection of concern. The I population comprises the infected individuals. These persons have the disease and can transmit it to the susceptible individuals. The R poulation cannot become infected and the individuals cannot transmit the disease to others. The model comprises a set of three ODEs:

$$\dot{S}(t) = -\frac{\beta}{N} S(t) I(t) \tag{7}$$

$$\dot{I}(t) = \frac{\beta}{N} S(t) I(t) - \gamma I(t) \tag{8}$$

$$\dot{R}(t) = \gamma I(t) \tag{9}$$

The model assumes a constant overall population $N = S(t) + I(t) + R(t)$ [2]. The interpretation of the parameters is that a disease carrier infects on average β individuals per day, for an average time of $1/\gamma$ days. The β parameter is called *disease transmission rate*, while γ – *recovery rate*. The average number of infections arising from an infected individual is then modelled by the number $R_0 = \frac{\beta}{\gamma}$, the *basic reproduction number*. Typical initial conditions are $S(0) = S_0, I(0) = I_0, R(0) = 0$ [2].

The model can be re-parametrized using normalized variables as

$$\dot{s} = -si \tag{10}$$

$$\dot{i} = si - gi, \quad g = \frac{\gamma}{\beta} = \frac{1}{R_0} \tag{11}$$

$$\dot{r} = gi, \tag{12}$$

subject to normalization $s + i + r = 1$ and time rescaling as $\tau = \beta t$.

4. The analytical solution

Since there is a first integral by construction, the system can be reduced to two differential equations in the phase plane:

$$\frac{di}{ds} = -1 + \frac{g}{s} \tag{13}$$

$$\frac{di}{dr} = \frac{s}{g} - 1 \tag{14}$$

In order to solve the model we will consider the two equations separately. Direct integration of the equation 13 gives

$$i = -s + g \log s + c \tag{15}$$

where the constant c can be determined from the initial conditions. The s variable can be represented explicitly in terms of the Lambert W function as in the above section

$$s = -g W_{\pm}\left(-\frac{e^{\frac{i-c}{g}}}{g} \right) \tag{16}$$

where the signs denote the two different real-valued branches of the function. Note, that both branches are of interest since the argument of the Lambert W function is negative. Therefore, the ODE 11 can be reduced to the first-order autonomous system

$$\dot{i} = -ig\left(W_{\pm}\left(-\frac{e^{\frac{i-c}{g}}}{g}\right)+1\right) \tag{17}$$

valid for two disjoined domains on the real line. The ODEs can be solved for the time τ as

$$-\int \frac{di}{i\left(W_{\pm}\left(-\frac{e^{\frac{i-c}{g}}}{g}\right)+1\right)} = g\tau \tag{18}$$

Finally, the r variable can also be conveniently expressed in terms of i. For this purpose we solve the differential equation

$$\frac{dr}{di} = \frac{g}{s-g} = \frac{-1}{1+W_{\pm}\left(-\frac{e^{\frac{i-c}{g}}}{g}\right)}$$

Therefore,

$$r = c_1 - g\log\left(-gW_{\pm}\left(-\frac{e^{\frac{i-c}{g}}}{g}\right)\right) = c_1 - g\log s$$

by Prop. 1. On the other hand,

$$g\log\left(-gW\left(-\frac{e^{\frac{i-c}{g}}}{g}\right)\right) = g\left(\log\left(\frac{e^{\frac{i-c}{g}}}{g}\right) - W\left(-\frac{e^{\frac{i-c}{g}}}{g}\right)\right) =$$

$$-gW\left(-\frac{e^{\frac{i-c}{g}}}{g}\right) + i - g\log g - c = s + i - g\log g - c$$

So that

$$r = gW\left(-\frac{e^{\frac{i-c}{g}}}{g}\right) - i + c_1$$

As an independent verification of the first integral 1 and a final remark we observe that

$$Y = \gamma + \frac{\dot{I}}{I} = S\alpha$$

in terms of the original variables. Therefore, eq. 1 can be re-expressed in Kudrashov et al's notation as

$$\alpha S - \gamma\log\alpha S + \alpha I = C$$

which is equivalent to eq. 15.

Appendix A. Useful integrals

Proposition 1.

$$\int \frac{dy}{1+W\left(-\frac{e^{\frac{y-c}{g}}}{g}\right)} = g\log\left(-gW\left(-\frac{e^{\frac{y-c}{g}}}{g}\right)\right) + C$$

Proof. We differentiate

$$g\left(\log W\left(-\frac{e^{\frac{y-c}{g}}}{g}\right)\right)' = -\frac{e^{\frac{y-c}{g}-W\left(-\frac{e^{\frac{y-c}{g}}}{g}\right)}}{gW\left(-\frac{e^{\frac{y-c}{g}}}{g}\right)\left(1+W\left(-\frac{e^{\frac{y-c}{g}}}{g}\right)\right)} = \frac{1}{1+W\left(-ge^{\frac{i-c}{g}}\right)}$$

□

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