Exact solution of All
Half-Integer Bessel LHODE Formula

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Abstract

A formula for all the Half-Integer Bessel ODE is produced using my article "Exact solution of ODEs Vector Space Transformation Technique, Part 2".

(In an earlier publication I produced an algorithm for the Bessel hallf-integer solutions, but advancements on the Vector Space Transformation Technique allowed an actual formula to be produced (as well as further results))

Corollary I.4: The half-integer Bessel ODE solutions may be written:

$$\text{If } Y_i'' + \frac{1}{x}Y_i' + \left[-\frac{\left(\frac{1+2m}{2}\right)^2}{x^2} - 1 \right] Y_i = 0 \quad , \quad (m \in \mathbb{N}):$$

then:

$$\Rightarrow \left\{ \begin{array}{l}
\boxed{\begin{array}{l} Y_{1m} = u_m J_{\frac{1}{2}} - v_m J_{-\frac{1}{2}} \\ Y_{2m} = v_m J_{\frac{1}{2}} + u_m J_{-\frac{1}{2}} \end{array}} \\
\boxed{\begin{array}{l} u_m = \sum_{k=0}^{\frac{m}{2}} a_{m(2k)} x^{-2k} + \sum_{k=0}^{\frac{m}{2}} a_{m(2k+1)} x^{-(2k+1)} \\ v_m = \sum_{k=0}^{\frac{m}{2}} b_{m(2k)} x^{-2k} + \sum_{k=0}^{\frac{m}{2}} b_{m(2k+1)} x^{-(2k+1)} \end{array}} \\
\textit{where}: \\
\left\{ \begin{array}{ll} b_{m0} = \frac{2}{m(m+1)} a_{m1} & \\ b_{m1} = -\frac{m(m+1)}{2} a_{m0} & , \quad (m = 1) \end{array} \right. \\
\left\{ \begin{array}{ll}
a_{m(2k)} = \left(-\frac{1}{4}\right)^k \frac{\left(\frac{(m+(2k-1))!}{-(m+1)!}\right)\left(\frac{(m+2k)!}{m!}\right)}{(2k)!} a_{m0} & , \quad (0 \le 2k \le m) \\
a_{m(1+2k)} = \left(-\frac{1}{4}\right)^k \frac{\left(\frac{(1-m+(2k-1))!}{-m!}\right)\left(\frac{(1+m+2k)!}{(1+m)!}\right)}{(1+2k)!} a_{m1} & , \quad (1 \le 2k \le m-1) \\
b_{m(2k)} = \frac{2(2k+1)}{(2k-m)(2k+m+1)} a_{m(2k+1)} & , \quad (0 \le 2k \le m-1) \\
b_{m(2k-1)} = -\frac{4k}{(2k-1-m)(2k+m)} a_{m(2k)} & , \quad (1 \le 2k \le m)
\end{array} \right.
\end{array} \right\}$$

Proof:

Given the half-integer Bessel ODE, from the main theorem (Theorem I.1) "Exact solution of ODEs Vector Space Transformation Technique, Part 2".:

$$\left\{ \begin{array}{c}
\begin{array}{|l|l|} \hline y_1 = J_{\frac{1}{2}} = \sqrt{\frac{2}{\pi}}\, x^{-\frac{1}{2}} \sin x & y_2 = J_{-\frac{1}{2}} = \sqrt{\frac{2}{\pi}}\, x^{-\frac{1}{2}} \cos x \\ \hline y_1' = J_{\frac{1}{2}}' = -\frac{1}{2}x^{-1}y_1 + y_2 & y_2' = J_{-\frac{1}{2}}' = -y_1 - \frac{1}{2}x^{-1}y_2 \\ \hline \end{array} \\
\Rightarrow y_i'' + \frac{1}{x}y_i' + \left(-\frac{\left(\pm\frac{1}{2}\right)^2}{x^2} - 1^2 \right) y_i = 0 \\
\begin{array}{|l|l|} \hline r_1 = -\frac{1}{2}x^{-1} & s_1 = 1 \\ \hline r_2 = -1 & s_2 = -\frac{1}{2}x^{-1} \\ \hline \end{array} \\
\begin{array}{|l|} \hline (r_2 + s_1) = 0 \\ \hline (r_1 - s_2) = 0 \\ \hline \end{array}
\end{array} \right.$$

$$\Rightarrow \left\{ \begin{array}{|l|} \hline 0 = -2(r_1 - s_2)u' + (\varphi_1 - \varphi_2)u + 2(r_2 + s_1)v' \\ \hline 0 = 2(r_1 - s_2)v' + (\varphi_1 - \varphi_2)v + 2(r_2 + s_1)u' \\ \hline 0 = u'' + [2r_1 + P_2]u' + \varphi_2 u - 2r_2 v' \\ \hline 0 = v'' + [2s_2 + P_2]v' + \varphi_2 v - 2s_1 u' \\ \hline \end{array} \right.$$

$$\left\{ \begin{array}{|l|} \hline 0 = \varphi_1 - \varphi_2 \\ \hline 0 = \varphi_1 - \varphi_2 \\ \hline 0 = u'' + \varphi_1 u + 2v' \\ \hline 0 = v'' + \varphi_1 v - 2u' \\ \hline 0 = 0 \\ \hline 0 = 0 \\ \hline \end{array} \right.$$

So:

$$-\frac{\left(\frac{1}{2}\right)^2}{x^2}+\varphi_1=-\frac{\left(\frac{1+2m}{2}\right)^2}{x^2}\Rightarrow\varphi_1=\frac{1}{x^2}\left(-\frac{1+4m+4m^2}{4}+\frac{1}{4}\right)=-\frac{m(m+1)}{x^2}$$

where $(m\in\mathbb{N})$ represent the half-integer $\lambda's$.

For each half-integer solution choose u_m & v_m as inhomogeneous functions:

$$\left\{\boxed{\begin{array}{l}0=u_m''+\varphi_1u_m+2v_m'\\ 0=v_m''+\varphi_1v_m-2u_m'\end{array}}\right\}\Rightarrow\left\{\boxed{\begin{array}{l}u_m''+\varphi_1u_m=-2v_m'\\ v_m''+\varphi_1v_m=2u_m'\end{array}}\right.$$

$$\Rightarrow\left\{\boxed{\begin{array}{l}u_m''-\dfrac{m(m+1)}{x^2}u_m=-2v_m'\\ v_m''-\dfrac{m(m+1)}{x^2}v_m=2u_m'\end{array}}\right.$$

Let: $u_m=\sum_{n=0}^{m}a_{mn}x^{-n}$, $v_m=\sum_{n=0}^{m}b_{mn}x^{-n}$:

$$\Rightarrow\begin{cases}\left(\sum_{n=0}^{m}a_{mn}x^{-n}\right)''-\dfrac{m(m+1)}{x^2}\left(\sum_{n=0}^{m}a_{mn}x^{-n}\right)=-2\left(\sum_{n=0}^{m}b_{mn}x^{-n}\right)'\\ \left(\sum_{n=0}^{m}b_{mn}x^{-n}\right)''-\dfrac{m(m+1)}{x^2}\left(\sum_{n=0}^{m}b_{mn}x^{-n}\right)=2\left(\sum_{n=0}^{m}a_{mn}x^{-n}\right)'\end{cases}$$

$$\Rightarrow\begin{cases}\left(\sum_{n=0}^{m}(-n)a_{mn}x^{-n-1}\right)'-m(m+1)\left(\sum_{n=0}^{m}a_{mn}x^{-n-2}\right)=-2\left(\sum_{n=0}^{m}(-n)b_{mn}x^{-n-1}\right)\\ \left(\sum_{n=0}^{m}(-n)b_{mn}x^{-n-1}\right)'-m(m+1)\left(\sum_{n=0}^{m}b_{mn}x^{-n-2}\right)=2\left(\sum_{n=0}^{m}(-n)a_{mn}x^{-n-1}\right)\end{cases}$$

$$\Rightarrow\begin{cases}\sum_{n=0}^{m}n(n+1)a_{mn}x^{-n-2}-m(m+1)\sum_{n=0}^{m}a_{mn}x^{-n-2}=2\sum_{n=0}^{m}nb_{mn}x^{-n-1}\\ \sum_{n=0}^{m}n(n+1)b_{mn}x^{-n-2}-m(m+1)\sum_{n=0}^{m}b_{mn}x^{-n-2}=-2\sum_{n=0}^{m}na_{mn}x^{-n-1}\end{cases}$$

$$\Rightarrow\begin{cases}\sum_{n=0}^{m}[n(n+1)-m(m+1)]a_{mn}x^{-n-2}=2\sum_{n=0}^{m}nb_{mn}x^{-n-1}\\ \sum_{n=0}^{m}[n(n+1)-m(m+1)]b_{mn}x^{-n-2}=-2\sum_{n=0}^{m}na_{mn}x^{-n-1}\end{cases}$$

$$\Rightarrow\begin{cases}\sum_{n=0}^{m}[n(n+1)-m(m+1)]a_{mn}x^{-n-2}=2\left(\sum_{n=1}^{m}nb_{mn}x^{-n-1}+0\bullet b_{m0}x^{-0-1}\right)\\ \sum_{n=0}^{m}[n(n+1)-m(m+1)]b_{mn}x^{-n-2}=-2\left(\sum_{n=1}^{m}na_{mn}x^{-n-1}+0\bullet a_{m0}x^{-0-1}\right)\end{cases}$$

$$\Rightarrow\begin{cases}\sum_{n=0}^{m}[n(n+1)-m(m+1)]a_{mn}x^{-n-2}=2\left(\sum_{k=0}^{m-1}(k+1)b_{m(k+1)}x^{-k-2}+0\bullet b_{m0}x^{-0-1}\right)\\ \sum_{n=0}^{m}[n(n+1)-m(m+1)]b_{mn}x^{-n-2}=-2\left(\sum_{k=0}^{m-1}(k+1)a_{m(k+1)}x^{-k-2}+0\bullet a_{m0}x^{-0-1}\right)\end{cases}$$

$$\Rightarrow\begin{cases}\sum_{n=0}^{m}[n(n+1)-m(m+1)]a_{mn}x^{-n-2}=2\sum_{n=0}^{m-1}(n+1)b_{m(n+1)}x^{-n-2}\\ \sum_{n=0}^{m}[n(n+1)-m(m+1)]b_{mn}x^{-n-2}=-2\sum_{n=0}^{m-1}(n+1)a_{m(n+1)}x^{-n-2}\end{cases}$$

$$\Rightarrow\begin{cases}\sum_{n=0}^{m-1}[n(n+1)-m(m+1)]a_{mn}x^{-n-2}+[m(m+1)-m(m+1)]a_{mm}x^{-m-2}=2\sum_{n=0}^{m-1}(n+1)b_{m(n+1)}x^{-n-2}\\ \sum_{n=0}^{m-1}[n(n+1)-m(m+1)]b_{mn}x^{-n-2}+[m(m+1)-m(m+1)]b_{mm}x^{-m-2}=-2\sum_{n=0}^{m-1}(n+1)a_{m(n+1)}x^{-n-2}\end{cases}$$

$$\Rightarrow\begin{cases}\sum_{n=0}^{m-1}[n(n+1)-m(m+1)]a_{mn}x^{-n-2}=2\sum_{n=0}^{m-1}(n+1)b_{m(n+1)}x^{-n-2}\\ \sum_{n=0}^{m-1}[n(n+1)-m(m+1)]b_{mn}x^{-n-2}=-2\sum_{n=0}^{m-1}(n+1)a_{m(n+1)}x^{-n-2}\end{cases}$$

$$\Rightarrow\begin{cases}\sum_{n=0}^{m-1}([n(n+1)-m(m+1)]a_{mn}-2(n+1)b_{m(n+1)})x^{-n-2}=0\\ \sum_{n=0}^{m-1}([n(n+1)-m(m+1)]b_{mn}+2(n+1)a_{m(n+1)})x^{-n-2}=0\end{cases}$$

$$\Rightarrow\begin{cases}[n(n+1)-m(m+1)]a_{mn}-2(n+1)b_{m(n+1)}=0\\ [n(n+1)-m(m+1)]b_{mn}+2(n+1)a_{m(n+1)}=0\end{cases}\quad(0\le n\le m-1)$$

$$\Rightarrow\begin{cases}a_{mn}=\dfrac{2(n+1)}{[n(n+1)-m(m+1)]}b_{m(n+1)}\\ b_{mn}=-\dfrac{2(n+1)}{[n(n+1)-m(m+1)]}a_{m(n+1)}\end{cases}\quad(0\le n\le m-1)$$

$$\Rightarrow \begin{cases} a_{mn} = \dfrac{2(n+1)}{(n-m)(n+m+1)} b_{m(n+1)} \\ b_{mn} = -\dfrac{2(n+1)}{(n-m)(n+m+1)} a_{m(n+1)} \\ \qquad\qquad (0 \le n \le m-1) \end{cases} \Rightarrow \begin{cases} a_{m0} = -\dfrac{2}{m(m+1)} b_{m1} \\ b_{m0} = \dfrac{2}{m(m+1)} a_{m1} \end{cases}$$

$$m = 1 \quad \Rightarrow \left.\begin{cases} a_{10} = -b_{11} \\ b_{10} = a_{11} \end{cases}\right\} \Rightarrow \left.\begin{cases} u_1 = a_{10}x^{-0} + a_{11}x^{-1} \\ v_1 = b_{10}x^{-0} + b_{11}x^{-1} = a_{11}x^{-0} - a_{10}x^{-1} \end{cases}\right\}$$

$$\Rightarrow \begin{cases} \boxed{Y_{11} = v_1 J_{\frac{1}{2}} + u_1 J_{-\frac{1}{2}} = (a_{11}x^{-0} - a_{10}x^{-1})J_{\frac{1}{2}} + (a_{10}x^{-0} + a_{11}x^{-1})J_{-\frac{1}{2}}} \\ \boxed{Y_{21} = u_1 J_{\frac{1}{2}} - v_1 J_{-\frac{1}{2}} = (a_{10}x^{-0} + a_{11}x^{-1})J_{\frac{1}{2}} - (a_{11}x^{-0} - a_{10}x^{-1})J_{-\frac{1}{2}}} \end{cases}$$

$$\Rightarrow \begin{cases} \boxed{Y_{11} = (a_{11} - a_{10}x^{-1})J_{\frac{1}{2}} + (a_{10} + a_{11}x^{-1})J_{-\frac{1}{2}}} \\ \boxed{Y_{21} = (a_{10} + a_{11}x^{-1})J_{\frac{1}{2}} - (a_{11} - a_{10}x^{-1})J_{-\frac{1}{2}}} \end{cases}$$

$$\left.\begin{cases} \boxed{Y_{11} = J_{-\frac{3}{2}}} \\ \boxed{Y_{21} = J_{\frac{3}{2}}} \end{cases}\right\} \Rightarrow \left.\begin{cases} \boxed{(a_{11} - a_{10}x^{-1})J_{\frac{1}{2}} + (a_{10} + a_{11}x^{-1})J_{-\frac{1}{2}} = c_2\left(J_{\frac{1}{2}} + x^{-1}J_{-\frac{1}{2}}\right)} \\ \boxed{(a_{10} + a_{11}x^{-1})J_{\frac{1}{2}} - (a_{11} - a_{10}x^{-1})J_{-\frac{1}{2}} = c_1\left(x^{-1}J_{\frac{1}{2}} - J_{-\frac{1}{2}}\right)} \end{cases}\right\}$$

$$\Rightarrow \left.\begin{cases} \boxed{a_{11} = c_2} \ \boxed{a_{10} = 0} \\ \boxed{a_{11} = c_1} \ \boxed{a_{10} = 0} \end{cases}\right\}$$

$$\left.\begin{cases} \boxed{Y_{11} = J_{\frac{3}{2}}} \\ \boxed{Y_{21} = J_{-\frac{3}{2}}} \end{cases}\right\} \Rightarrow \left.\begin{cases} \boxed{(a_{11} - a_{10}x^{-1})J_{\frac{1}{2}} + (a_{10} + a_{11}x^{-1})J_{-\frac{1}{2}} = c_1\left(x^{-1}J_{\frac{1}{2}} - J_{-\frac{1}{2}}\right)} \\ \boxed{(a_{10} + a_{11}x^{-1})J_{\frac{1}{2}} - (a_{11} - a_{10}x^{-1})J_{-\frac{1}{2}} = c_2\left(J_{\frac{1}{2}} + x^{-1}J_{-\frac{1}{2}}\right)} \end{cases}\right\}$$

$$\Rightarrow \left.\begin{cases} \boxed{a_{11} = 0} \ \boxed{-a_{10} = c_1} \\ \boxed{a_{11} = 0} \ \boxed{a_{10} = c_2} \end{cases}\right\}$$

$$m = 2 \quad \Rightarrow \left.\begin{cases} a_{20} = -\dfrac{2}{2(2+1)} b_{21} = -\dfrac{1}{3} b_{21} \quad a_{21} = \dfrac{2(1+1)}{(1-2)(1+2+1)} b_{22} = -b_{22} \\ b_{20} = \dfrac{2}{2(2+1)} a_{21} = \dfrac{1}{3} a_{21} \qquad b_{21} = -\dfrac{2(1+1)}{(1-2)(1+2+1)} a_{22} = a_{22} \end{cases}\right\}$$

$$\Rightarrow \left\{ a_{22} = b_{21} = -3a_{20} \right\}$$

$$\Rightarrow \left.\begin{cases} u_2 = a_{20}x^{-0} + a_{21}x^{-1} + a_{22}x^{-2} = a_{20}x^{-0} + a_{21}x^{-1} - 3a_{20}x^{-2} = a_{20}(x^{-0} - 3x^{-2}) + a_{21}x^{-1} \\ v_2 = b_{20}x^{-0} + b_{21}x^{-1} + b_{22}x^{-2} = \dfrac{1}{3}a_{21}x^{-0} - 3a_{20}x^{-1} - a_{21}x^{-2} = \dfrac{1}{3}a_{21}(x^{-0} - 3x^{-2}) - 3a_{20}x^{-1} \end{cases}\right\}$$

$$\Rightarrow \left.\begin{cases} \boxed{J_{\frac{5}{2}} = c_1\left[\left(\dfrac{3}{x^2} - 1\right)J_{\frac{1}{2}} - \dfrac{3}{x}J_{-\frac{1}{2}}\right]} \\ \boxed{J_{-\frac{5}{2}} = c_2\left[-\dfrac{3}{x}J_{\frac{1}{2}} - \left(\dfrac{3}{x^2} - 1\right)J_{-\frac{1}{2}}\right]} \end{cases}\right\}$$

$$\left.\begin{cases} \boxed{Y_{11} = J_{-\frac{5}{2}}} \\ \boxed{Y_{21} = J_{\frac{5}{2}}} \end{cases}\right\}$$

$$\Rightarrow \left.\begin{cases} \boxed{\left(\dfrac{1}{3}a_{21}(x^{-0} - 3x^{-2}) - 3a_{20}x^{-1}\right)J_{\frac{1}{2}} + (a_{20}(x^{-0} - 3x^{-2}) + a_{21}x^{-1})J_{-\frac{1}{2}} = c_1\left[\left(\dfrac{3}{x^2} - 1\right)J_{\frac{1}{2}} - \dfrac{3}{x}J_{-\frac{1}{2}}\right]} \\ \boxed{(a_{20}(x^{-0} - 3x^{-2}) + a_{21}x^{-1})J_{\frac{1}{2}} - \left(\dfrac{1}{3}a_{21}(x^{-0} - 3x^{-2}) - 3a_{20}x^{-1}\right)J_{-\frac{1}{2}} = c_2\left[-\dfrac{3}{x}J_{\frac{1}{2}} - \left(\dfrac{3}{x^2} - 1\right)J_{-\frac{1}{2}}\right]} \end{cases}\right\}$$

$$\Rightarrow \begin{cases} \boxed{\dfrac{1}{3}a_{21} = -c_1} \ \boxed{a_{20} = 0} \\ \boxed{-a_{21} = 3c_1} \end{cases} \quad \boxed{a_{20} = 0} \ \boxed{a_{21} = -3c_1} \quad \boxed{a_{20} = 0} \ \boxed{a_{21} = -3c_2} \quad \begin{array}{|c|c|} \hline \dfrac{1}{3}a_{21} = -c_2 & \\ \hline -a_{21} = 3c_2 & a_{20} = 0 \\ \hline \end{array}$$

$$m = 3 \quad \Rightarrow \begin{cases} a_{30} = -\dfrac{2}{3(3+1)} b_{31} = -\dfrac{1}{6} b_{31} \quad a_{31} = \dfrac{2(1+1)}{(1-3)(1+3+1)} b_{32} = -\dfrac{2}{5} b_{32} \quad a_{32} = \dfrac{2(2+1)}{(2-3)(2+3+1)} b_{33} = -b_{33} \\ b_{30} = \dfrac{2}{3(3+1)} a_{31} = \dfrac{1}{6} a_{31} \qquad b_{31} = -\dfrac{2(1+1)}{(1-3)(1+3+1)} a_{32} = \dfrac{2}{5} a_{32} \quad b_{32} = -\dfrac{2(2+1)}{(2-3)(2+3+1)} a_{33} = a_{33} \end{cases}$$

$$\Rightarrow \left\{ a_{33} = b_{32} = -\dfrac{5}{2}a_{31} \quad a_{32} = \dfrac{5}{2}b_{31} = \dfrac{5}{2} \cdot 6a_{30} = 15a_{30} \right\}$$

$$\Rightarrow \left.\begin{cases} u_3 = a_{30}x^{-0} + a_{31}x^{-1} + a_{32}x^{-2} + a_{33}x^{-3} \\ \quad = a_{30}x^{-0} + a_{31}x^{-1} - 15a_{30}x^{-2} - \dfrac{5}{2}a_{31}x^{-3} = a_{30}(x^{-0} - 15x^{-2}) + a_{31}\left(x^{-1} - \dfrac{5}{2}x^{-3}\right) \\ v_3 = b_{30}x^{-0} + b_{31}x^{-1} + b_{32}x^{-2} + b_{33}x^{-3} \\ \quad = \dfrac{1}{6}a_{31}x^{-0} - 6a_{30}x^{-1} - \dfrac{5}{2}a_{31}x^{-2} + 15a_{30}x^{-3} = -6a_{30}\left(x^{-1} - \dfrac{5}{2}x^{-3}\right) + \dfrac{1}{6}a_{31}(x^{-0} - 15x^{-2}) \end{cases}\right\}$$

$$\Rightarrow \left.\begin{cases} \boxed{J_{\frac{7}{2}} = c_1\left[\left(\dfrac{15}{x^3} - \dfrac{6}{x}\right)J_{\frac{1}{2}} + \left(\dfrac{15}{x^2} - 1\right)J_{-\frac{1}{2}}\right]} \\ \boxed{J_{-\frac{7}{2}} = c_2\left[\left(\dfrac{15}{x^2} - 1\right)J_{\frac{1}{2}} - \left(\dfrac{15}{x^3} - \dfrac{6}{x}\right)J_{-\frac{1}{2}}\right]} \end{cases}\right\}$$

$$\left.\begin{cases} \boxed{Y_{11} = J_{-\frac{7}{2}}} \\ \boxed{Y_{21} = J_{\frac{7}{2}}} \end{cases}\right\}$$

$$\Rightarrow\left\{\begin{array}{|c|}\hline \left[-6a_{30}\left(x^{-1}-\frac{5}{2}x^{-3}\right)+\frac{1}{6}a_{31}(x^{-0}-15x^{-2})\right]J_{\frac{1}{2}}+\left[a_{30}(x^{-0}-15x^{-2})+a_{31}\left(x^{-1}-\frac{5}{2}x^{-3}\right)\right]J_{-\frac{1}{2}}= \\ \hline =c_1\left[\left(\frac{15}{x^3}-\frac{6}{x}\right)J_{\frac{1}{2}}-\left(\frac{15}{x^2}-1\right)J_{-\frac{1}{2}}\right] \\ \hline \left[a_{30}(x^{-0}-15x^{-2})+a_{31}\left(x^{-1}-\frac{5}{2}x^{-3}\right)\right]J_{\frac{1}{2}}-\left[-6a_{30}\left(x^{-1}-\frac{5}{2}x^{-3}\right)+\frac{1}{6}a_{31}(x^{-0}-15x^{-2})\right]J_{-\frac{1}{2}}= \\ \hline =c_2\left[\left(\frac{15}{x^2}-1\right)J_{\frac{1}{2}}+\left(\frac{15}{x^3}-\frac{6}{x}\right)J_{-\frac{1}{2}}\right] \\ \hline\end{array}\right\}$$

$$\Rightarrow\left\{\begin{array}{|l|l|}\hline -6a_{30}=-6c_1 & a_{31}=0 \\ \hline 15a_{30}=15c_1 & \\ \hline a_{30}=c_1 & -15a_{30}=-15c_1 \\ \hline a_{31}=0 & \\ \hline\end{array}\quad\begin{array}{|l|l|}\hline a_{30}=-c_2 & a_{31}=0 \\ \hline -15a_{30}=15c_2 & \\ \hline 6a_{30}=-6c_2 & \\ \hline -15a_{30}=15c_2 & a_{31}=0 \\ \hline\end{array}\right\}$$

Continuing:

$$\left\{\begin{array}{l} a_{mn}=\frac{2(n+1)}{(n-m)(n+m+1)}b_{m(n+1)} \\ b_{mn}=-\frac{2(n+1)}{(n-m)(n+m+1)}a_{m(n+1)} \\ \qquad (0\le n\le m-1)\end{array}\right.\quad\Rightarrow\left\{\begin{array}{l} a_{m0}=-\frac{2}{m(m+1)}b_{m1} \\ b_{m0}=\frac{2}{m(m+1)}a_{m1}\end{array}\right.$$

$$\Rightarrow\left\{\begin{array}{l} a_{m(n-1)}=\frac{2((n-1)+1)}{((n-1)-m)((n-1)+m+1)}b_{mn} \\ \qquad (0\le n-1\le m-2) \\ b_{mn}=-\frac{2(n+1)}{(n-m)(n+m+1)}a_{m(n+1)} \\ \qquad (0\le n\le m-1)\end{array}\right.$$

$$\Rightarrow\left\{\begin{array}{l} a_{m(n-1)}=-\frac{4n(n+1)}{(n-1-m)(n+m)(n-m)(n+m+1)}a_{m(n+1)} \\ \qquad (1\le n\le m-1) \\ b_{mn}=-\frac{2(n+1)}{(n-m)(n+m+1)}a_{m(n+1)} \\ \qquad (0\le n\le m-1)\end{array}\right.$$

$$\Rightarrow\left\{\begin{array}{l} a_{m(n-1)}=-\frac{4n(n+1)}{(n+m)(n-m)(n+(m+1))(n-(m+1))}a_{m(n+1)} \\ \qquad (1\le n\le m-1) \\ b_{mn}=-\frac{2(n+1)}{(n-m)(n+m+1)}a_{m(n+1)} \\ \qquad (0\le n\le m-1)\end{array}\right.$$

$$\Rightarrow\left\{\begin{array}{l} a_{m(n-1)}=-\frac{4n(n+1)}{(n^2-m^2)\left(n^2-(m+1)^2\right)}a_{m(n+1)} \\ \qquad (1\le n\le m-1) \\ b_{mn}=-\frac{2(n+1)}{(n-m)(n+m+1)}a_{m(n+1)} \\ \qquad (0\le n\le m-1)\end{array}\right.$$

$$\Rightarrow\left\{\begin{array}{l} b_{m0}=\frac{2}{m(m+1)}a_{m1} \\ b_{m1}=-\frac{m(m+1)}{2}a_{m0} \\ \qquad\qquad (m=1) \\ a_{mh}=-\frac{4(h+1)((h+1)+1)}{\left((h+1)^2-m^2\right)\left((h+1)^2-(m+1)^2\right)}a_{m((h+1)+1)}\qquad (h=n-1) \\ \qquad (1\le h+1\le m-1) \\ b_{mn}=-\frac{2(n+1)}{(n-m)(n+m+1)}a_{m(n+1)} \\ \qquad (0\le n\le m-1)\end{array}\right.$$

$$\Rightarrow\left\{\begin{array}{l} b_{m0}=\frac{2}{m(m+1)}a_{m1} \\ b_{m1}=-\frac{m(m+1)}{2}a_{m0} \\ \qquad\qquad (m=1) \\ a_{mh}=-\frac{4(h+1)(h+2)}{\left((h+1)^2-m^2\right)\left((h+1)^2-(m+1)^2\right)}a_{m((h+1)+1)}\qquad (h=n-1) \\ \qquad (0\le h\le m-2) \\ b_{mn}=-\frac{2(n+1)}{(n-m)(n+m+1)}a_{m(n+1)} \\ \qquad (0\le n\le m-1)\end{array}\right.$$

$$\Rightarrow \begin{cases} b_{m0} = \dfrac{2}{m(m+1)} a_{m1} \\ b_{m1} = -\dfrac{m(m+1)}{2} a_{m0} \\ \qquad\qquad\qquad (m = 1) \\ a_{m(n+2)} = -\dfrac{\left((n+1)^2 - m^2\right)\left((n+1)^2 - (m+1)^2\right)}{4(n+1)(n+2)} a_{mn} \\ \qquad (0 \le n \le m-2) \\ b_{mn} = -\dfrac{2(n+1)}{(n-m)(n+m+1)} a_{m(n+1)} \\ \qquad (0 \le n \le m-1) \end{cases}$$

Let:

$$W_{mn} \equiv \frac{\left((n+1)^2 - m^2\right)\left((n+1)^2 - (m+1)^2\right)}{4(n+1)(n+2)} \quad , \quad (0 \le n \le m-2)$$

$$= \frac{(n-m)(n-m+1)(n+m+1)(n+m+2)}{4(n+1)(n+2)} \quad , \quad (0 \le n \le m-2)$$

$$\Rightarrow a_{m(n+2)} = (-1)^1 W_{mn} a_{mn}$$

$$= (-1)^1 \frac{(n-m)(n-m+1)(n+m+1)(n+m+2)}{4(n+1)(n+2)} a_{mn} \quad , \quad (0 \le n \le m-2)$$

$$\Rightarrow a_{m(n+4)} = (-1)^1 W_{m(n+2)} a_{m(n+2)} \quad , \quad (0 \le n \le m-4)$$

$$= (-1)^1 W_{m(n+2)} W_{mn} a_{mn} \quad , \quad (0 \le n \le m-4)$$

$$= (-1)^1 \frac{((n+2)-m)((n+2)-m+1)((n+2)+m+1)((n+2)+m+2)}{4((n+2)+1)((n+2)+2)} a_{m(n+2)}$$

$$, \quad (0 \le n \le m-4)$$

$$= (-1)^1 \frac{(n-m+2)(n-m+3)(n+m+3)(n+m+4)}{4(n+3)(n+4)} a_{m(n+2)} \quad , \quad (0 \le n \le m-4)$$

$$= (-1)^1 \frac{(n-m+2)(n-m+3)(n+m+3)(n+m+4)}{4(n+3)(n+4)} (-1)^1 \frac{(n-m)(n-m+1)(n+m+1)(n+m+2)}{4(n+1)(n+2)} a_{mn}$$

$$, \quad (0 \le n \le m-4)$$

$$= \left(-\frac{1}{4}\right)^2 \frac{\left(\dfrac{(n-m+(4-1))!}{(n-m-1)!}\right)\left(\dfrac{(n+m+4)!}{(n+m)!}\right)}{\left(\dfrac{(n+4)!}{n!}\right)} a_{mn} \quad , \quad (0 \le n \le m-4)$$

$$\Rightarrow a_{m(n+6)} = -W_{m(n+4)} a_{m(n+4)} = (-1)^3 W_{m(n+4)} W_{m(n+2)} W_{mn} a_{mn} \quad , \quad (0 \le n \le m-6)$$

$$= \left(-\frac{1}{4}\right)^3 \frac{\left(\dfrac{(n-m+(6-1))!}{(n-m-1)!}\right)\left(\dfrac{(n+m+6)!}{(n+m)!}\right)}{\left(\dfrac{(n+6)!}{n!}\right)} a_{mn} \quad , \quad (0 \le n \le m-6)$$

$$\vdots$$

$$\Rightarrow a_{m(n+2k)} = -W_{m(n+2(k-1))} a_{m(n+2(k-1))} = (-1)^k W_{m(n+2(k-1))} \cdots W_{mn} a_{mn} \quad , \quad (0 \le n \le m-2k)$$

$$= \left(-\frac{1}{4}\right)^k \frac{\left(\dfrac{(n-m+(2k-1))!}{(n-m-1)!}\right)\left(\dfrac{(n+m+2k)!}{(n+m)!}\right)}{\left(\dfrac{(n+2k)!}{n!}\right)} a_{mn} \quad , \quad (0 \le n \le m-2k)$$

So:

$$a_{m(0+2k)} = \left(-\frac{1}{4}\right)^k \frac{\left(\dfrac{(0-m+(2k-1))!}{(0-m-1)!}\right)\left(\dfrac{(0+m+2k)!}{(0+m)!}\right)}{\left(\dfrac{(0+2k)!}{0!}\right)} a_{m0} \quad , \quad (0 \le m-2k)$$

$$= \left(-\frac{1}{4}\right)^k \frac{\left(\dfrac{(m+(2k-1))!}{-(m+1)!}\right)\left(\dfrac{(m+2k)!}{m!}\right)}{\left(\dfrac{(2k)!}{0!}\right)} a_{m0} \quad , \quad (2k \le m)$$

$$\Rightarrow a_{m(2k)} = \left(-\frac{1}{4}\right)^k \frac{\left(\dfrac{(m+(2k-1))!}{-(m+1)!}\right)\left(\dfrac{(m+2k)!}{m!}\right)}{(2k)!} a_{m0} \quad , \quad (2k \le m)$$

$$a_{m(1+2k)} = \left(-\frac{1}{4}\right)^k \frac{\left(\dfrac{(1-m+(2k-1))!}{(1-m-1)!}\right)\left(\dfrac{(1+m+2k)!}{(1+m)!}\right)}{\left(\dfrac{(1+2k)!}{1!}\right)} a_{m1} \quad , \quad (1+2k \le m)$$

$$\Rightarrow a_{m(1+2k)} = \left(-\frac{1}{4}\right)^k \frac{\left(\dfrac{(1-m+(2k-1))!}{-m!}\right)\left(\dfrac{(1+m+2k)!}{(1+m)!}\right)}{(1+2k)!} a_{m1} \quad , \quad (1+2k \le m)$$

$$\Rightarrow b_{m(2k)} = \frac{2(2k+1)}{(2k-m)(2k+m+1)} a_{m(2k+1)} \quad , \quad (0 \le 2k \le m-1)$$

$$\Rightarrow b_{m(2k)} = \frac{2(2k+1)}{(2k-m)(2k+m+1)} \left(-\frac{1}{4}\right)^k \frac{\left(\dfrac{(1-m+(2k-1))!}{-m!}\right)\left(\dfrac{(1+m+2k)!}{(1+m)!}\right)}{(1+2k)!} a_{m1}$$

$$, \quad (0 \le 2k \le m-1)$$

$$\Rightarrow b_{m(2k)} = \left(-\frac{1}{4}\right)^k \frac{2\left(\dfrac{(1-m+(2k-1))!}{-m!}\right)\left(\dfrac{(m+2k)!}{(1+m)!}\right)}{(2k-m)(2k)!} a_{m1}$$

$$, \quad (0 \le 2k \le m-1)$$

$$\Rightarrow b_{m(2k-1)} = -\frac{4k}{(2k-1-m)(2k+m)} a_{m(2k)} \quad , \quad (1 \le 2k \le m)$$

$$\Rightarrow b_{m(2k-1)} = -\frac{4k}{(2k-1-m)(2k+m)}\left(-\frac{1}{4}\right)^k \frac{\left(\frac{(m+(2k-1))!}{-(m+1)!}\right)\left(\frac{(m+2k)!}{m!}\right)}{(2k)!} a_{m0} \quad , \quad (1 \le -2k \le m)$$

$$\Rightarrow b_{m(2k-1)} = -\left(-\frac{1}{4}\right)^k \frac{2\left(\frac{(m+(2k-1))!}{-(m+1)!}\right)\left(\frac{(m+2k-1)!}{m!}\right)}{(2k-1-m)(2k-1)!} a_{m0} \quad , \quad (1 \le 2k \le m)$$

$$\Rightarrow \left\{ \begin{array}{l} \left\{ \boxed{\begin{array}{ll} b_{m0} = \frac{2}{m(m+1)} a_{m1} & \\ b_{m1} = -\frac{m(m+1)}{2} a_{m0} & , \quad (m = 1) \end{array}} \right. \\ \left\{ \boxed{\begin{array}{ll} a_{m(2k)} = \left(-\frac{1}{4}\right)^k \frac{\left(\frac{(m+(2k-1))!}{-(m+1)!}\right)\left(\frac{(m+2k)!}{m!}\right)}{(2k)!} a_{m0} & , \quad (0 \le 2k \le m) \\ a_{m(1+2k)} = \left(-\frac{1}{4}\right)^k \frac{\left(\frac{(1-m+(2k-1))!}{-m!}\right)\left(\frac{(1+m+2k)!}{(1+m)!}\right)}{(1+2k)!} a_{m1} & , \quad (1 \le 2k \le m-1) \\ b_{m(2k)} = \left(-\frac{1}{4}\right)^k \frac{2\left(\frac{(1-m+(2k-1))!}{-m!}\right)\left(\frac{(m+2k)!}{(1+m)!}\right)}{(2k-m)(2k)!} a_{m1} & , \quad (0 \le 2k \le m-1) \\ b_{m(2k-1)} = -\left(-\frac{1}{4}\right)^k \frac{2\left(\frac{(m+(2k-1))!}{-(m+1)!}\right)\left(\frac{(m+2k-1)!}{m!}\right)}{(2k-1-m)(2k-1)!} a_{m0} & , \quad (1 \le 2k \le m) \end{array}} \right. \end{array} \right\}$$

So, the half-integer Bessel ODE solutions may be written:

$$\Rightarrow \left\{ \begin{array}{l} \boxed{Y_{1m} = u_m J_{\frac{1}{2}} - v_m J_{-\frac{1}{2}}} \\ \boxed{Y_{2m} = v_m J_{\frac{1}{2}} + u_m J_{-\frac{1}{2}}} \\ \boxed{u_m = \sum_{k=0}^{\frac{m}{2}} a_{m(2k)} x^{-2k} + \sum_{k=0}^{\frac{m}{2}} a_{m(2k+1)} x^{-(2k+1)}} \\ \boxed{v_m = \sum_{k=0}^{\frac{m}{2}} b_{m(2k)} x^{-2k} + \sum_{k=0}^{\frac{m}{2}} b_{m(2k+1)} x^{-(2k+1)}} \\ \text{where}: \\ \left\{ \boxed{\begin{array}{ll} b_{m0} = \frac{2}{m(m+1)} a_{m1} & \\ b_{m1} = -\frac{m(m+1)}{2} a_{m0} & , \quad (m = 1) \end{array}} \right. \\ \left\{ \boxed{\begin{array}{ll} a_{m(2k)} = \left(-\frac{1}{4}\right)^k \frac{\left(\frac{(m+(2k-1))!}{-(m+1)!}\right)\left(\frac{(m+2k)!}{m!}\right)}{(2k)!} a_{m0} & , \quad (0 \le 2k \le m) \\ a_{m(1+2k)} = \left(-\frac{1}{4}\right)^k \frac{\left(\frac{(1-m+(2k-1))!}{-m!}\right)\left(\frac{(1+m+2k)!}{(1+m)!}\right)}{(1+2k)!} a_{m1} & , \quad (1 \le 2k \le m-1) \\ b_{m(2k)} = \frac{2(2k+1)}{(2k-m)(2k+m+1)} a_{m(2k+1)} & , \quad (0 \le 2k \le m-1) \\ b_{m(2k-1)} = -\frac{4k}{(2k-1-m)(2k+m)} a_{m(2k)} & , \quad (1 \le 2k \le m) \end{array}} \right. \end{array} \right\}$$

□

The transformation: $z = \nu x$
of the HLODE:

$$\frac{d^2}{dz^2} Y_i(z) + \frac{1}{z}\frac{d}{dz} Y_i'(z) + \left[-\frac{\lambda^2}{z^2} - 1\right] Y_i(z) = 0$$

yields the general Bessel HLODE:

$$\frac{d^2}{dx^2} Y_i + \frac{1}{x}\frac{d}{dx} Y_i' + \left[-\frac{\lambda^2}{x^2} - \nu^2\right] Y_i = 0$$

Thus, the above simplified Bessel HLODE is equivalent to transforming to the general Bessel HLODE - for any λ ; so, in particular, for $\lambda = \frac{1}{2}$.