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ASSUMING $c < rad^2(abc)$ IMPLIES THE ABC CONJECTURE TRUE

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Abstract. In this paper about the *abc* conjecture, assuming the condition $c < rad^2(abc)$ holds, and the constant $K(\epsilon)$ is a smooth function, having a derivative for $\epsilon \in]0,1[$, then we give the proof of the *abc* conjecture.

To the memory of my Father who taught me arithmetic
To my wife ***Wahida****, my daughter* ***Sinda*** *and my son* ***Mohamed Mazen***

1. Introduction and notations

Let a positive integer $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as :

$$a = \prod_i a_i^{\alpha_i} = rad(a).\prod_i a_i^{\alpha_i - 1} \tag{1}$$

We note:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{2}$$

The *abc* conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) [1]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the *abc* conjecture is given below:

Conjecture 1.1. *(**abc Conjecture**): Let a, b, c positive integers relatively prime with $c = a + b$, then for each $\epsilon > 0$, there exists a constant $K(\epsilon)$ such that :*

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \tag{3}$$

$K(\epsilon)$ depending only of ϵ.

The idea to try to write a paper about this conjecture was born after the publication of an article in Quanta magazine, in September 2018, about the remarks of professors Peter Scholze of the University of Bonn and Jakob Stix

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of Goethe University Frankfurt concerning the proof of Shinichi Mochizuki [2]. The difficulty to find a proof of the *abc* conjecture is due to the incomprehensibility how the prime factors are organized in c giving a, b with $c = a + b$.

We know that numerically, $\dfrac{Logc}{Log(rad(abc))} \leq 1.629912$ [1]. A conjecture was proposed that $c < rad^2(abc)$ [3]. It is the key to resolve the *abc* conjecture. In my paper, we assume that the last conjecture holds, and the constant $K(\epsilon)$ for $\epsilon \in]0,1[$ is a smooth function having a derivative for $\epsilon \in]0,1[$. The paper is organized as follows: in the second section, we begin by presenting some properties of the constant $K(\epsilon)$, then we give the proof of the *abc* conjecture.

2. The Proof of the *abc* Conjecture

Let a, b, c positive integers relatively prime with $c = a + b, a > b, b \geq 2$. We denote $R = rad(abc), I =]0,1[$. For $c < R$, it is trivial that the *abc* conjecture holds. In the following, we consider the triples (a, b, c) with a, b, c relatively coprime and $c > R$. As we assume that $c < R^2$, it follows that $\forall\, \epsilon \geq 1$, it suffices to take $K(\epsilon) = 1$ and c satisfies $c < K(\epsilon)R^{1+\epsilon}$ and the *abc* conjecture is true.

2.1. Properties of the constant $K(\epsilon)$

- From the definition of the *abc* conjecture, above, the constant $K(\epsilon)$ is a positive real number, and for every $\epsilon > 0$, it exists a number $K(\epsilon)$ dependent only of ϵ.

- In the following, we consider that $\epsilon \in I$. We can say that K is a function $K : \epsilon \in I \longrightarrow K(\epsilon) \in]0, +\infty[$, so that $c < K(\epsilon)R^{1+\epsilon}$ holds, if the *abc* conjecture is true. Assuming that $c < R^2$ is satisfying, we can adopt that $K(\epsilon = 1) = 1$, because $c < K(1)R^{1+1}$. Then we choose $K(\epsilon)$ so that $lim_{\epsilon \longrightarrow 1^- = K(1)}$

- We obtain also that $K(\epsilon) > 1$ if $\epsilon \in I$. If not, we consider the example $9 = 8 + 1$, we take $\epsilon = 0.2$, then $c < K(0.2)R^{1.02} < 1.R^{1.2}$. But $c = 9 > 6^{1.2} \approx 8.58$, then the contradiction.

- In 1996, A. Nitaj had confirmed that the constant $K(\epsilon)$ verifies [4]:

$$lim_{\epsilon \longrightarrow 0} K(\epsilon) = +\infty \tag{4}$$

It follows that the function $K(\epsilon)$ is very large when ϵ is very small.

2.2. The proof of the *abc* conjecture

Proof. Let us suppose that $K(\epsilon)$ is a smooth function having a derivative in every point $\in]0,1[$. Let a, b, c positive integers relatively prime with $c = a + b, c > R$. We denote :

$$Y_c(\epsilon) = LogK(\epsilon) + (1 + \epsilon)LogR - Logc \tag{5}$$

We obtain $lim_{\epsilon\to 1} Y_c(\epsilon) = 2LogR - Logc = y_1 > 0$, assuming $c < R^2$, and $lim_{\epsilon\longrightarrow 0} Y_c(\epsilon) = +\infty$. The derivative of $Y_c(\epsilon)$ gives:

$$Y_c'(\epsilon) = \frac{K'(\epsilon)}{K(\epsilon)} + LogR \tag{6}$$

We have the following cases:

i)- If $Y_c'(\epsilon) > 0$ for all $\epsilon \in]0,1[$, then Y is an increasing function of ϵ. It follows the contradiction because $lim_{\epsilon\longrightarrow 0} Y_c(\epsilon) = +\infty$.

ii) - If $Y_c'(\epsilon) < 0$ for all $\epsilon \in]0,1[$, then Y is a decreasing function of ϵ. It follows $\forall\,\epsilon,\ Y_c(\epsilon) > 0 \Longrightarrow c < K(\epsilon)R^{1+\epsilon}$ is satisfied. As c is an arbitrary integer with the condition $c > R$, we deduce that the abc conjecture is true.

iii) - If $Y_c'(\epsilon) = 0$ for some $\epsilon_0 \in]0,1[$. ϵ_0 is a solution of the equation :

$$-\frac{K'(\epsilon_0)}{K(\epsilon_0)} = LogR$$

We remark that ϵ_0 depends of R, then of a, b, c.

* If $Y_c(\epsilon_0)$ is positive, then $0 < Y_c(\epsilon_0) \leq Y_c(\epsilon) \Longrightarrow Y_c(\epsilon) > 0$. As above, we deduce that the abc conjecture holds for the triplet (a, b, c).

** If $Y_c(\epsilon_0)$ is negative, then it exists two values ϵ_1, ϵ_2 with $0 < \epsilon_1 < \epsilon_0 < \epsilon_2 < 1$, so that $Y_c(\epsilon_1) = Y_c(\epsilon_2) = 0$. It follows for example, that $c = K(\epsilon_1)R^{\epsilon_1}.rad(abc)$. Suppose that $K(\epsilon_1)R^{\epsilon_1}$ is an integer, we obtain that a, b, c are not coprime. Then the contradiction and this case to reject.

Then, we have obtained that the abc conjecture holds for $\forall\epsilon \in I$ for the triplet (a, b, c), as it is chosen arbitrary with the condition $c > rad(abc)$. It follows that the abc conjecture is true, assuming that $c < R^2$. □

Q.E.D
End of the mystery!

3. Conclusion

Finally, assuming $c < R^2$, and choosing the constant $K(\epsilon)$ as a smooth function, having a derivative for $\epsilon \in]0,1[$, we have given an elementary proof that the abc conjecture is true.

We can announce the important theorem:

Theorem 3.1. *Let a, b, c positive integers relatively prime with $c = a + b$, assuming $c < rad^2(abc)$, then for each $\epsilon > 0$, there exists $K(\epsilon)$ such that :*

$$c < K(\epsilon).rad(abc)^{1+\epsilon} \tag{7}$$

where $K(\epsilon)$ is a constant depending only of ϵ and varying smoothly, having a derivative.

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