22 Hilbert machine

Draft chapter of the book *Infinity Put to the Test* by Antonio León (next publication).

Abstract.Hilbert's machine is a theoretical device, inspired by the emblematic Hilbert's Hotel, whose functioning leads to a contradiction involving the consistency of the hypothesis of the actual infinity subsumed into the axiom of infinity.

Keywords: Hilbert Hotel, Hilbert machine, w-order, inconsistency of the actual infinity.

Hilbert's Hotel

P1 In the next discussion we will make use of a supermachine inspired by the emblematic Hilbert's Hotel. But before beginning, let us relate some of the prodigious, and suspicious, abilities of the illustrious Hotel.

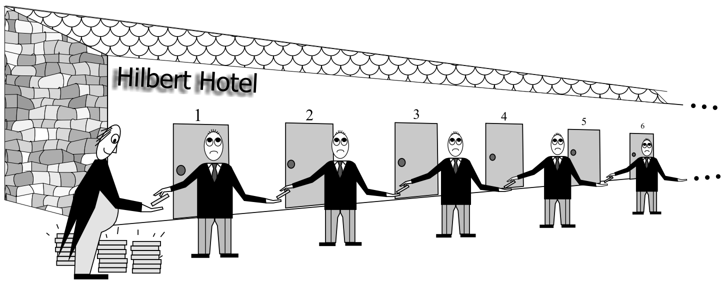

Figura 22.1 – The power of the ellipsis: An infinitist way of making money.

P2 Its director, for example, has discovered a fantastic way of getting rich: he demands one euro to R_1 (the guest of the room 1); R_1 recovers his euro by demanding one euro to R_2 (the guest of the room 2); R_2 recovers his euro by demanding one euro to R_3 (the guest of the room 3); and so on. Finally all guests recover his euro, because there is not a last guest losing his money. Our crafty director then demands a second euro to R_1 which recovers again his euro by demanding one euro to R_2, which recovers again his euro by demanding one euro to R_3, and so on and on. Thousands of euros coming from the (infinitist) nothingness to the pocket of the fortunate director.

P3 Hilbert's Hotel is even capable of violating the laws of thermodynamics by making it possible the functioning of a perpetuum mobile: in fact we would only have to power the appropriate machine with the calories obtained from the successive rooms of the prodigious hotel in the same way its director gets the euros.

P4 Incredible as it may seem, infinitists justify all those absurd pathologies, and many others, in behalf of the *peculiarities* of the actual infinity. They prefer to assume any pathological behaviour of the world before examining the consistency of the pathogene. In the next discussion, however, we will come to a contradiction that cannot be easily justified by the picturesque nature of the actual infinity.

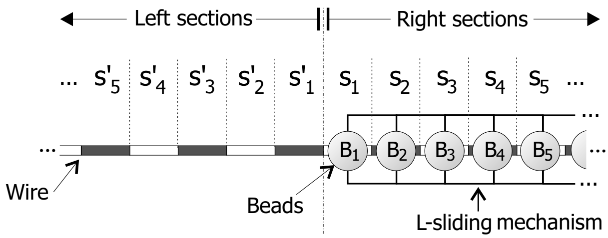

Figura 22.2 – Hilbert's machine just before performing the first L-sliding.

Definitions

P5 In the following conceptual discussion we will make use of a theoretical device, inspired by the emblematic Hilbert Hotel, that will be referred to as *Hilbert machine*, composed of the following elements (see Figure 22.2):

1) An infinite horizontal wire divided into two infinite parts, the left and the right side:

    a) The right side in turn is divided into an ω-ordered sequence of disjoint and adjacent sections $\langle S_i \rangle$ of equal length labeled from left to right as S_1, S_2, S_3, ... They will be referred to as right sections.

    b) The left side is also divided into an ω-ordered sequence of disjoint and adjacent sections $\langle S'_i \rangle$ of equal length, the same length as the right sections, and labeled now from right to left as ..., S'_3, S'_2, S'_1; being S'_1 adjacent to S_1. They will be referred to as left sections.

2) An ω-ordered sequence of labeled beads $\langle b_n \rangle$ strung on the wire, so that they can slide on the wire as the beads of an abacus, being

the center of each bead b_i initially placed on the center of the right section S_i.

3) All beads are mechanically linked by an sliding mechanism that slides simultaneously all beads the same distance along the wire.

4) The sliding mechanism is adjusted in such a way that it slides simultaneously each bead exactly one, and only one, section to the left (L-sliding).

P6 Obviously, Hilbert's machine is a theoretical artifact, and its functioning is a simple thought experiment that illustrates a formal argument to test ω-order, the order type of the well ordered set $\mathbb{N}$ of the natural numbers in their natural order of precedence, whose ordinal number is ω, the least transfinite ordinal [2, §15, Theorem A, p.160]. This is not, therefore, a discussion on the physical restrictions and consequences of performing a particular sequence of physical actions.

P7 Since the sections $\langle S'_i \rangle$ of the left side of the wire are ω-ordered, each section S'_n has an immediate successor section S'_{n+1} just on its left (ω-successiveness). In accord with the hypothesis of the actual infinity all those infinitely many left sections exist as a complete totality in spite of the fact that there is no last section completing the sequence.

P8 We will assume Hilbert's machine always works according to the following:

Restriction P8.-*An L-sliding will be carried out if, and only if, after being performed all beads remain strung on the wire. Otherwise, the L-sliding will be undone so that every bead recover its previous position and then the machine stops.*

P9 Let us begin by proving that for each $v \in \mathbb{N}$ the first v L-slidings can be carried out according to Restriction P8. Assume this assertion is not true. There will be a natural number $u \leq v$ such that it is impossible to perform the uth L-sliding according to Restriction P8. But this is impossible because whatsoever be the left section occupied by b_1 just before performing the uth L-sliding, there always be a left section contiguous to that section (ω-successiveness), so that b_1 can L-slide to that section (otherwise b_1 would be in the impossible last left section), and every ball $b_{i,i>1}$ can move to the section previously occupied by b_{i-1}. Therefore, the uth L-sliding can be carried out according to Restriction P8. Consequently our assumption is not true, and for each $v \in \mathbb{N}$ it is possible to carry out the first v L-slidings according to Restriction P8.

P10 The following inductive argument leads to the same conclusion as the previous one P9 (Modus Tollens). It is clear that the first L-sliding can be performed: b_1 slides to S'_1 and every $b_{i;i>1}$ to the section previously occupied by b_{i-1}. Suppose that, for any natural number n, the first n L-slidings can be carried out. Since each L-sliding moves each ball one, and only one, section to the left, all balls will have been moved n sections to the left, so that b_1 will be in the left section S'_n, since S'_n is n sections to the left of the S_1, the section initially occupied by b_1. And since S'_n has an adjacent left section S'_{n+1} (ω-successiveness), b_1 can slide to S'_{n+1} and each $b_{i;i>1}$ to the section previously occupied by b_{i-1}. So, if for any n the first n L-slidings can be carried out, the first $n+1$ L-slidings can also be carried out. And since the first L-sliding can be carried out, we conclude that for any $v \in \mathbb{N}$ the first v L-slidings can be carried out.

Hilbert machine contradiction

P11 Assume that while the successive L-slidings can be carried out, they are carried out. It is immediate to prove the following:

Theorem P11a.-*Once performed all possible L-slidings all balls remain strung on the wire.*

Proof.-It is an immediate consequence of Restriction P8: if an L-sliding removes a bead from the wire that L-sliding would be undone and the machine stops with every ball strung on the wire in the section occupied just before that L-sliding. In addition, since an L-sliding simultaneously moves each ball one, and only one, section to the left and the first ball to the left of all balls is b_1, it had to be b_1, and only b_1, the ball that came out of the wire by one L-sliding. Otherwise, if the first n balls were simultaneously removed from the wire by one L-sliding, then each ball $b_{i,i\leq n}$ would have been moved i sections to the left by one L-sliding, which is impossible. In consequence, if b_1 is removed from the wire, b_2 would have to be in the impossible last section of an ω-ordered collection $\langle S'_i \rangle$ of sections. So, once all the possible L-slides are done, all the balls remain strung on the wire. □

Theorem P11b.-*Once performed all possible L-slidings no bead remains strung on the wire.*

Proof.-Let b_v be any bead and assume that once performed all possible L-slidings it is strung on the right section S_k. It must be $k < v$ because all L-slidings are towards the left, the direction towards which the indexes of $\langle S_i \rangle$ decrease. Since b_v was initially on S_v only a finite number $v - k$ of

L-slidings would have been performed, and then it would not have been possible to perform the the first $v-k+1$ L-slidings, which goes against P9 and P10, because $v-k+1$ is a natural number. A similar reasoning can be applied if b_v were finally strung on a left section S'_n, being now the number of performed L-slidings exactly $v+n-1$ and then it would not have been possible to perform the first $v+n$ L-slidings, which also goes against P9 and P10, because $v+n$ is also a natural number. Thus, since b_v is any bead, if all possible L-slidings have been performed, then no bead remains strung on the wire. Note this is not a question of indeterminacy but of impossibility: the set of possible sections any ball b_v could be finally occupying is the empty set. □

P12 A point of note on the above argument is that it is only necessary to know that, under the hypothesis of the actual infinity, all possible L-slidinigs have been carried out. A corollary of the theorem P11b that the reader will be able to prove is that all the balls stop being inserted in the wire at the same instant t_b, an instant at which L-sliding are no longer performed.

Discussion

P13 Let us compare the functioning of the above Hilbert machine (H_ω from now on) with the functioning of a finite version of the machine (symbolically H_n). This finite machine has a finite number n of both right and left sections (Figure 22.3). A finite sequence of n beads are initially strung on the right side of the wire, the center of each bead b_i placed on the center of the right section S_i. It is immediate to prove that H_n can only perform n L-slidings because not having a left section S'_{n+1}, Restriction P8 will stop the machine with each left section S'_i occupied by the bead b_{n-i+1} and all right sections empty, and this is all. No contradiction is derived from the functioning of H_n. Thus for any natural number n, the corresponding machine H_n is a consistent theoretical artifact. Only the infinite Hilbert's machine H_ω is inconsistent.

P14 What contradiction P11a-P11b proves is not the inconsistent functioning of a supermachine. What it proves is the inconsistency of ω-order itself (Principle of Autonomy) because of ω-successiveness. Perhaps we should not be surprised by this conclusion. After all, an ω-ordered sequence is one which is both complete (as the actual infinity requires) and uncompletable (there is not a last element that completes the sequence). On the other hand, and as Cantor proved [1, 2], ω-order is an inevitable consequence of assuming the existence of infinite sets as complete totalities. An existence

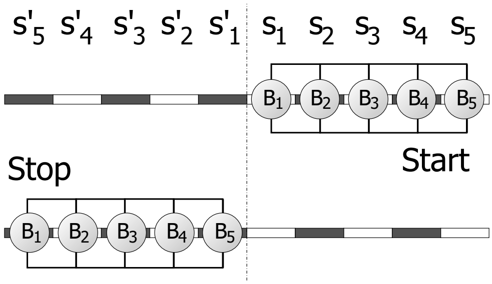

Figura 22.3 – A finite machine of five sections.

axiomatically stated in our days by the Axiom of Infinity, in all axiomatic set theories including its most popular versions ZFC and BNG [4, 3]. It is, therefore, that axiom the ultimate cause of contradiction P11a-P11b.

Bibliografía

[1] Georg Cantor, *Beiträge zur Begründung der transfiniten Mengenlehre*, Mathematishe Annalen **XLIX** (1897), 207 – 246.

[2] ______, *Contributions to the founding of the theory of transfinite numbers*, Dover, New York, 1955.

[3] Robert R. Stoll, *Set Theory and Logic*, Dover, New York, 1979.

[4] Patrrick Suppes, *Axiomatic Set Theory*, Dover, New York, 1972.