Chern-Simons Theory and Its Applications

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Abstract
Chern-Simons theory is a gauge theory in $2+1$ dimensional spaceime. This theory does not depend on additional structures, like a metric structure, thus it is a topological quantum theory that measures topological invariants like linking numbers, Jones polynomial, and other quantum invariants for knots and 3-manifolds. The equations of motion of Chern-Simons action is vanishing of the curvature $F = 0$. No metric is used in forming the action principle. One might expect the path integral to be a topological invariant of 3 manifolds. The difference for the equation of motion with the Maxwell theory is that the Maxwell theory has non-trivial solution of curvature $F\ne0$ in absence of matter, while the Chern-Simons theory has solution only with $F=0$. The Chern-Simons theory has non-trivial solution with $F \ne0$ only when the gauge field couples with matter. Since the action functional of the Chern-Simons theory is first order in space-time derivatives, its Legendre transform gives the trivial Hamiltonian $H=0$. So there is no dynamics, and the only dynamics would be inherited from coupling to dynamical matter fields.

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Introduction

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We can use the Chern-Simons theory with $U(1)$-gauge group for interpretation of linking numbers in knot theory, this number is topological invariant for links in 3-manifolds. The non-trivial linking numbers of link arise from non-trivial flat connections in Chern-Simons for which the spatial surface to have non-trivial topology.

Since the Chern-Simons action does not depend on the metric, its energy-momentum tensor is zero,

$$T^{\mu\nu} = -2\frac{\delta S_{CS}}{\delta g_{\mu\nu}} = 0.$$

The sourceless stress energy tensor therefore vanishes, so this theory is completely invariant under space-time diffeomorphisms and can therefore be called topological theory, i.e. independent of local geometry. But if we couple the gauge field with particles, then the energy levels and spins of particles can be shifted from their classical values.

1 Chern-Simons theory of $U(1)$

Let $P = M_4 \times U(1)$ be the trivial principal $U(1)$-bundle on a compact oriented smooth 4-manifold M_4 with boundary $M_3 = \partial M_4$. Since P is trivial, we may regard the connection A as $\mathfrak{u}(1)$-valued 1-form over M_4. In this case, $U(1)$ group, we can easily obtain the Chern-Simons term $\int_{M_3} A \wedge dA$ by using Stokes theorem from the topological action $\int_{M_4} F \wedge F$ with the curvature F of a connection A on P. The equation of motion of the Chern-Simons action requires flat connections on the boundary M_3, so that the restriction $P|_{M_3}$ of P to M_3 is flat principal $U(1)$-bundle. (But in over the bounding 4-manifold M_4, this topological term depends only on the boundary value of the curvature and in particular not flat solutions over the interior of M_4, and so we have the boundary condition $F(A)|_{\partial M_4} = 0$. This condition is precisely the equation of motion of Chern-Simons action on the boundary $M_3 = \partial M_4$). We start with the topological term

$$\int_{M_4} F \wedge F = \int_{M_4} dA \wedge dA = \int_{M_4} d\left(A \wedge dA\right).$$

This term is gauge invariant since F is invariant under $U(1)$-gauge transformations, and the cohomology class $[F \wedge F] \in H^2_{DR}(M)$ is also invariant under one-parameter family of diffeormorphisms since $dF = 0$ and so it changes by addition of an exact form under diffeomorphisms. By integrating it over M_4 with boundary $M_3 = \partial M_4$, we get

$$\int_{M_4} F \wedge F = \int_{M_4} d\left(A \wedge dA\right) = \int_{M_3} A \wedge dA.$$

Definition 1.1. Let $\mathcal{A}$ be the space of all smooth $U(1)$-connections on the trivial principal $U(1)$-bundle $P = M_3 \times U(1) \rightarrow M_3$. Then the $U(1)$-Chern-Simons functional $S_{CS} : \mathcal{A} \to \mathbb{R}$ is defined to be

$$S_{CS}(A) = \frac{k}{2\pi} \int_{M_3} A \wedge dA = \frac{k}{2\pi} \int_{M_3} A \wedge F, \tag{1}$$

where k is some constant factor which will be discussed later for non-abelian gauge group.

Let $\mathcal{G}$ be the group of all $U(1)$-gauge transformations $g : P \to P$. Now the $U(1)$-gauge transformation can be identified with a map $g \in Map(M_3, U(1))$. Then $\mathcal{G}$ acts on $\mathcal{A}$ by $g^*A = g^{-1}Ag + g^{-1}dg$.

Proposition 1.1. *The $U(1)$-Chern-Simons functional $S_{CS} : \mathcal{A} \to \mathbb{R}$ is invariant under $U(1)$-gauge transformations, $S_{CS}(g^*A) = S_{CS}(A)$ for all $A \in \mathcal{A}$ and $g \in \mathcal{G}$.*

Proof. Under infinitesimal gauge transformations, $A \to A+d\lambda$, with $\lambda \in \Omega^0(M_3; \mathfrak{u}(1))$, by using $dF = 0$, we obtain

$$S_{CS}(A+d\lambda) = \frac{k}{2\pi} \int_{M_3} (A + d\lambda) \wedge d(A + d\lambda) = \frac{k}{2\pi} \int_{M_3} A \wedge F + \frac{k}{2\pi} \int_{M_3} (d\lambda) \wedge F$$

$$= \frac{k}{2\pi} \int_{M_3} A \wedge F + \frac{k}{2\pi} \int_{M_3} d(\lambda F).$$

If $\partial M_3 = \emptyset$, we have $\int_{M_3} d(\lambda F) = 0$, therefore $S_{CS}(A + d\lambda) = S_{CS}(A)$ for infinitesimal gauge transformations in which $A' - A = d\lambda$ is exact. In large gauge transformations, $g^*A = g^{-1}Ag + \omega$ for not necessarily exact $\omega = g^{-1}dg \in \Omega^1(M_3; \mathfrak{u}(1))$ with $g : M_3 \to U(1)$, the action changes as (F is gauge invariant)

$$S_{CS}(g^*A) = S_{CS}(A) + \frac{k}{2\pi} \int_{M_3} \omega \wedge F,$$

we write

$$\int_{M_3} \omega \wedge F = \int_{M_3} \omega \wedge dA = - \int_{M_3} d(\omega \wedge A) = - \int_{\partial M_3} \omega \wedge A = 0.$$

□

Proposition 1.2. *The critical points set of $S_{CS} : \mathcal{A} \to \mathbb{R}$ is precisely the set $\mathcal{F}$ of all flat connections A. In fact A is given by $A = g^{-1}dg$ over a contractible open set U for some smooth map $g : U \to U(1)$ called pure gauge solution.*

Proof. From the action (1), we obtain (by ignoring the boundary, or take $\partial M_3 = \emptyset$)

$$\begin{aligned}
\frac{d}{dt}\bigg|_{t=0} S_{CS}(A+t\alpha) &= \frac{k}{2\pi}\frac{d}{dt}\bigg|_{t=0}\int_{M_3}(A+t\alpha)\wedge F_{A+t\alpha} \\
&= \frac{k}{2\pi}\int_{M_3}\frac{d}{dt}\bigg|_{t=0}(A+t\alpha)\wedge(F_A+td\alpha) \\
&= \frac{k}{2\pi}\int_{M_3}\frac{d}{dt}\bigg|_{t=0}(A\wedge F_A+tA\wedge d\alpha+t\alpha\wedge F_A+t^2\alpha\wedge d\alpha) \\
&= \frac{k}{2\pi}\int_{M_3}(\alpha\wedge F_A+A\wedge d\alpha),
\end{aligned}$$

then we use $d(A\wedge\alpha) = dA\wedge\alpha - A\wedge d\alpha$ and $F_A = dA$ to get

$$\frac{d}{dt}\bigg|_{t=0} S_{CS}(A+t\alpha) = \frac{k}{\pi}\int_{M_3}\alpha\wedge F_A,$$

and therefore

$$\frac{d}{dt}\bigg|_{t=0} S_{CS}(A+t\alpha) = 0 \text{ for all } \alpha\in\Omega^1(M_3;\mathfrak{g}) \text{ implies } F_A = 0,$$

thus we have flat connection, $F_A = 0$. Then the A-parallel transport give a trivialization $P|_U \cong U\times U(1)$ and let $g: U\to U(1)$ be the transition function to the original trivialization $P|_U = U\times U(1)$ then $A = g^*0 = g^{-1}dg$. □

The holonomy $Hol_\gamma(A) = \exp\int_\gamma A$ of the flat connection A on M_3 depends only on the homology class $[\gamma]\in H_1(M_3;\mathbb{Z})$ of loops γ based at x_0 inducing a homomorphim $\rho: H_1(M_3;\mathbb{Z})\to U(1)$. In fact if $[\gamma_1] = [\gamma_2]\in H_1(M_3;\mathbb{Z})$, then there exist a 2-chain S such that

$$\int_{\gamma_2}A - \int_{\gamma_1}A = \int_{\partial S}A = \int_S dA = \int_S F_A = 0.$$

Conversely, if a representation $\rho: H_1(M_3;\mathbb{Z})\to U(1)$ is given, let $\tilde{M}_3$ be the universal covering $\pi:\tilde{M}_3\to M_3$ of M_3, and $\tilde{P}\to\tilde{M}_3$ be the pullback bundle

$\tilde{P} = \pi^* P \to M_3$, then $P = \tilde{P}/H_1(M_3;\mathbb{Z})$, where the acting of the homology group $H_1(M_3;\mathbb{Z})$ on $\tilde{M}_3 \times U(1)$ is given by

$$[\gamma]\cdot(\tilde{x}, g) = (\gamma\cdot\tilde{x}, \rho(\gamma)g)\,, \text{ for } \tilde{x}\in\tilde{M}_3, \quad [\gamma]\in H_1(M_3;\mathbb{Z}).$$

Since the parallel transport gives a trivialization $\tilde{P} \cong \tilde{M}_3 \times U(1)$, the trivial connection on $\tilde{P}$ induces a flat connection A on P.

Now we introduce a current $J \in \Omega^2(M_3;\mathfrak{u}(1))$ to add a source term to $S_{CS}(A)$ as follows ([1])

$$S_{CS}(A,J) = \frac{k}{2\pi}\int_{M_3} A\wedge dA + \int_{M_3} A\wedge J. \tag{2}$$

Then the equation of motion becomes

$$\frac{\delta}{\delta A(x)} S_{CS}(A,J) = \frac{k}{\pi}F(x) + J(x) = 0 \Rightarrow F = -\frac{\pi}{k}J,$$

this is linear relation between the curvature F and the source $J \in \Omega^2(M_3;\mathfrak{u}(1))$, also since $dF = 0$, the source is also closed $dJ = 0$ and not necessarily exact for non-trivial principal $U(1)$-bundle.

The Chern-Simons theory is used for interpretation or definition of topological invariants, like Wilson loops, Jones polynomial, or other quantum invariants for knots or 3-manifolds. We see this by using the path integral of Chern-Simons theory to calculate the topological invariants ([2]). We give an example about calculating the linking numbers of Wilson loops.

Let γ be an oriented loop in M_3 and N be a tubular neighborhood of γ with a trivialization $\varphi : N \cong \gamma \times D^2$ called framing of γ. Let $J \in \Omega^2(M_3)$ be a smooth 2-form with support $\operatorname{supp} J \subset N$ of the form $J = \varphi^*(f r dr \wedge d\theta)$ with an oriented coordinates t of γ and polar coordinates (r,θ) of D^2 for a smooth function f of $\operatorname{supp} f \subset D^2$ satisfying $\int_{D^2} f r dr \wedge d\theta = 1$. This 2-form J is called a flux tube of γ with framing φ. Then we have the following

Theorem 1.1. *Let M_3 be a closed oriented 3-manifold with $H_1(M_3;\mathbb{Z}) = \{0\}$. Let $\{\gamma_\alpha\}$ be a mutually disjoint oriented loops in M_3 with framings. Let $J = \sum_\alpha n_\alpha J_\alpha \in \Omega^2(M_3)$ be a smooth 2-form on M_3 associated with a 1-cycle $\gamma = \sum_\alpha n_\alpha \gamma_\alpha$ in M_3 for flux tubes J_α of γ_α with mutually disjoint supports each other. If A is a solution of the equation $F = -\frac{\pi}{k}J$, then $\int_{M_3} A\wedge dA = \frac{\pi^2}{k^2}\sum_{\alpha,\beta} n_\beta n_\alpha L(\gamma_\alpha,\gamma_\beta)$.*

Proof. Here for simplicity we replace the flux tubes J_α with de Rham currents of degree 2 on M_3. Let $\{\gamma_\alpha\}$ be a set of loops in M_3, $\gamma_\alpha : [0,1] \to M_3$, the Wilson operator(gauge invariant) on the loop γ_α is

$$W(n_\alpha,\gamma_\alpha) = \exp i n_\alpha \int\limits_{\gamma_\alpha} A,$$

where $n_\alpha \in \mathbb{Z}$ is charge. Now the action becomes

$$S_{CS}(A,J) = \frac{k}{2\pi}\int_{M_3} A\wedge dA + \sum\nolimits_\alpha n_\alpha \int\limits_{\gamma_\alpha} A.$$

Let $J_\alpha \in \Omega(M_3)$ be the closed current of degree 2 representing the loop $\gamma_\alpha \subset M_3$, given by

$$J_\alpha(x) = \frac{1}{2}dx^i dx^j \varepsilon_{ijk}\oint\limits_{\gamma_\alpha}\frac{dx^k(t)}{dt}\delta^3\left(x - x(t)\right),$$

where $\frac{dx^k(t)}{dt}$ is the tangent vector to γ_α at the point $x(t)$.

Since $\int\limits_{\gamma_\alpha} A = \int\limits_{M_3} A\wedge J$, the equation of motion becomes

$$F(x) = -\frac{\pi}{k}\sum\nolimits_\alpha n_\alpha J_\alpha = -\frac{\pi}{k}\sum\nolimits_\alpha n_\alpha \frac{1}{2}dx^i dx^j \varepsilon_{ijk}\oint\limits_{\gamma_\alpha} dt\frac{dx^k(t)}{dt}\delta^3\left(x - x(t)\right),$$

so

$$dA(x) = -\frac{\pi}{k}\sum\nolimits_\alpha n_\alpha \frac{1}{2}dx^i dx^j \varepsilon_{ijk}\oint\limits_{\gamma_\alpha} dt\frac{dx^k(t)}{dt}\delta^3\left(x - x(t)\right).$$

Let $\{\gamma_\beta\}$ be a set of mutually disjoint loops γ_β in M_3. Since $H_1(M_3;\mathbb{Z}) = \{0\}$, there exists a compact oriented surface D_β with boundary $\partial D_\beta = \gamma_\beta$ embedded in M_3 and we can calculate Wilson link of $\{\gamma_\beta\}$ as follows

$$W(\{\gamma_\beta\}) = \exp i \sum\nolimits_\beta n_\beta \int\limits_{\gamma_\beta} A = \exp i \sum\nolimits_\beta n_\beta \int\limits_{D_\beta} dA = \exp \frac{-i\pi}{k} \sum\nolimits_{\alpha,\beta} n_\beta n_\alpha \int\limits_{D_\beta} J_\alpha.$$

Since J_α is a δ-function valued 2-form in the direction normal to $\frac{dx^k(t)}{dt}$, the integration $\int\limits_{D_\beta} J_\alpha$ is the linking number $L(\gamma_\alpha, \gamma_\beta)$ of the two loops γ_α and γ_β, and therefore

$$W(\{\gamma_\beta\}) = \exp \frac{-i\pi}{k} \sum\nolimits_{\alpha,\beta} n_\alpha n_\beta L(\gamma_\alpha, \gamma_\beta).$$

Here we have a problem of defining self-linking $L(\gamma_\alpha, \gamma_\alpha)$. To treat the self-linking properly, we need to discuss with flux tubes.

Doing same thing,

$$\begin{aligned}
\int\limits_{M_3} A \wedge dA &= \int\limits_{M_3} A \wedge \left(\frac{-\pi}{k} \sum\nolimits_\alpha n_\alpha J_\alpha \right) = \frac{-\pi}{k} \sum\nolimits_\alpha n_\alpha \int\limits_{M_3} A \wedge J_\alpha \\
&= \frac{-\pi}{k} \sum\nolimits_\alpha n_\alpha \int\limits_{\gamma_\alpha} A = \frac{-\pi}{k} \sum\nolimits_\alpha n_\alpha \int\limits_{D_\alpha} dA \\
&= \frac{-\pi}{k} \sum\nolimits_\alpha n_\alpha \int\limits_{D_\alpha} \frac{-\pi}{k} \sum\nolimits_\beta n_\beta J_\beta \\
&= \frac{\pi^2}{k^2} \sum\nolimits_\alpha n_\alpha \sum\nolimits_\beta n_\beta \int\limits_{D_\alpha} J_\beta = \frac{\pi^2}{k^2} \sum\nolimits_{\alpha,\beta} n_\alpha n_\beta L\left(\gamma_\alpha, \gamma_\beta\right).
\end{aligned}$$

We find that $\int\limits_{M_3} AdA$ also depends linearly on the linking number $L(\gamma_\alpha, \gamma_\beta)$ when A satisfies the equation of motion $F = -\frac{\pi}{k} J$. $\square$

We can define a nonabelian version of the Chern-Simons action on a compact connected smooth oriented 3-manifold M_3 ([3]). Let G be a Lie group and $P = M_3 \times G \to M_3$ be the trivial principal G-bundle. The gauge field A on P takes values in the Lie algebra $\mathfrak{g} = Lie(G)$ of G. We write $A = A^a T^a \in \Omega^1(M_3, \mathfrak{g})$ where the $\{T^a\}$ are the generators of $\mathfrak{g}$, for $a = 1, \cdots, \dim(\mathfrak{g})$, satisfying the commutation relations $[T^a, T^b] = if^{ab}{}_c T^c$, and the normalization $Tr(T^a T^b) = \eta^{ab}$, where η^{ab} is the Killing form on the Lie algebra of G. The Chern-Simons action is defined as follows.

Definition 1.2. Let $\mathcal{A}$ be the space of all smooth G-connections on the trivial principal G-bundle $P = M_3 \times G \to M_3$. Then the Chern-Simons action functional $S_{CS} : \mathcal{A} \to \mathbb{R}$ is defined to be

$$S_{CS}(A) = \frac{k}{2\pi} \int_{M_3} Tr\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right), \tag{3}$$

for any $A \in \mathcal{A} \cong \Omega^1(M_3, \mathfrak{g})$.

Then we have the following.

Proposition 1.3. *The critical point set of $S_{CS} : \mathcal{A} \to \mathbb{R}$ is the set $\mathcal{F}$ of all flat connections A on P. A is locally given by $A = g_U^{-1} dg_U$ for smooth map $g_U : U \to G$ on a contractible open set U in M_3.*

Proof. The variation of $S_{CS}(A)$ under arbitrary $\delta A \in \Omega^1(M_3; \mathfrak{g})$ is

$$\begin{aligned}\delta S_{CS}(A) = \frac{k}{2\pi} \int_{M_3} & Tr((\delta A) \wedge dA + A \wedge d\delta A + \frac{2}{3} \delta A \wedge A \wedge A \\ & + \frac{2}{3} A \wedge \delta A \wedge A + \frac{2}{3} A \wedge A \wedge \delta A),\end{aligned}$$

by using properties of the trace, this gives

$$\delta S_{CS}(A) = \frac{k}{2\pi} \int_{M_3} Tr\left(d(\delta A \wedge A) + 2\delta A \wedge dA + 2\delta A \wedge A \wedge A\right).$$

The first term is boundary term $\int_{\partial M_3} Tr(\delta A \wedge A)$, (it relates to sympletic form $\int_{\partial M_3} Tr(\delta A \wedge \delta A)$), so it does not contribute to the equation of motion under the condition of variation $\delta A|_{\partial M_3} = 0$. Thus we obtain

$$\delta S_{CS}(A) = \frac{k}{2\pi} \int_{M_3} Tr\left(2\delta A \wedge (dA + A \wedge A)\right) = \frac{k}{2\pi} \int_{M_3} Tr\left(2\delta A \wedge F(A)\right).$$

The condition $\delta S_{CS}(A) = 0$ implies $F(A) = 0$, this is just vanishing the curvature (there is no dynamics in $S_{CS}(A)$). The source-free equations $F(A) = 0$ has pure gauge solutions (flat connections) $A = g_U^{-1} dg_U$, for $g_U : U \to G$, since A-parallel transport gives a trivialization of P over a contractible open set $U \subset M_3$ and the transition function g_U to the original trivialization gives $A = g_U^* 0 = g_U^{-1} g_U$.

□

Let $\mathcal{G}$ be the group of all smooth automorphisms(gauge transformation) $g : P \to P$ of the G-bundle P. Since the bundle P is trivial $P = M_3 \times G$, we can write $g : M_3 \to G$. $\mathcal{G}$ acts on $\mathcal{A}$ by $g^* A = g^{-1} A g + g^{-1} dg$, for $g \in \mathcal{G}$, $A \in \mathcal{A}$. Then we have the next.

Proposition 1.4. *$S_{CS} : \mathcal{A} \to \mathbb{R}$ behaves $S_{CS}(g^* A) = S_{CS}(A) - 4\pi k w(g)$ under the action of $g \in \mathcal{G}$, where $w(g) = \int_{M_3} \frac{1}{24\pi^2} Tr\left((g^{-1}dg)^3\right)$ is the winding number of $g : M_3 \to G$, M_3 is a compact connected smooth oriented closed 3-manifold.*

Proof. The gauge transformation $A^g = g^{-1} A g + g^{-1} dg$ in $S_{CS}(A)$ produces a boundary term on ∂M. This boundary term makes $e^{iS_{CS}(A)}$ not gauge invariant. In order to get an invariance we add WZW actions on the boundary term of the Chern-Simons action.

To calculate $S_{CS}(A^g)$, set $g^{-1}dg = \alpha$ and $g^{-1}Ag = w$ to calculate the Lagrangian of $S_{CS}(w + \alpha)$,

$$\begin{aligned} L(A^g) = L(w+\alpha) &= Tr\left((w+\alpha)d(w+\alpha) + \frac{2}{3}(w+\alpha)^3\right) \\ &= Tr(w+\alpha)d(w+\alpha) + \frac{2}{3}Tr\left((w+\alpha)^3\right), \end{aligned}$$

we use trace cyclic properties, and since the three form $tr(AAA)$ is also invariant under cyclic reordering, $Tr\left((w+\alpha)^3\right)$ behaves as commuting between one-forms w and α. Thus we get

$$L(A^g) = Tr(wdw + wd\alpha + \alpha dw + \alpha d\alpha) + \frac{2}{3}Tr(w^3 + \alpha^3 + 3w^2\alpha + 3w\alpha^2)$$

$$= Tr(wdw) + Tr(wd\alpha) + Tr(\alpha dw) + Tr(\alpha d\alpha) + \frac{2}{3}Tr(w^3) + \frac{2}{3}Tr(\alpha^3) + 2Tr(w^2\alpha) + 2Tr(w\alpha^2).$$

Using $w = g^{-1}Ag$, we obtain

$$\begin{aligned} L(A^g) =& Tr\left(g^{-1}Agd(g^{-1}Ag)\right) + Tr\left(g^{-1}Agd\alpha\right) + Tr\left(\alpha d(g^{-1}Ag)\right) + Tr(\alpha d\alpha) \\ &+ \frac{2}{3}Tr\left(g^{-1}Ag\right)^3 + \frac{2}{3}Tr(\alpha^3) + 2Tr\left((g^{-1}Ag)^2\alpha\right) + 2Tr\left((g^{-1}Ag)\alpha^2\right). \end{aligned}$$

This becomes (Appendix B (8))

$$L(A^g) = L(A) + L(\alpha) - dTr\left(\alpha g^{-1}Ag\right),$$

therefore

$$L(A^g) - L(A) = L(\alpha) - dTr\left(\alpha g^{-1}Ag\right).$$

The first term of the right hand side is

$$\begin{aligned} L(\alpha) = Tr(\alpha d\alpha) + \frac{2}{3}Tr(\alpha^3) &= Tr\left(g^{-1}dgd(g^{-1}dg)\right) + \frac{2}{3}Tr\left((g^{-1}dg)^3\right) \\ &= -\frac{1}{3}Tr\left((g^{-1}dg)^3\right). \end{aligned}$$

Therefore the changing of $S_{CS}(A)$ under the gauge transformation $g : M_3 \to G$ is

$$\delta S_{CS}(A) = -\frac{k}{2\pi}\int\limits_{M_3}\frac{1}{3}Tr\left((g^{-1}dg)^3\right) - \frac{k}{2\pi}\int\limits_{M_3} d\left(Tr(\alpha g^{-1}Ag)\right),$$

and using

$$Tr\left(\alpha g^{-1}Ag\right) = Tr\left(g^{-1}dgg^{-1}Ag\right) = Tr\left(dgg^{-1}A\right) = -Tr\left(A(dg)g^{-1}\right),$$

this becomes

$$\delta S_{CS}(A) = -\frac{k}{2\pi}\int\limits_{M_3}\frac{1}{3}Tr\left((g^{-1}dg)^3\right) + \frac{k}{2\pi}\int\limits_{M_3} d\left(TrAdgg^{-1}\right).$$

The winding number of the group valued $g : M_3 \to G$ is

$$w(g) = \int_{M_3}\frac{1}{24\pi^2}Tr\left((g^{-1}dg)^3\right) \in \mathbb{Z},$$

so

$$-\frac{k}{2\pi}\int\limits_{M_3}\frac{1}{3}Tr\left((g^{-1}dg)^3\right) = -4\pi k w(g),$$

Therefore it must be $k \in \frac{1}{2}\mathbb{Z}$ in order to keep the contribution of the first term in $\exp(i\delta S_{CS})$ trivial. But $\exp(iS_{CS})$ is not invariant when the manifold M_3 has boundary $\Sigma = \partial M_3$. We will see the term $TrA(dg)g^{-1}$ in exponential solution of $F(A)\psi(A) = 0$ on Σ, where $\psi(A)$ is quantum state. $\square$

2 Chern-Simons theory over the torus T^2

We study the Chern-Simons theory of group $U(1)$ on space-time $T^2 \times \mathbb{R}$, with torus $T^2 = S^1 \times S^1$, and its quantization. In the following discussion we will see that the corresponding Hilbert space is a complex vector space of dimension $2k$. By taking a gauge transformation on $P \to T^2 \times \mathbb{R}$ if necessary, we may assume that the dt-component of the connection A is zero, so it is in the form $A(t,x) = a_1(t,x)dx^1 + a_2(t,x)dx^2$, $a_i(t,x) \in C^\infty(T^2 \times \mathbb{R}; \mathfrak{u}(1))$ ([4, 5]).

Let (x^1, x^2) be the standard coordinate on $\mathbb{R}^2$. Then the coordinate (x^1, x^2) gives a local coordinate on torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ through the map

$$\phi : \mathbb{R}^2 \to T^2 = \mathbb{R}^2/\mathbb{Z}^2, \quad \phi(x^1, x^2) = [(x^1, x^2)],$$

with the class $[(x^1, x^2)] = [(x^1 + m^1, x^2 + m^2)], (m^1, m^2) \in \mathbb{Z}^2$.

There are two generators of π_1 and we can take a representatives

$$\alpha : [0,1] \to T^2 = \mathbb{R}^2/\mathbb{Z}^2, \quad \alpha(t) = [t, 0],$$
$$\beta : [0,1] \to T^2 = \mathbb{R}^2/\mathbb{Z}^2, \quad \beta(t) = [0, t].$$

Remark 2.1. The gauge equivalence classes of flat G-connections on a manifold M are in one-to-one correspondence with the conjugancy classes of homomorphisms $\pi_1(M) \to G$.

Let $\gamma_s(t)$ be a homotopy of loops from $\gamma_0(t)$ to $\gamma_1(t)$ based on x_0, and $0 \leq s \leq 1$. Take a connection $A \in \mathcal{A}$, then consider the A-holonomy map from space of all loops $\Omega_{x_0}(M)$ based on $x_0 \in M$ to the group G,

$$Hol(A) : \Omega_{x_0}(M) \to G,$$

given by

$$Hol_{\gamma_s}(A) = P \exp \int_{\gamma_s} A,$$

where P is path ordered. If the connection A is flat, $F(A) = 0$, we have $\frac{d}{ds} Hol_{\gamma_s}(A) = 0$, and hence $Hol_\gamma(A)$ depends only on the homotopy class of γ. It follows that $Hol(A)$ induces a group homomorphism $Hol(A) : \pi(M, x_0) \to G$ ([6]).

Therefore, the gauge equivalence classes of flat $U(1)$-connections are described by homomorphisms $\pi_1(T^2) \to U(1)$, and these are given by the holonomies around the α, β cycles

$$U_1 = Hol_\alpha(A) = \mathrm{e}^{i \oint_\alpha A}, \quad U_2 = Hol_\beta(A) = \mathrm{e}^{i \oint_\beta A},$$

from which we can define logarithms

$$U_i = e^{i a_i}, \quad [a_i(t, x)] \in \mathbb{R}/2\pi\mathbb{Z}.$$

Let $\mathcal{A}_{temp}$ be the space of all smooth $U(1)$-connections on $P = \mathbb{R} \times T^2 \times U(1) \to \mathbb{R} \times T^2$ in temporal gauge

$$A(t, x) = a_1(t, x)dx^1 + a_2(t, x)dx^2 \in \Omega^1(\mathbb{R} \times T^2; \mathfrak{u}(1)),$$

and let $\mathcal{G}_{temp}$ be the group of gauge transformation $Map(\mathbb{R} \times T^2; U(1))$ which are constant along $\mathbb{R}$, in order that $\mathcal{G}_{temp}$ acts on $\mathcal{A}_{temp}$. Let $g = \exp 2\pi i(m_1 x^1 + m_2 x^2) \in \mathcal{G}_{temp}$, then

$$\begin{aligned} g \cdot A = A + g^{-1}dg &= a_1 dx^1 + a_2 dx^2 + g^{-1}\partial_1 g dx^1 + g^{-1}\partial_2 g dx^2 \\ &= (a_1 + 2\pi i m_1)dx^1 + (a_2 + 2\pi i m_2)dx^2. \end{aligned}$$

If the connection A on $\mathbb{R} \times T^2$ is flat, then

$$0 = dA = dt \wedge \partial_t A + F(A_2),$$

so that

$$\partial_t A = 0, \quad and \quad F(A_2) = 0,$$

in the temporal gauge.

We regard $\mathcal{A}_{temp}/\mathcal{G}_{temp}$ as $Map(\mathbb{R}, \mathcal{A}/\mathcal{G})$, where $\mathcal{A}$ is the space of $U(1)$-connections on $P \to T^2$ and $\mathcal{G} = Map(T^2, U(1))$. We write the $U(1)$-connection $A \in \Gamma\left(T^*T^2 \otimes \mathfrak{u}(1)\right)$ with respect to the trivialization $P = T^2 \times U(1)$ as

$$A = a_1(x)dx^1 + a_2(x)dx^2,$$

where we must identify

$$A = a_1(x)dx^1 + a_2(x)dx^2 \sim A' = a'_1(x)dx^1 + a'_2(x)dx^2, \text{ if and only if } a'_i(x) - a_i(x) \in 2\pi\mathbb{Z},$$

because of large gauge transformations $g \cdot A = A + g^{-1}dg$ over T^2. This identification is necessary for quantization of states $C^\infty(\mathcal{A}/\mathcal{G})$ on T^2. We consider such connections in Chern-Simons action of $U(1)$ with temporal gauge $A_0 = 0$. Then the Chern-Simons action reduces to

$$\begin{aligned} CS(A) &= \frac{k}{2\pi} \int\limits_{\mathbb{R}\times T^2} A \wedge dA = \frac{k}{2\pi} \int\limits_{\mathbb{R}\times T^2} a_i \dot{a}_j dx^i \wedge dt \wedge dx^j \\ &= -\frac{k}{2\pi} \int\limits_{\mathbb{R}\times T^2} a_i \dot{a}_j dt \wedge dx^i \wedge dx^j = -\frac{k}{2\pi} \int\limits_{\mathbb{R}\times T^2} \varepsilon^{0ij} a_i \dot{a}_j dt dx^1 dx^2 \\ &= -\frac{k}{2\pi} \int\limits_{\mathbb{R}\times T^2} \varepsilon^{ij} a_i \dot{a}_j dt dx^1 dx^2 = -\frac{k}{2\pi} \int\limits_{\mathbb{R}\times T^2} (-a_1 \dot{a}_2 + a_2 \dot{a}_1) dt dx^1 dx^2 \\ &= \frac{k}{2\pi} \int\limits_{\mathbb{R}\times T^2} (a_1 \dot{a}_2 - a_2 \dot{a}_1) dt dx^1 dx^2. \end{aligned} \tag{4}$$

We used the Levi-Civita anti-symmetric tensor $\varepsilon^{ij} = -\varepsilon^{ji}$ on T^2 with $\varepsilon^{12} = 1$ and $\varepsilon^{0ij} = \varepsilon^{ij}$. We need to find a symplectic form on the space $\mathcal{A}$ of connections A.

In general, we find a symplectic form on a manifold M, by finding a map

$$\omega : T_pM \to T_p^*M,$$

if we let $\omega \in \Omega^2(M)$, then for every $v \in \Gamma(T_pM)$, we have a pairing $\omega(v) \in \Omega^1(M)$. When $H^1_{DR}(M) = 0$, every one-form is exact, so that $\omega(v) = df$ for some scalar function $f \in \Omega^0(M)$.

In our case the scalar function is an action. To find a symplectic form, we take the action with a Hamiltonian $H(p^i, q^i) \in C^\infty(T^*M)$. The variables $(q^1, ..., q^n)$ are coordinates on M and $z = (q^1, ..., q^n, p_1, ..., p_n)$ are coordinates on T^*M. Let $\gamma : I \to M$ and $\tilde{\gamma} : I \to T^*M$ be a smooth paths with $\pi \circ \tilde{\gamma} = \gamma$.

Proposition 2.1. *The action*

$$S(\tilde{\gamma}) = \int_{\tilde{\gamma}} p_i dq^i - H(p_i, q^i)dt \in \mathbb{R}, \quad \tilde{\gamma} : [t_1, t_2] \to T^*M \tag{5}$$

is stationary on the path $\tilde{\gamma}$ if and only if the trajectory $\tilde{\gamma}$ satisfies the Hamiltonian equation.

Proof. Take a variation of $S(\tilde{\gamma})$ the path $\tilde{\gamma}$. Let $\{\tilde{\gamma}_s\}_{s\in I}$ be the one-parameter family of paths $\tilde{\gamma}_s : I \to T^*M$,

$$\gamma_s(t) = (q^i(t) + s\delta q^i(t), p_i(t) + s\delta p_i(t)),$$

with $\delta q^i(t_1) = \delta q^i(t_2) = 0$. Then

$$\frac{d}{ds}\bigg|_{s=0} S(\tilde{\gamma}) = \int_{t_1}^{t_2} \left((\delta p_i)\dot{q}^i + p_i \delta \dot{q}^i - \frac{\partial H(p_i, q^i)}{\partial p_i}\delta p_i - \frac{\partial H(p_i, q^i)}{\partial q^i}\delta q^i \right) dt,$$

where $\dot{q}^i$ is the time derivative of q^i. Integrating the second term by parts, we obtain

$$\frac{d}{ds}\bigg|_{s=0} S(\tilde{\gamma}) = \int_{t_1}^{t_2} \left((\delta p_i)\dot{q}^i - \dot{p}_i \delta q^i - \frac{\partial H(p_i, q^i)}{\partial p_i}\delta p_i - \frac{\partial H(p_i, q^i)}{\partial q_i}\delta q_i \right) dt + \int_{t_1}^{t_2} \frac{d}{dt}(p_i \delta q^i) dt,$$

or

$$\frac{d}{ds}\bigg|_{s=0} S(\tilde{\gamma}) = \int_{t_1}^{t_2} \left(\left(\dot{q}^i - \frac{\partial H(p_i, q^i)}{\partial p_i} \right) \delta p_i - \left(\dot{p}_i + \frac{\partial H(p_i, q^i)}{\partial q_i} \right) \delta q^i \right) dt + \int_{t_1}^{t_2} \frac{d}{dt}(p_i \delta q^i) dt.$$

This vanishes on the path $\tilde{\gamma}$, on which $\delta q^i(t_2) = \delta q^i(t_1) = 0$, when

$$\frac{dp^i}{dt} = -\frac{dH}{dq^i} \quad \text{and} \quad \frac{dq^i}{dt} = \frac{dH}{dp^i},$$

these are the equations of motion. $\square$

We see that $H(p_i, q^i)$ has a constant value on the path $\tilde{\gamma}$, this is $d(H \circ \tilde{\gamma}) = \tilde{\gamma}^*(dH) = 0$. From

$$dH = \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q^i} dq^i \in \Omega^1(T^*M),$$

we obtain

$$\Omega^1(I) \ni \tilde{\gamma}^*(dH) = \left(\frac{\partial H}{\partial p_i} \frac{\partial p_i}{\partial t} + \frac{\partial H}{\partial q^i} \frac{\partial q^i}{\partial t} \right) dt, \tag{6}$$

by using the equations of motion $\frac{dp_i}{dt} = -\frac{dH}{dq^i}$ and $\frac{dq^i}{dt} = \frac{dH}{dp_i}$, we find that $\gamma^*(dH) = 0$.

We have the boundary term

$$\left. \frac{d}{ds} \right|_{s=0} S(\tilde{\gamma}) = \int_{t_1}^{t_2} \frac{d}{dt}(p_i \delta q^i) dt = \int_{\tilde{\gamma}} d(p_i \delta q^i) = p_i(t_2) \delta q^i(t_2) - p_i(t_1) \delta q^i(t_1),$$

from which we conclude that the 2-form $\omega = dp_i \wedge dq^i \in \Omega^2(T^*M)$ is a symplectic form, since $\delta S = 0$ on the path $\tilde{\gamma}$ of constant $H(p^i, q^i)$ and so $\omega(v_H) = 0$, where v_H is vector field on $H(p^i, q^i) = constant$, that is $(dH)(v_H) = 0$. From the equations $\frac{dp^i}{dt} = -\frac{dH}{dq^i}$ and $\frac{dq^i}{dt} = \frac{dH}{dp^i}$ on $\tilde{\gamma}$, we find

$$v_H = \frac{\partial p^i}{\partial t} \frac{\partial}{\partial p^i} + \frac{\partial q^i}{\partial t} \frac{\partial}{\partial q^i} = -\frac{\partial H}{\partial q^i} \frac{\partial}{\partial p^i} + \frac{\partial H}{\partial p^i} \frac{\partial}{\partial q^i} \in \Gamma\left(T(T^*M)\right),$$

and

$$\omega(v_H) = \left(dp_i \wedge dq^i\right) \left(-\frac{\partial H}{\partial q^i} \frac{\partial}{\partial p^i} + \frac{\partial H}{\partial p^i} \frac{\partial}{\partial q^i} \right) = \frac{\partial H}{\partial q^i} dp_i + \frac{\partial H}{\partial p^i} dp^i = dH,$$

so $\tilde{\gamma}^*(dH) = 0$ according to (6).

We define the symplectic form associated with the action $S_{CS}(A)$ (4) by comparing it with the action , we find that $H = 0$ in Chern-Simons action (4). The moduli space $\mathcal{F}/\mathcal{G}$ of flat $U(1)$-connection on P can be identified by the torus as follows.

$$\begin{aligned}\mathcal{F}/\mathcal{G} &\cong H^1_{DR}(T^2)/H^1_{DR}(T^2)_{int} \\ &\cong \{a_1dx^1 + a_2dx^2 | a_1, a_2 \in \mathbb{R}\}/\{m_1dx^1 + m_2dx^2 | m_1, m_2 \in \mathbb{Z}\} \cong \mathbb{R}^2/\mathbb{Z}^2.\end{aligned}$$

Let (a_1, a_2) be the standard coordinates on the universal covering $\mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$, define the symplectic form ω on the moduli space $\mathcal{F}/\mathcal{G}$ of flat connections by comparing (4) with (5) to be

$$\omega = \frac{k}{\pi} da_1 \wedge da_2. \tag{7}$$

therefore $\omega = \frac{1}{2}\omega_{ij} da^i \wedge da^j$ with $\omega_{12} = \omega^{12} = \frac{\pi}{k}$. The Poisson brackets are

$$\{f, g\} = \omega^{ij} \frac{\partial f}{\partial a_i} \frac{\partial g}{\partial a_j}.$$

In our case, we get

$$\{a_1, a_2\} = \frac{\pi}{k}.$$

The canonical quantization procedure lets us to define the commutation relation

$$[\hat{a}_1, \hat{a}_2] = -i\frac{\pi}{k},$$

for some corresponding operators $\hat{a}_1$, $\hat{a}_2$ on a Hilbert space. Let us choose a_1 to be the canonical coordinate, so that $p_{a_1} = a_2$ is the corresponding canonical momentum. Then a wavefunction $\psi(a_1)$ is a smooth function $\psi : \mathbb{R} \to \mathbb{C}$ which comes from the pull-back of a smooth function $\tilde{\psi} : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ by the covering map $\pi : \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$. A wavefunction $\psi(a_1)$ must be periodic under $a_1 \sim a_1 + 2\pi n$, $n \in \mathbb{Z}$. From the commutation relation, we have

$$\hat{a}_2 = i\frac{\pi}{k}\frac{\partial}{\partial a_1}.$$

This is translation operator on the coordinate a_1, it can be written in exponential form with some real parameter α as

$$e^{-i\frac{k}{\pi}\alpha\hat{a}_2},$$

this acts on $\psi(a_1)$ and translates it to $\psi(a_1+\alpha)$, we can see this by the Taylor expansion of ψ around a_1 formally,

$$e^{-i\frac{k}{\pi}\alpha\hat{a}_2}\psi(a_1) = \psi(a_1+\alpha).$$

The wavefunction $\psi(a_1)$ must be periodic under $a_1 \sim a_1 + 2\pi$, therefore

$$e^{-i2k\hat{a}_2}\psi(a_1) = \psi(a_1).$$

By taking the Fourier transform on $L^2(\mathbb{R}) \ni \psi(a_1) \leftrightarrow \tilde{\psi}(a_2) \in L^2(\mathbb{R}^*)$ with the kernel $e^{-i\frac{k}{\pi}a_1a_2}$, we get the eigenvalues equation

$$e^{-i2ka_2}\tilde{\psi}(a_2) = \tilde{\psi}(a_2).$$

$\tilde{\psi}(a_2) \neq 0$ if and only if $a_2 = n\pi/k$ for some $n \in \mathbb{Z}$. But k is an integer and $a_2 \sim a_2 + 2\pi n$, so we choose $a_2(n) = (n\pi/k) \mod (2\pi\mathbb{Z})$, these are

$$a_2 = 0, \frac{\pi}{k}, \frac{2\pi}{k}, ..., \frac{(2k-1)\pi}{k}.$$

Therefore the general wavefunction on the momentum coordinate a_2 is of the form

$$\tilde{\psi}(a_2) = \sum_{n=0}^{2k-1} c_n \sum_{m\in\mathbb{Z}} \delta\left(a_2 - \frac{\pi n}{k} - 2\pi m\right), \quad c_n \in \mathbb{C}.$$

Then we take the inverse Fourier transform back to obtain

$$\tilde{\psi}_n = \tilde{\psi}\left(a_2(n)\right) = \frac{1}{2\pi}\int_0^{2\pi} e^{-i\frac{k}{\pi}a_2(n)a_1}\psi(a_1)da_1 = \frac{1}{2\pi}\int_0^{2\pi} e^{-ina_1}\psi(a_1)da_1,$$

with

$$\psi(a_1) = \sum_{n=0}^{2k-1} e^{ina_1}\tilde{\psi}_n = \sum_{n=0}^{2k-1} c_n e^{ina_1}, \quad c_n \in \mathbb{C}. \tag{8}$$

This is the physical Hilbert space $\mathcal{H}_{phy} = L^2(\mathbb{R}/\mathbb{Z})$ of states of $U(1)$ flat connections $\mathcal{F}/\mathcal{G} = \mathbb{R}^2/\mathbb{Z}^2$ on the torus T^2, its dimension is $2k$, k is integer as we saw in previous section. We can choose other canonical coordinate and its momentum (a'_1, a'_2) on the universal cover $\mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2 = \mathcal{F}/\mathcal{G}$ by the transformation

$$\begin{pmatrix} a'_1 \\ a'_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix},$$

with

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL\left(2, \mathbb{Z}\right).$$

This transformation preserves $a_i \sim a_i + 2\pi n$, since the sums and multiplying by an integer preserve this equivalence relation. This gives another representations for the solutions (8).

3 Quantizing Chern-Simons and WZW model

For the canonical quantization of a system on the space $\mathbb{R} \times \Sigma$, we first find the phase space (the canonical variables "coordinates" and "momenta") which is independent of time. The phase spae can be obtained by requiring the Lagrangian be invariant under the time reparametrizion, and this gives both phase space variables and constraint equations. Then we find a sympletic form to construct a Poisson brackets. The quantum states are obtained by converting the Poisson brackets to commutators and can be used in solving the constraint equations. In general, the constraints generate a canonical transformations of the phase space, called the constraint group. Therefore the physical phase space of a constrained system is the space of solutions of the constraint equations modulo the action of the constraint group. In Chern-Simons theory, the constraints are vanishing of the curvatures on the principal G-bundle over the space M_3 ([7, 8]). The solutions of the constraint equation $F(A) = 0$ are flat connections on P. The flat connections on $P \to M_3$ are controlled by homomorhisms $\pi_1(M_3) \to G$.

The Chern-Simons action of G is ([9])

$$S_{CS}(A) = \frac{k}{2\pi} \int\limits_{M_3} Tr\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right),$$

where the connection $A = A^i T^i \in \Omega^1(\mathfrak{g})$ takes values in Lie algebra $\mathfrak{g}$ of G, $\{T^i\}$ are generators of $\mathfrak{g}$ with normalization $Tr(T^i T^j) = \delta^{ij}$. Let M_3 be the product of the real line $\mathbb{R}$ with a two dimensional closed oriented manifold Σ. Take a local coordinates (t, x^1, x^2) on $\mathbb{R} \times \Sigma$. By writing the connection as $A = A_0 dt + A_a dx^a$, where $(x^a)_{a=1,2}$ are coordinates on Σ, we obtain

$$\frac{2}{3} Tr\left(A \wedge A \wedge A\right) = \frac{2}{3} Tr\left(A_0 dt \wedge A \wedge A + A \wedge A_0 dt \wedge A + A \wedge A \wedge A_0 dt\right)$$

By using the circlic property of the trace, we obtain

$$\frac{2}{3} Tr\left(A \wedge A \wedge A\right) = 2 Tr\left(A_0 dt \wedge A \wedge A\right), \tag{9}$$

and

$$\begin{aligned} Tr\left(A \wedge dA\right) &= Tr\left(A_0 dt \wedge dA + A \wedge dt \wedge \partial_0 A + A \wedge dA_0 \wedge dt\right) \\ &= dt \wedge Tr\left(A_0 dA - A \wedge \partial_0 A + A \wedge dA_0\right). \end{aligned}$$

Then we integrate the third term on Σ by parts, and since Σ is closed, the boundary term vanishes

$$\begin{aligned} \int\limits_{M_3} Tr\left(A \wedge dA\right) &= \int\limits_{M_3} dt \wedge Tr\left(A_0 dA - A \wedge \partial_0 A + A_0 dA\right) \\ &= \int\limits_{M_3} dt \wedge Tr\left(-A \wedge \partial_0 A + 2 A_0 dA\right). \end{aligned} \tag{10}$$

From (9) and (10), we obtain

$$\begin{aligned} S_{CS}(A) &= \frac{k}{2\pi} \int\limits_{M_3} dt \wedge Tr\left(-A \wedge \partial_0 A + 2A_0 dA + 2A_0 A \wedge A\right) \\ &= \frac{k}{2\pi} \int\limits_{M_3} dt \wedge Tr\left(-A \wedge \partial_0 A + 2A_0 (dA + A \wedge A)\right) \\ &= \frac{k}{2\pi} \int\limits_{\mathbb{R}\times\Sigma} dt \wedge Tr\left(-A \wedge \partial_0 A + 2A_0 F(A_{(2)})\right), \end{aligned}$$

where $F(A_{(2)}) = dA_{(2)} + A_{(2)} \wedge A_{(2)} = (\partial_a A_b + A_a A_b) dx^a \wedge dx^b$ is the curvature of the connection $A_{(2)} = A_1(t,x)dx^1 + A_2(t,x)dx^2$ on $P|_{\{t\}\times\Sigma}$. We have

$$\begin{aligned} &Tr(-dt \wedge A \wedge \partial_0 A) = Tr(A \wedge dt \wedge \partial_0 A) = Tr(\varepsilon^{a0b} A_a \partial_0 A_b) dx^1 \wedge dt \wedge dx^2 \\ &= Tr(\varepsilon^{0ab} A_a \partial_0 A_b) dt \wedge dx^1 \wedge dx^2 = Tr(\varepsilon^{ab} A_a \partial_0 A_b) dt \wedge dx^1 \wedge dx^2, \end{aligned}$$

where ε^{ab} is inverse of the Levi-Civita anti-symmetric tensor ε_{ab} on Σ. By comparing with the discussion in (4) and (7), we conclude that $A_{(2)} = (A_a)_{a=1,2} \in \mathcal{A}(P|_{\{t\}\times\Sigma})$ are the variables of the phase space and $F(A_{(2)}) = (F_{ab})_{a,b=1,2} = 0$ is the constraint which satisfies the gauge invariance since $F(g \cdot A_{(2)}) = g^{-1} F(A_{(2)}) g$. The Poisson brackets are

$$\{A_a^i(x), A_b^j(y)\} = \frac{2\pi}{k} \varepsilon_{ab} \delta^{ij} \delta^2(x-y), \quad (x,y) \in \Sigma.$$

For the canonical quantization, let $\hat{A}_a^i(x)$ $(x \in \Sigma)$ be a linear operator on a Hilbert space over the space of connections $\mathcal{A}(P|_{\{t\}\times\Sigma})$. Then replace the Poisson brackets with the commutator

$$[\hat{A}_a^i(x), \hat{A}_b^j(y)] = -i\frac{2\pi}{k} \varepsilon_{ab} \delta^{ij} \delta^2(x-y).$$

Let $E = P \times_\rho \mathbb{C}^r \to \Sigma$ be complex vector bundle associates with the principle G-bundle and a representation $\rho : G \to GL(r, \mathbb{C})$. It is suitable to assume a complex structure on the base space Σ with a complex local coordinates $z = x + iy$ and $\bar{z} = x - iy$. That is Σ is assumed to be Riemann surface and consider the complexification $G^{\mathbb{C}}$ of the structure group G so that the vector bundle $E \to \Sigma$ is assumed to be a holomorphic vector bundle over Σ. Let $\mathcal{A}(E)$ be the space of all Hermitian connections on $E \to \Sigma$ compatible with the holomorphic structure on E. By writing the connection as $A = A_z dz + A_{\bar{z}} d\bar{z}$, the commutators become

$$[\hat{A}_z^i(x), \hat{A}_{\bar{z}}^j(y)] = \frac{\pi}{k} \delta^{ij} \delta^2(x-y).$$

If we choose A_z as the canonical coordinates on the connection space $\mathcal{A}(E)$, and let $A_{\bar{z}}$ be canonical momenta, then by the canonical quantization procedure,

$$\hat{A}_{\bar{z}}^i(z, \bar{z}) = -\frac{\pi}{k} \frac{\delta}{\delta A_z^i(z, \bar{z})}.$$

The curvature is $F^i_{\bar{z}z} = \partial_{\bar{z}} A^i_z - \partial_z A_{\bar{z}} + [A_{\bar{z}}, A_z]$, we regard it as operator, so

$$\hat{F}^i_{z\bar{z}} = \partial_{\bar{z}} A^i_z(z,\bar{z}) + \partial_z \left(\frac{\pi}{k} \frac{\delta}{\delta A^i_z(z,\bar{z})} \right) - \left[\frac{\pi}{k} \frac{\delta}{\delta A^i_z(z,\bar{z})}, A_z(z,\bar{z}) \right].$$

Since we have chosen A_z as coordinates, then the wave function (state) ψ is written as $\psi(A_z) \in C^\infty(\mathcal{A})$, thus we have the constraint equation $\hat{F}_A \psi(A) = 0$ as requirement for the quantization. The equation $F_A = 0$ is invariant under gauge transformation, the equation $\hat{F}_A \psi(A) = 0$ is also required to be invariant under gauge transformation $A \mapsto g^* A$, therefore $\psi(A_z) \mapsto \psi(A^g_z) = U(g)\psi(A_z)$, for $U : G \to \mathbb{C}$. We have the constraint equation

$$\hat{F}^i_{\bar{z}z} \psi(A_z) = \partial_{\bar{z}} A^i_z \psi(A_z) + \partial_z \left(\frac{\pi}{k} \frac{\delta \psi(A_z)}{\delta A^i_z} \right) - \left[\frac{\pi}{k} \frac{\delta \psi(A_z)}{\delta A^i_z}, A_z \right] = 0.$$

If we regard $\partial_z \left(\frac{\delta \psi(A_z)}{\delta A^i_z} \right) + \left[A_z, \frac{\delta \psi(A_z)}{\delta A^i_z} \right]$ as covariant derivative of $\frac{\delta \psi(A_z)}{\delta A^i_z}$, the last equation becomes

$$\partial_{\bar{z}} A^i_z \psi(A_z) + \frac{\pi}{k} D_z \left(\frac{\delta \psi(A_z)}{\delta A^i_z} \right) = 0.$$

The infinitesimal gauge transformation of $\psi(A_z)$ under infinitesimal parameter $\varepsilon \in \Omega^0(\mathfrak{g})$ is

$$\begin{aligned} \psi(A_z + D_z \varepsilon) &= \psi(A_z) + \int_\Sigma Tr \left((D_z \varepsilon) \wedge \frac{\delta}{\delta A_z(x)} \psi(A_z) \right) + \cdots \\ &= \psi(A_z) - \int_\Sigma Tr \left(\varepsilon D_z \frac{\delta}{\delta A_z(x)} \psi(A_z) \right) + \cdots, \end{aligned}$$

where we integrated by parts and used the fact that Σ is closed. We can write this transformation in exponential form with parameter $\varepsilon \in \Omega^0(\mathfrak{g})$ corresponding to $g \in G$, $g = e^\varepsilon$, we get

$$\psi(A^g_z) = e^{-\int_\Sigma Tr\left(\varepsilon D_z \frac{\delta}{\delta A_z}\right)} \psi(A_z),$$

then using $\frac{\pi}{k} D_z \left(\frac{\delta}{\delta A^i_z(x)} \right) = -\partial_{\bar{z}} A^i_z(x)$, we obtain

$$\psi(A^g_z) = e^{\frac{k}{\pi} \int_\Sigma Tr\left(\varepsilon \partial_{\bar{z}} A^i_z\right)} \psi(A_z).$$

The surface Σ is closed, so integrating by parts, it becomes

$$\psi\left(A_z^g\right) = e^{-\frac{k}{\pi}\int_\Sigma Tr\left(\partial_{\bar z}\varepsilon\wedge A_z^i\right)}\psi\left(A_z\right) = e^{\frac{k}{\pi}\int_\Sigma Tr\left(A_z^i\wedge\partial_{\bar z}\varepsilon\right)}\psi\left(A_z\right).$$

But $g^{-1}\partial_{\bar z}g = \varepsilon + O(\varepsilon^2)$, therefore

$$\psi\left(A_z^g\right) = e^{\frac{k}{\pi}\int_\Sigma Tr\left(A_z^i\wedge g^{-1}\partial_{\bar z}g + O(\varepsilon^2)\right)}\psi\left(A_z\right).$$

Let us write $\frac{k}{\pi}\int_\Sigma Tr\left(A_z^i\wedge g^{-1}\partial_{\bar z}g + O(\varepsilon^2)\right) = f(A,g)$, the function $f(A,g)$ has to satisfy

$$f(A_z,g_1) + f(A^{g_1},g_2) = f(A_z,g_1g_2) \quad \mathrm{mod}\ (2\pi iZ), \tag{11}$$

in order to get $U(g_2)U(g_1)\psi = U(g_2g_1)\psi$, where $\left(U(g)\psi\right)(A_z) = \psi(A_z^g)$ with $A^g = g^{-1}Ag + g^{-1}dg$ and $(A^{g_1})^{g_2} = A^{g_1g_2}$. We find that

$$f(A_z,g) = \frac{k}{2\pi}\int_\Sigma TrA^{1,0}(\bar\partial g)g^{-1} + kS^{\pm}_{WZW}(g),$$

satisfies (11), where $S^{\pm}_{WZW} : Map(\Sigma,G)\to\mathbb{R}$ is called WZW action defined to be

$$S^{\pm}_{WZW}(g) = \frac{c}{4\pi}\int_\Sigma Tr(g^{-1}\partial g g^{-1}\bar\partial g) \pm \frac{c}{12\pi}\int_M Tr(g^{-1}dg)^3, \tag{12}$$

with $\partial M = \Sigma$, and c is constant. The condition (11) follows from the fact that the WZW action $S^{\pm} := S^{\pm}_{WZW}$ satisfies ([7])

$$S^{\pm}(g_1g_2) = S^{\pm}(g_1) + S^{\pm}(g_2) + \frac{1}{\pi}\int_\Sigma Tr(g_1^{-1}\partial g_1 g_2^{-1}\bar\partial g_2).$$

Finally, we have

$$f^{+}(A_z,g) = -kS^{+}(g) + \frac{k}{2\pi}\int_\Sigma Tr(A^{1,0}g^{-1}\bar\partial g),$$

and

$$f^{-}(A_z,g) = +kS^{-}(g) + \frac{k}{2\pi}\int_\Sigma Tr(A^{1,0}\bar\partial g g^{-1}).$$

These two functions satisfy (11). Thus we obtain the path integral of a two-dimensional WZW action for the group G with the source term $\frac{k}{2\pi}Tr(A^{1,0}\bar\partial g g^{-1})$. We get classical solutions $Map(\Sigma,G)$ by requiring $\frac{\delta}{\delta g}f(A^{1,0},g) = 0$.

Theorem 3.1. *WZW action (12) for the group G has a conformal symmetries $G_L \times G_R$, the corresponding currents are holomorphic and anti-holomorphic currents.*

We see this clearly when we consider the transformation functions $\Lambda \in G$ as holomorphic or anti-holomorphic functions on the coordinates $(z, \bar{z})$, these functions act on g leading to conformal transformations of WZW action (12). The inavariance of WZW action under the transformations $g \mapsto \Lambda(z)g$ and $g \mapsto \Lambda(\bar{z})g$ gives conserved holomorphic and anti-holomorphic currents (Appendix C (9)).

4 WZW model on 3d SU(2) algebra and on 4d Heisenberg algebra

As we saw, WZW model is a solution of Chern-Simons on $2d$ surface Σ without boundary immersed in manifold M_G of Lie group G. Here we see that the WZW model induces metric on group manifold M_G by using the Killing form on Lie algebra of G and the metric on Σ. In some groups, like Heisenberg group, that metric is Lorentz metric. While in other groups that metric is Taub–NUT metric, like $SU(2)$ group ([10]). We apply WZW model on $SU(2)$ algebra which has 3 dimensions, three generators and on Heisenberg algebra which has 4 dimensions, four generators ([11]). The WZW model for a group G depends on the metric on $Lie(G) \cong T_eG$ and metric on $2d$ surface that immersed in manifold M_G.

Definition 4.1. The Heisenberg group H_4 is generated by $\{a, a^+, N = a^+a, I\}$, with the commutation relations

$$[a, a^+] = I,$$
$$[N, a^+] = a^+,$$
$$[N, a] = -a.$$

Let $t_1 = a^+$, $t_2 = a$, $t_3 = N$, $t_4 = I$, then the commutation relations becomes

$$[t_1, t_2] = -t_4, \quad \text{so} \quad {f_{12}}^4 = -1$$

$$[t_3, t_1] = t_1, \quad \text{so} \quad {f_{31}}^1 = 1$$

$$[t_3, t_2] = -t_2, \quad \text{so} \quad {f_{32}}^2 = -1.$$

A Killing form η is invariant so it has to satisfy

$$\eta_{cd}{f_{ab}}^d + \eta_{bd}{f_{ac}}^d = 0, \quad \text{so} \quad f_{abc} + f_{acb} = 0,$$

where we use η for raising and lowering indices in adjoint representation. This has to keep f_{abc} anti-symmetric tensor, it can given by non-degenerate, like

$$(\eta_{ab}) = \begin{pmatrix} 0 & a & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & b & -a \\ 0 & 0 & -a & 0 \end{pmatrix}, \left(\eta^{ab}\right) = \begin{pmatrix} 0 & a^{-1} & 0 & 0 \\ a^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -a^{-1} \\ 0 & 0 & -a^{-1} & -ba^{-2} \end{pmatrix}.$$

Therefore

$$f_{123} = \eta_{34}{f_{12}}^4 = a \times 1 = a, \quad f_{321} = \eta_{12}{f_{32}}^2 = -a, \quad f_{312} = \eta_{21}{f_{31}}^1 = a.$$

The elements of H_4 are written as

$$g = e^{iqa + i\bar{q}a^+} e^{iuN + ivI},$$

q and $\bar{q}$ are complex coordinates and u and v are real coordinates on the manifold of Heisenberg group H_4. We have the relations

$$\begin{aligned} e^{iqa+i\bar{q}a^+} &= e^{iqa}e^{i\bar{q}a^+}e^{-[iqa, i\bar{q}a^+]/2} = e^{iqa}e^{i\bar{q}a^+}e^{(q\bar{q})/2} \\ e^{-iuN}ae^{iuN} &= \left(\prod_{i=0}^{\infty} e^{-iu_iN}\right) a \left(\prod_{i=0}^{\infty} e^{iu_iN}\right) = e^{iu}a, \\ e^{-i\bar{q}a^+}ae^{i\bar{q}a^+} &= \left(\prod_{i=0}^{\infty} e^{-i\bar{q}_ia^+}\right) a \left(\prod_{i=0}^{\infty} e^{i\bar{q}_ia^+}\right) = a + i\frac{1}{2}\bar{q}I, \end{aligned} \tag{13}$$

with $|u_i| \prec\prec 1$ and $u \sum_{i=0}^{\infty} u_i = u$, same thing for $\bar{q}$. The one-form $\theta = g^{-1}dg$ is

$$\begin{aligned} g^{-1}dg = {} & e^{-iuN-ivI}e^{-iqa-i\bar{q}a^+}d\left(e^{iqa+i\bar{q}a^+}e^{iuN+ivI}\right) \\ &= e^{-iuN-ivI}e^{-iqa-i\bar{q}a^+}(iadq + ia^+d\bar{q})e^{iqa+i\bar{q}a^+}e^{iuN+ivI} \\ &+ e^{-iuN-ivI}e^{-iqa-i\bar{q}a^+}e^{iqa+i\bar{q}a^+}(iNdu + iIdv)e^{iuN+ivI}, \end{aligned}$$

this gives

$$
\begin{aligned}
g^{-1}dg = {} & e^{-iuN-ivI}e^{-iqa-i\bar{q}a^{+}}(iadq+ia^{+}d\bar{q})e^{iqa+i\bar{q}a^{+}}e^{iuN+ivI} \\
& + e^{-iuN-ivI}(iNdu+iIdv)e^{iuN+ivI}, \\
= {} & e^{-iuN-ivI}\left(i(a+i\frac{1}{2}\bar{q}I)dq+i(a^{+}-i\frac{1}{2}qI)d\bar{q}\right)e^{iuN+ivI}+(iNdu+iIdv) \\
= {} & e^{-iuN-ivI}i\left(a+i\frac{1}{2}\bar{q}I\right)dqe^{iuN+ivI}+e^{-iuN-ivI}i\left(a^{+}-i\frac{1}{2}qI\right)d\bar{q}e^{iuN+ivI} \\
& + (iNdu+iIdv) \\
= {} & i\left(e^{iu}a+i\frac{1}{2}\bar{q}I\right)dq+i\left(e^{-iu}a^{+}-i\frac{1}{2}qI\right)d\bar{q}+(iNdu+iIdv),
\end{aligned}
$$

therefore

$$
\begin{aligned}
g^{-1}dg &= ie^{iu}dqa-\frac{1}{2}\bar{q}dqI+ie^{-iu}d\bar{q}a^{+}+\frac{1}{2}qd\bar{q}I+iNdu+iIdv \\
&= ie^{iu}dqa+ie^{-iu}d\bar{q}a^{+}+iduN+\left(-\frac{1}{2}\bar{q}dq+\frac{1}{2}qd\bar{q}+idv\right)I \\
&= e^{1}+e^{2}+e^{3}+e^{4}.
\end{aligned}
$$

We define one-forms flat connections $\Omega^{1}(M)\otimes Lie(G)$

$$
\begin{aligned}
e^{1} &= ie^{iu}dq, \\
e^{2} &= ie^{-iu}d\bar{q}, \\
e^{3} &= idu, \\
e^{4} &= i\left(\frac{i}{2}\bar{q}dq-\frac{i}{2}qd\bar{q}+dv\right).
\end{aligned}
$$

These are one-forms connections, if we use the metric η on lie algebra, and metric g on $2d$ dimensions, we obtain

$$
\begin{aligned}
& tr\left(g^{-1}dg\wedge *g^{-1}dg\right)\rightarrow \\
& g^{ij}e_{i}^{a}e_{j}^{b}\eta_{ab}=-\left(2ag^{ij}\partial_{i}q\partial_{j}\bar{q}-2ag^{ij}\left(\frac{i}{2}\bar{q}\partial_{i}q-\frac{i}{2}q\partial_{i}\bar{q}+\partial_{i}v\right)\partial_{j}u+bg^{ij}\partial_{i}u\partial_{j}u\right),
\end{aligned}
$$

and anti-symmetric term is

$$
\begin{aligned}
tr\left(g^{-1}dg\right)^{3}\rightarrow & \epsilon^{ijk}e_{i}^{a}e_{j}^{b}e_{k}^{c}tr(T_{a}T_{b}T_{c})=\frac{1}{2}\epsilon^{ijk}e_{i}^{a}e_{j}^{b}e_{k}^{c}tr([T_{a},T_{b}]T_{c})=\frac{1}{2}\epsilon^{ijk}e_{i}^{a}e_{j}^{b}e_{k}^{c}f_{ab}{}^{e}tr(T_{e}T_{c}) \\
&= \frac{1}{2}\epsilon^{ijk}e_{i}^{a}e_{j}^{b}e_{k}^{c}f_{ab}{}^{e}\eta_{ec}=\frac{1}{2}\epsilon^{ijk}f_{abc}e_{i}^{a}e_{j}^{b}e_{k}^{c}=\frac{1}{2}\epsilon^{ijk}6f_{123}e_{i}^{1}e_{j}^{2}e_{k}^{3}=3a\epsilon^{ijk}e_{i}^{1}e_{j}^{2}e_{k}^{3} \\
&= -3ai\epsilon^{ijk}\partial_{i}q\partial_{j}\bar{q}\partial_{k}u=-3ai\epsilon^{ijk}\partial_{k}(u\partial_{i}q\partial_{j}\bar{q}).
\end{aligned}
$$

The S_{WZW} is

$$S_{WZW} = \frac{1}{2\pi} \int\limits_{\Sigma} tr\left(g^{-1}dg\right) \wedge * \left(g^{-1}dg\right) + \frac{1}{6\pi} \int\limits_{X} tr\left(g^{-1}dg\right)^3,$$

the Wedge star operator is with respect to metric on Σ. This becomes

$$S_{WZW} = \frac{-1}{2\pi} \int\limits_{\Sigma} d^2\sigma \left(2ag^{ij}\partial_i q\partial_j \bar{q} - 2ag^{ij}\left(\frac{i}{2}\bar{q}\partial_i q - \frac{i}{2}q\partial_i\bar{q} + \partial_i v\right)\partial_j u + bg^{ij}\partial_i u\partial_j u\right)$$
$$- \frac{ai}{2\pi} \int\limits_{X} \epsilon^{ijk}\partial_k(u\partial_i q\partial_j\bar{q}),$$

then

$$S_{WZW} =$$
$$\frac{-1}{2\pi} \int\limits_{\Sigma} d^2\sigma \left(2ag^{ij}\partial_i q\partial_j \bar{q} - 2ag^{ij}\left(\frac{i}{2}\bar{q}\partial_i q - \frac{i}{2}q\partial_i\bar{q} + \partial_i v\right)\partial_j u + bg^{ij}\partial_i u\partial_j u + \frac{ai}{\pi}\epsilon^{ij}u\partial_i q\partial_j\bar{q}\right),$$

or

$$S_{WZW} =$$
$$\frac{-a}{\pi} \int\limits_{\Sigma} d^2\sigma \left(g^{ij}\partial_i q\partial_j \bar{q} - g^{ij}\left(\frac{i}{2}\bar{q}\partial_i q - \frac{i}{2}q\partial_i\bar{q} + \partial_i v\right)\partial_j u + \frac{b}{2a}g^{ij}\partial_i u\partial_j u + \frac{i}{2\pi}\epsilon^{ij}u\partial_i q\partial_j\bar{q}\right),$$

The background space-time metric, in the coordinate $(q, \bar{q}, u, v)$ is

$$ds^2 = dqd\bar{q} - \left(\frac{i}{2}\bar{q}dq - \frac{i}{2}qd\bar{q} + dv\right)du + \beta^2 du^2,$$

with $\beta^2 = b/2a$. And antisymmetric field is $B = iudq \wedge d\bar{q}/2$.

By introducing polar coordinates $q = Re^{i\theta}$, $\bar{q} = Re^{-i\theta}$, the metric turns out to be

$$\begin{aligned} ds^2 &= dR^2 + R^2 d\theta^2 - \left(dv - R^2 d\theta\right)du + \beta^2 du^2 \\ &= -\left(dv - R^2 d\theta\right)du + dR^2 + R^2 d\theta^2 + \beta^2 du^2. \end{aligned}$$

The signature of this metric in the orthonormal basis

$$e^0 = \tfrac{1}{2\theta}\left(dv - R^2 d\theta\right)du$$
$$e^1 = dR$$
$$e^2 = Rd\theta$$
$$e^3 = \beta du - e^0$$

is $(-+++)$, this is Lorentzain metric on the manifold of Heisenberg group H_4.

We apply WZW-model on $SU(2)$. We parametrize an element of $SU(2)$ by Euler angles,

$$g = e^{\varphi T_3} e^{\theta T_2} e^{\chi T_3},$$

and

$$g = \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix} \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \begin{pmatrix} e^{i\chi/2} & 0 \\ 0 & e^{-i\chi/2} \end{pmatrix},$$

with

$$0 \leq \chi \prec 4\pi, \quad 0 \leq \theta \prec \pi, \quad 0 \leq \varphi \prec 2\pi.$$

The commutation relations of the generators $\{T_1, T_2, T_3\}$ are

$$[T_i, T_j] = i\varepsilon_{ijk} T_k, \quad (T_i)^+ = T_i.$$

This then means that θ, φ are the usual angular coordinates describing the unit radius sphere S^2, and χ describes some circle S^1. The parametrization expresses the fact that topologically $SU(2)$ is S^3, which due to a mapping devised by Hopf, is equal to a S^2 fibered by and S^1. By writing

$$g^{-1}dg = i\sigma^i T_i \to \sigma^i = \frac{1}{i} tr\left(T^i g^{-1} dg\right),$$

this gives

$$\begin{aligned}
\sigma^1 &= \cos(\phi)d\theta + \sin(\phi)\sin(\theta)d\chi, \\
\sigma^2 &= -\sin(\phi)d\theta + \cos(\phi)\sin(\theta)d\chi, \\
\sigma^3 &= d\phi + \cos(\theta)d\chi.
\end{aligned}$$

From the commutation relations, the metric on $\mathfrak{su}(2)$ is just Kronecker's delta.

The symmetric term in S_{WZW} is

$$\frac{1}{2\pi}\int_X tr\left(g^{-1}dg\right)^2 = \frac{1}{2\pi}\int_X \delta_{ij}\sigma^i\otimes\sigma^j$$
$$= \frac{1}{2\pi}\int_X \left(d\chi^2 + d\theta^2 + d\phi^2 + 2\cos(\theta)d\chi d\phi\right)$$
$$= \frac{1}{2\pi}\int_X \left((d\chi+\cos(\theta)d\phi)^2 + d\theta^2 + \sin^2(\theta)d\phi^2\right).$$

This is induced metric on manifold S^3 of the group $SU(2)$, it has form of so-called Taub-NUT metric. Usually Taub-NUT metrics will contain a factor $f(r,t)$ like

$$(dt + Nf(r,t)\cos(\theta)d\phi)^2,$$

where N is the so-called Taub-NUT charge which has the interpretation of a gravitational instanton. The anti-symmetric term in S_{WZW} is

$$\frac{1}{6\pi}\int_X tr\left(g^{-1}dg\right)^3 = \frac{1}{6\pi}\int_X tr\left(\sigma^i T_i\sigma^j T_j\sigma^k T_k\right) = \frac{1}{6\pi}\int_X \sigma^i\sigma^j\sigma^k tr\left(T_iT_jT_k\right)$$
$$= \frac{1}{3\pi}\int_X \sigma^i\sigma^j\sigma^k tr\left([T_i,T_j]T_k\right) = \frac{1}{3\pi}\int_X \sigma^i\sigma^j\sigma^k\varepsilon_{ij}{}^{\ell} tr\left(T_\ell T_k\right)$$
$$= \frac{1}{3\pi}\int_X \sigma^i\sigma^j\sigma^k\varepsilon_{ij}{}^{\ell}\eta_{\ell k} = \frac{1}{3\pi}\int_X \sigma^i\sigma^j\sigma^k\varepsilon_{ijk} = \frac{2}{\pi}\int_X \varepsilon_{123}\sigma^1\wedge\sigma^2\wedge\sigma^3$$
$$= \frac{2}{\pi}\int_X sin(\theta)d\theta\wedge d\chi\wedge d\phi = \frac{-2}{\pi}\int_X d\left(cos(\theta)\wedge d\chi\wedge d\phi\right)$$
$$= \frac{-2}{\pi}\int_\Sigma cos(\theta)d\chi\wedge d\phi,$$

the anti-symmetric form of S_{WZW} is $\frac{-2}{\pi}cos(\theta)d\chi\wedge d\phi$.

5 Chern-Simons theory of gravity

In (2+1)-dimensional spacetime there are reasons for the simplicity of general relativity, one of them is that the curvature tensor $R_{\mu\nu\rho\sigma}$ is written in terms of a scalar curvature R, and Ricci tensor $R_{\mu\nu}$ as ([12])

$$R_{\mu\nu\rho\sigma} = g_{\mu\rho}R_{\nu\sigma} + g_{\nu\sigma}R_{\mu\rho} - g_{\nu\rho}R_{\mu\sigma} - g_{\mu\sigma}R_{\nu\rho} - \frac{1}{2}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right)R,$$

where $(x^\mu) = (x^0, x^1, x^2)$ with $g = g_{\mu\nu}dx^\mu dx^\nu$ is type $(-,+,+)$. This implies that any solution of the vacuum Einstein field equations is a flat metric, and that any solution of the field equations with a cosmological constant,

$$R_{\mu\nu} = 2\Lambda g_{\mu\nu}$$

has a solution of a metric of constant curvature. Physically, a (2+1)-dimensional spacetime has no local degrees of freedom: there are no gravitational waves in the classical theory, and no gravitons in the quantum theory.

The vanishing of the curvature implies that M can be covered by a set of contractible coordinate patches U_i, each isometric to the Minkowski space $\mathbb{M}^{2,1}$ of dimension $2+1$ with the standard Minkowski metric $\eta_{\mu\nu}$. In general these patches must be glued together by coordinate transform ϕ_{ij} on the intersections $U_i \cap U_j$, which determine how points are identified. Since the metrics on U_i and U_j are identical, these transition functions must be isometries for the metric $\eta_{\mu\nu}$, that is, elements of the Poincaré group $ISO(2,1)$. Also global isometries of the (2+1)-dimensional spacetime M can be given in terms of in the local Lorentz transformations and translations.

Let M be a connected compact oriented smooth 3-manifold possibly with boundary. A flat geometry on M determines a holonomy space $\mathcal{M}$

$$\mathcal{M} = Hom(\pi_1(M), ISO(2,1))/\!\sim,$$

where $\rho_1 \sim \rho_2$ for two representations $\rho_1, \rho_2 : \pi_1(\mathcal{M}) \to ISO(2,1)$ if $\rho_2 = g^{-1}\rho_1 g$ for some $g \in ISO(2,1)$. Conversely, we can get flat geometry on M by acting an $ISO(2,1)$-representation $\rho : \pi_1(M) \to ISO(2,1)$ of the fundamental group $\pi_1(M)$ acting properly discontinuously on a region $W \subset \mathbb{M}^{2,1}$ of Minkowski space, since a flat metric is determined by its holonomies. In fact M is isometric to $W/\!\sim$, where $x \sim y$ for $x, y \in W$ if $y = \rho([\gamma])x$ for a homotopy class $[\gamma]$ of path γ. This is a generalization of quotient construction for the flat torus. And if $M \cong \mathbb{R} \times \Sigma$, the fundamental group $\pi_1(M)$ is isomorphic to $\pi_1(\Sigma)$. If the cosmological constant Λ is nonzero, then the (2+1)-dimensional spacetime has constant curvature, and the coordinate patches U_i will be isometric to de Sitter space (for $\Lambda > 0$) or anti-de Sitter space (for $\Lambda < 0$). The gluing isometries become elements of $SO(3,1)$ (for $\Lambda > 0$) or $SO(2,2)$ (for $\Lambda < 0$), and the holonomies are now elements of one of these groups.

In order to construct a gravity theory in a formulation without using a metric, Palatini considered the variables (e^I, ω^{IJ}) instead of $(g_{\mu\nu}, \Gamma^{\rho}_{\mu\nu})$. To construct a gauge theory of gravity, the variable e^I is supposed to be a component of a connection in the Poincaré group in addition to the Lorentz group connection ω^{IJ}. Using the Chern-Simons theory for the Poincaré group in $2+1$ dimensions we recover the Einstein-Hilbert action and the same equations of motion for the gravity. Since we have the same equations of motion, this theory regarding the gravitational field e^I as a connection is identical to Einstein's theory of general relativity in (2+1)-dimension at the classical level. In this way we have a gravity theory, with only connections e^I, ω^{IJ} as variables, which depends only on the topology of the manifold M.

In general we can obtain Chern-Simons theory by integrating the Pontryagin topological term $Tr(F \wedge F \wedge \cdots)$ over a contractible manifold M_d with boundary $M_{d-1} = \partial M_d$ and by using the Poincaré lemma, $dTr(F \wedge F \wedge \cdots) = 0$ so there is non-closed $d-1$ form θ satisfying $Tr(F \wedge F \wedge \cdots) = d\theta$. Since the Pontryagin topological term $Tr(F \wedge F \wedge \cdots)$ is even form, the manifold M_d should be even dimensional with odd dimensional boundary manifold $M_{d-1} = \partial M_d$. Thus the Chern-Simons theory is formulated on odd dimensional manifold, and the theory in three dimensions with the structure group $ISO(2,1)$ is identical to the gravity theory for $(2+1)$-dimensions ([13]).

Let M be a flat three dimensional spacetime manifold and $\tilde{M}$ be its universal cover. Then $\tilde{M}$ is the Minkowski space $\mathbb{M}^{2,1}$ with the symmetry group $ISO(2,1)$, let $\Gamma \subset ISO(2,1)$ be a discrete subgroup with $\Gamma \cong \pi_1(M)$ acting properly discontinuously on $\mathbb{M}^{2,1}$. This amounts to give a representation $\rho : \pi_1(M) \to ISO(2,1)$ up to conjugation. Since the group $ISO(2,1)$ preserves the flat structure of $\tilde{M}$, the quotient space $M = \tilde{M}/\Gamma$ also has flat structure and its tangent space can be identified with the Minkowski space $\mathbb{M}^{2,1}$. The corresponding connection on the principal $ISO(2,1)$-bundle is flat and induces a holonomy representation $\rho : \pi_1(M) \to ISO(2,1)$.

Let us consider the 2+1-spacetime $M = \mathbb{R}\times\Sigma$, where Σ is a connected oriented closed surface of genus g, and $\mathbb{R}$ be the real line (time). Since $\mathbb{R}$ is contractible, $\pi_1(\mathbb{R}\times\Sigma) = \pi_1(\Sigma)$, and $\Sigma = \tilde{\Sigma}/\pi_1(\Sigma)$. Thus the flat structures on $M = \mathbb{R}\times\Sigma$ correspond to representations $\rho : \pi_1(\Sigma) \to G := ISO(2,1)$.

The fundamental group $\pi_1(M)$ of M is naturally described with $2g$ generators $\{a_i, b_i\}_{1,\cdots,g}$ and one relation $a_1 b_1 a_1^{-1} a_i^{-1} \cdots a_g b_g a_g^{-1} a_g^{-1} = 1$. $Hom(\pi_1(M), G)$ can be identified with the subspace of G^{2g} defined by the equation $\prod_{i=1}^{g}\left(A_i B_i A_i^{-1} B_i^{-1}\right) = 1$, $(A_i, B_i \in G)$ whose dimension is $(2g-1)\dim(G)$, we have the following identification as the quotient space

$$Hom(\pi_1(M), G)/\!\sim\; \cong \left\{ (A_i, B_i, \cdots, A_g, B_g) \in G^{2g} \,\middle|\, \prod_{i=1}^{g}\left(A_i B_i A_i^{-1} B_i^{-1}\right) = 1 \right\} \Big/ G$$

and two homomorphisms of them are equivalent if they are conjugated by an element of the group G. Therefore heuristically the dimension of $Hom(\pi_1(M), G)/\!\sim$ is $(2g-2)\dim(G)$. In fact $G = ISO(2,1)$ is six dimensional, so the space of flat structures has dimension $(2g-2)\dim(G) = 12g - 12$. The solutions of Einstein's equations in the vacuum in 3D are flat connections, $F = 0$, and hence homomorphisms $\rho : \pi_1(\Sigma) \to ISO(2,1)$ correspond to the solutions of Einstein's equations, the equation of motion of the $ISO(2,1)$-Chern-Simons action.

The Poincaré group $ISO(d-1,1)$ is the group of all isometries of Minkowski space $\mathbb{M}^{d-1,1}$, whose Lie algebra is generated by Lorentz generators J^{ab} and translations generators P^a, with $a, b = 1, \cdots, d$. In $3d$ space-time, the Chern-Simons Lagrangian for the gauge group $ISO(d-1,1)$ contains the term $Tr(AdA) = Tr(T^aT^b)A^a dA^b$, so we need an invariant non-degenerate bilinear form $d^{ab} = Tr(T^aT^b)$. But a non-degenerate bilinear form on the Lie algebra of $ISO(d-1,1)$ exists only in $d = 3$, so there would be no reasonable Chern-Simons three form for $ISO(d-1,1)$ for general d. In $d = 3$, we can define $ISO(2,1)$ invariant and non-degenerate bilinear form $W = \epsilon_{abc}P^aJ^{bc}$. Therefore, a reasonable Chern-Simons action in $d = 3$ for $ISO(2,1)$ may exist. It is convenient to write $J^a = \frac{1}{2}\epsilon^{abc}J_{bc}$, so that $W = \eta_{ab}P^aJ^b = P_aJ^a$ is a non-degenerate bilinear form on the Lie algebra of $ISO(2,1)$, and η is Minkowski metric, so we use it in raising and lowering indices of $\mathfrak{g} := \mathfrak{iso}(2,1)$ in the adjoint representation. Thus the inner product on $\mathfrak{g}$ is

$$\left\langle J^a, P^b \right\rangle = \eta^{ab}, \quad \left\langle J^a, J^b \right\rangle = \left\langle P^a, P^b \right\rangle = 0. \tag{14}$$

We identify $\mathfrak{g}^* = Hom(\mathfrak{g}, \mathbb{R})$ with $\mathfrak{g}$ in the adjoint representation by using the Minkowski metric η. The commutation relations of $ISO(2,1)$ are

$$[J_a, J_b] = \epsilon_{abc}J^c, \quad [J_a, P_b] = \epsilon_{abc}P^c \quad [P_a, P_b] = 0. \tag{15}$$

We check here that bilinear form above are invariant,

$$[J_a, P_bJ^b] = P^b[J_a, J_b] + [J_a, P_b]J^b = P^b\epsilon_{abc}J^c + \epsilon_{abc}P^cJ^b = \epsilon_{abc}P^bJ^c - \epsilon_{acb}P^cJ^b = 0$$

$$[P_a, P_bJ^b] = P^b[P_a, J_b] + [P_a, P_b]J^b = P^b\epsilon_{abc}P^c = \epsilon_{abc}P^bP^c = 0.$$

Let $P_4 \to M_4$ be an $ISO(2,1)$-principal bundle over a connected oriented compact smooth 4-manifold M_4 with boundary $M_3 = \partial M_4$. Let $\mathcal{A}_{ISO(2,1)}$ be the space of all $ISO(2,1)$-connections $A \in \Omega^1(P_4; \mathfrak{g})$ on P_4. Take a local coordinate neighborhood (U, x^i) with a trivialization $P_4|_U \cong U \times G$, $G := ISO(2,1)$. Then the gauge field is locally a one-form with values in Lie-algebra $\mathfrak{g}$ of G with basis $\{J_a, P_a\}$,

$$A = e_i^a P_a dx^i + \omega_i^a J_a dx^i \in \Gamma(T^*U \otimes \mathfrak{g}).$$

Since $M^{2,1} \triangleleft ISO\left(2,1\right)$ and $ISO\left(2,1\right)\big/\mathbb{M}^{2,1} \cong SO\left(2,1\right)$, the $ISO(2,1)$-connection A regarding as an $\mathfrak{iso}(2,1)$-valued 1-form $A \in \Omega^1(P;\mathfrak{iso}(2,1))$ on P induces a $SO(2,1)$-connection $\underline{\omega} \in \Omega^1(\mathbb{Q};\mathfrak{so}(2,1))$ on the induced principal $SO(2,1)$-bundle $\mathbb{Q} = P\big/\mathbb{M}^{2,1} \to M_3$ and an $\mathbb{M}^{2,1}$-valued 1-form $e := A - \omega \in \Omega^1(P;\mathbb{M}^{2,1})$, where $\omega = \pi^*_{\mathbb{M}^{2,1}}\underline{\omega}$ for the projection $\pi_{M^{2,1}} : P \to P/\mathbb{M}^{2,1}$.

Let $\mathcal{G}$ be the group of all gauge transformations of P, i.e, automorphisms $g : P \to P$ of G-bundles P. Then $\mathcal{G}$ acts on $\mathcal{A}_{ISO(2,1)}$ by $g^*A = g^{-1}Ag + g^{-1}dg$ locally. An infinitesimal gauge transformation will be an element of Lie algebra of $\mathcal{G}$ and is $u = \rho^a P_a + \tau^a J_a \in \Omega^0(\mathfrak{g})$, with $\rho^a, \tau^a \in C^\infty(M_4)$. The infinitesimal gauge transformation of $A_i = e^a_i P_a + \omega^a_i J_a$ is

$$\delta A_i := \left.\frac{d}{dt}\right|_{t=0} \left(e^{tu} \cdot A_i\right) = -D_i u,$$

where D_i is the covariant derivative with respect to the connection A_i

$$D_i u = \partial_i u + [A_i, u],$$

and $\delta A_i = \delta e^a_i P_a + \delta\omega^a_i J_a$ is given by

$$\begin{aligned}
\delta A_i = \delta e^a_i P_a + \delta\omega^a_i J_a &= -D_i u = -D_i\left(\rho^a P_a + \tau^a J_a\right) \\
&= -\partial_i\left(\rho^a P_a + \tau^a J_a\right) - \left[A_i, \rho^a P_a + \tau^a J_a\right] \\
&= -\partial_i\left(\rho^a P_a + \tau^a J_a\right) - \left[e^a_i P_a + \omega^a_i J_a, \rho^b P_b + \tau^b J_b\right] \\
&= -\partial_i\left(\rho^a P_a + \tau^a J_a\right) - e^a_i\tau^b\left[P_a, J_b\right] - \omega^a_i\rho^b\left[J_a, P_b\right] - \omega^a_i\tau^b\left[J_a, J_b\right]
\end{aligned}$$

and by using the commutation relation (15), we obtain

$$\begin{aligned}
\delta A_i &= -\partial_i\left(\rho^a P_a + \tau^a J_a\right) + e^a_i\tau^b\epsilon_{bac}P^c - \omega^a_i\rho^b\epsilon_{abc}P^c - \omega^a_i\tau^b\epsilon_{abc}J^c \\
&= \left(-\partial_i\rho_c + e^a_i\tau^b\epsilon_{bac} - \omega^a_i\rho^b\epsilon_{abc}\right)P^c - \left(\partial_i\tau_c + \omega^a_i\tau^b\epsilon_{abc}\right)J^c.
\end{aligned}$$

Therefore

$$\begin{aligned}
\delta e^a_i &= -\partial_i\rho^a - \epsilon^{abc}e_{ib}\tau_c - \epsilon^{abc}\omega_{ib}\rho_c, \\
\delta\omega^a_i &= -\partial_i\tau^a - \epsilon^{abc}\omega_{ib}\tau_c.
\end{aligned} \tag{16}$$

Proposition 5.1. *The curvature tensor $F(A)$ of the connection A is $F(A) = De + F(\omega)$.*

Proof.

$$
\begin{aligned}
F(A) &= dA + \frac{1}{2}[A, A] = de + d\omega + \frac{1}{2}[e, e] + [e, \omega] + \frac{1}{2}[\omega, \omega] \\
&= (de^a) P_a + (d\omega^a) J_a + \frac{1}{2} e^a \wedge e^b [P_a, P_b] + e^a \wedge \omega^b [P_a, J_b] + \frac{1}{2}\omega^a \wedge \omega^b [J_a, J_b] \\
&= (de^a) P_a + (d\omega^a) J_a + \omega^a \wedge e^b \epsilon_{abc} P^c + \frac{1}{2}\omega^a \wedge \omega^b \epsilon_{abc} J^c \\
&= \left(de^a + \epsilon^a{}_{bc}\omega^b \wedge e^c\right) P_a + \left(d\omega^a + \frac{1}{2}\epsilon^a{}_{bc}\omega^b \wedge \omega^c\right) J_a \\
&= (de + [\omega \wedge e]) + (d\omega + [\omega \wedge \omega]/2),
\end{aligned}
\tag{17}
$$

the components F_{ij} on M_4 are

$$
\begin{aligned}
F_{ij}(A) = &\left(\partial_i e^a_j - \partial_j e^a_i + \epsilon^{abc}(e_{ib}\omega_{jc} + \omega_{ib} e_{jc})\right) P_a \\
&+ \left(\partial_i \omega^a_j - \partial_j \omega^a_i + \epsilon^{abc}\omega_{ib}\omega_{jc}\right) J_a.
\end{aligned}
$$

□

If M_4 is closed, $\partial M_4 = \emptyset$, the de Rham cohomology class $[\langle F_A \wedge F_A \rangle] \in H^4_{DR}(M_4)$ does not depend on choice of the connection A.

We study $ISO(2,1)$ gauge field on a M_4 using integral of Pontryagin form $\langle F \wedge F \rangle = F^a \wedge F^b d_{ab}$ on M_4, where d_{ab} is an invariant quadratic form on the Lie algebra of $ISO(2,1)$. By using the quadratic form d_{ab} defined in (14) and the curvature (17), we obtain the invariant

$$
\begin{aligned}
\int_{M_4} U = \int_{M_4} \langle F(A) \wedge F(A) \rangle &= \int_{M_4} \langle (de + [\omega \wedge e]) \wedge (d\omega + [\omega \wedge \omega]/2) \rangle \\
&= \int_{M_4} \left(de^a + \epsilon^a{}_{bc}\omega^b \wedge e^c\right) \wedge \left(d\omega^b + \frac{1}{2}\epsilon^b{}_{ef}\omega^e \wedge \omega^f\right) d_{ab} \\
&= \int_{M_4} \epsilon^{ijkl}(\partial_i e^a_j - \partial_j e^a_i + \epsilon^{abc}(\omega_{ib} e_{jc} + e_{ib}\omega_{jc}))(\partial_k \omega_{la} - \partial_l \omega_{ka} + \epsilon_{ade}\omega^d_k \omega^e_l).
\end{aligned}
$$

$$ \tag{18} $$

In Palatini formalism of gravity theory in $3d$ spacetime, we regard $e^a = e^a_i dx^i$ as gravitational field, not a connection. We define the gravitational field as local trivialization map of TM_4,

$$ e : U \times \mathbb{R}^4 \to TU, $$

over a coordinate neighborhood of U in M_4. This sends the Minkowski metric $\eta = (- + ++)$ of $\mathbb{R}^4$ to a metric g on M by $g(u,v) = \eta\left(e^{-1}(u), e^{-1}(v))\right)$, $u, v \in TU$, or $g_{\mu\nu} = \eta_{IJ} e^I_\mu e^J_\nu$, the components e^I_μ are inverse of e^μ_I which are determined by mapping sections $p \mapsto (p, \boldsymbol{e}_I) \in \Gamma\left(U \times \mathbb{R}^4\right)$, $I = 0, 1, 2, 3$ to sections $e(p, \boldsymbol{e}_I)$ in TU, by $e(p, \boldsymbol{e}_I) = e^\mu_I (\partial_\mu)_p$, where $\{\boldsymbol{e}_I\}_{I=1,\cdots,4}$ is the standard frame of $\mathbb{R}^4$.

Therefore the transformation generated by $\{P^a\}$ in $ISO(2,1)$ correspond to diffeomorphisms, and the covariant derivative with respect to connection ω^a_i corresponds to local Lorentz symmetry $SO(2,1)$ with generators $\{J^a\}$, this covariant derivative is $D := D_\omega = d + \omega$. Therefore the second part of the curvature (17) is a curvature with respect to the connection ω, so it has to satisfy the Bianchi identity:

$$ DF^a(\omega) = dF^a(\omega) + \epsilon^{abc}\omega_b F_c(\omega) = 0. $$

In fact,

$$ \begin{aligned} & d(d\omega^a + \frac{1}{2}\epsilon^{abc}\omega_b\omega_c) + \epsilon^{abc}\omega_b F_c = d^2\omega^a + \frac{1}{2}\epsilon^{abc} d(\omega_b\omega_c) + \epsilon^{abc}\omega_b(d\omega_c + \frac{1}{2}\epsilon_{cde}\omega^d\omega^e) \\ & = d^2\omega^a + \frac{1}{2}\epsilon^{abc} d(\omega_b\omega_c) + \epsilon^{abc}\omega_b d\omega_c + \epsilon^{abc}\omega_b \frac{1}{2}\epsilon_{cde}\omega^d\omega^e, \end{aligned} $$

using $d^2\omega^a = 0$ and

$$ \begin{aligned} \epsilon^{abc}\omega_b d\omega_c &= \frac{1}{2}\epsilon^{abc}\omega_b d\omega_c + \frac{1}{2}\epsilon^{abc}\omega_b d\omega_c = \frac{1}{2}\epsilon^{abc} d\omega_c\omega_b + \frac{1}{2}\epsilon^{abc}\omega_b d\omega_c \\ &= -\frac{1}{2}\epsilon^{acb} d\omega_c\omega_b + \frac{1}{2}\epsilon^{abc}\omega_b d\omega_c = -\frac{1}{2}\epsilon^{abc}\left((d\omega_b)\omega_c + \omega_b d\omega_c\right) = -\frac{1}{2}\epsilon^{abc} d(\omega_b\omega_c) \end{aligned} $$

also

$$\epsilon^{abc}\omega_b\epsilon_{cde}\omega^d\omega^e = \epsilon^{abc}_{cde}\omega_b\omega^d\omega^e = (\delta^a_d\delta^b_e - \delta^b_d\delta^a_e)\omega_b\omega^d\omega^e = 0,$$

we find

$$DF^a(\omega) = dF^a(\omega) + \epsilon^{abc}\omega_b F_c(\omega) = 0.$$

We can show that $U := F^a \wedge F^b d_{ab}$ is an exact form, $F^a \wedge F^b d_{ab} = dV$ for some $V \in \Omega^3(M_4)$. Therefore, if the four manifold M_4 has a boundary three manifold M_3, (18) reduces to an integral on M_3. Using the covariant derivative $D := D_\omega = d + \omega$, we write the curvature (17) of $A = e + \omega$ as

$$F(A) = (de + [\omega \wedge e]) + (d\omega + [\omega \wedge \omega]/2) = De + F(\omega),$$

therefore

$$\begin{aligned} U &= \int\limits_{M_4} \langle F(A) \wedge F(A)\rangle = \int\limits_{M_4} \langle (D_\omega e + F(\omega)) \wedge (D_\omega e + F(\omega))\rangle \\ &= \int\limits_{M_4} \langle De \wedge F(\omega)\rangle = \int\limits_{M_4} d\,\langle e \wedge F(\omega)\rangle = \int\limits_{M_3} \langle e \wedge F(\omega)\rangle, \end{aligned}$$

where we used $DF(\omega) = 0$, and $M_3 = \partial M_4$.

Let $P_3 = P_4|_{M_3} \to M_3$ be an $ISO(2,1)$-principal bundle over $M_3 = \partial M_4$, and $\mathcal{A}_{ISO(2,1)}(P_3)$ be the space of all $ISO(2,1)$-connections $A \in \Omega^1(P_3; \mathfrak{g})$ on P_3.

Definition 5.1. The $ISO(2,1)$-Chern-Simons action functional $I_{CS} : \mathcal{A}_{ISO(2,1)}(P_3) \to \mathbb{R}$ is defined to be

$$\begin{aligned} I_{CS} &= \int_{M_3} \left\langle A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right\rangle = \int\limits_{M_3} \langle e \wedge F(\omega)\rangle \\ &= \int\limits_{M_3} e_a \wedge (d\omega^a + \frac{1}{2}\epsilon^{abc}\omega_b \wedge \omega_c) \\ &= \int\limits_{M_3} \epsilon^{ijk} e_{ia}(\partial_j\omega_{ka} - \partial_k\omega_{ja} + \epsilon_{abc}\omega^b_j\omega^c_k). \end{aligned} \tag{19}$$

Theorem 5.1. *The Chern-Simons action I_{CS} of the connection $A = e + \omega$ can be identical to the Einstein-Hilbert action in $2+1$ spacetime.*

We see this clearly if we regard e^a as a gravitational field(tensor) and ω^{ab} as connection of Lorentz gauge group in $2+1$ spacetime, but we have to verify that this action is both Lorentz and diffeomorphism invariant ([13]).

Since $S(A) = \int\limits_{M_4} F^a \wedge F^b d_{ab}$ is gauge invariant, $S(g \cdot A) = S(A)$, so the boundary integral $\int\limits_{M_3} \langle e \wedge F(\omega) \rangle$ is gauge invaraint. The action (19) is gravity in $3d$ spacetime. This means that the gravity in $3d$ spacetime is a Chern-Simons gauge theory of group $ISO(2,1)$, but not only Lorentz symmetry which is generated by $\tau^a J_a$ but also there is symmetry generated by $\rho^a P_a$ in (16). But the gravity is described by local Lorentz symmetry, and a gravitational field e_i^a is an isomorphism between tangent spaces on M_3 at each fixed $p \in M_3$,

$$e : T_p M_3 \to \mathbb{R}^n,$$

which maps arbitrary metric $g(p)$ on $T_p M_3$ to the Minkowski metric $\eta = (-+\cdots+)$ on $\mathbb{R}^n$. Thus we relate the transformation δe_i^a in (16) to a diffeomorphism and so regarding e^a as gravitational field (tensor). We can see this by taking the transformations (16) generated by $\rho^a P_a$ with $\tau^a J_a = 0$:

$$\delta e = -d\rho - [\omega, \rho], \quad \delta\omega = 0,$$

or

$$\delta e_i^a = -\partial_i \rho^a - \epsilon^{abc} \omega_{ib} \rho_c, \quad \delta \omega_i^a = 0. \tag{20}$$

We compare this transformation with the transformations under an infinitesimal diffeomorphism generated by a vector field $v = v^i \partial_i$. The field $e^a = e_i^a dx^i$ is a one-form, so it changes under a diffeomorphism is the derivative:

$$\delta_v e = \mathcal{L}_v(e) = i_v(de) + d(i_v(e)),$$

or

$$\delta_v e^a = \partial_i e^a_j i_v(dx^i dx^j) + d(i_v(e^a)).$$

Then we use $i_v(dx^i dx^j) = i_v(dx^i)dx^j - i_v(dx^j)dx^i$, and $i_v(dx^i) = v^i$, we obtain

$$\delta_v e^a = \mathcal{L}_v(e^a) = \partial_i e^a_j(v^i dx^j - v^j dx^i) + \partial_i(v^j e^a_j)dx^i = \partial_j e^a_i v^j dx^i - \partial_i e^a_j v^j dx^i + \partial_i(v^j e^a_j)dx^i.$$

Thus

$$\delta_v e^a_i = \mathcal{L}_v(e^a) = (\partial_j e^a_i - \partial_i e^a_j)v^j + \partial_i(v^j e^a_j). \tag{21}$$

Same thing we do for the connection ω^a_i, to get

$$\delta_v \omega = \mathcal{L}_v(\omega) = i_v(d\omega) + d(i_v(\omega)),$$

or

$$\delta_v \omega^a_i = (\partial_j \omega^a_i - \partial_i \omega^a_j)v^j + \partial_i(v^j \omega^a_j).$$

We need to identical (20) with (21), we let $\rho^a = v^j e^a_j$ and replace v by $-v$, this is

$$\delta_v e^a_i = \mathcal{L}_v(e^a) = -(\partial_j e^a_i - \partial_i e^a_j)v^j - \partial_i \rho^a.$$

Then we take the difference $\delta_v e^a_i - \delta e^a_i$,

$$\begin{aligned}\delta_v e^a_i - \delta e^a_i &= -(\partial_j e^a_i - \partial_i e^a_j)v^j - \partial_i \rho^a + \partial_i \rho^a + \epsilon^{abc}\omega_{ib}\rho_c \\ &= -(\partial_j e^a_i - \partial_i e^a_j)v^j + \epsilon^{abc}\omega_{ib} v^j e_{jc},\end{aligned} \tag{22}$$

or

$$\begin{aligned}\delta_v e^a_i - \delta e^a_i &= -(\partial_j e^a_i - \partial_i e^a_j + \epsilon^{abc}\omega_{jb}e_{ic} - \epsilon^{abc}\omega_{ib}e_{jc})v^j \\ &\quad + (\epsilon^{abc}\omega_{jb}e_{ic} - \epsilon^{abc}\omega_{ib}e_{jc})v^j + \epsilon^{abc}\omega_{ib}v^j e_{jc}.\end{aligned} \tag{23}$$

The $\partial_j e^a_i + \epsilon^{abc}\omega_{jb}e_{ic} = D_j e^a_i$ is covariant derivative with respect to Lorentz connection ω^a, so

$$\delta_v e^a_i - \delta e^a_i = -(D_j e^a_i - D_i e^a_j)v^j + (\epsilon^{abc}\omega_{jb}e_{ic})v^j.$$

The first terms vanish by the equation of motion $De^a = 0$, which we get from

$$F(\omega + t\delta\omega) = F(\omega) + tD(\delta\omega) + \frac{t^2}{2}[\delta\omega \wedge \delta\omega],$$

for

$$\left.\frac{d}{dt}\right|_{t=0} \int_{M_3} \langle e \wedge F(\omega + t\delta\omega)\rangle = \int_{M_3} \langle e \wedge D(\delta\omega)\rangle = \int_{M_3} d\langle e \wedge (\delta\omega)\rangle - \int_{M_3} \langle (De) \wedge \delta\omega\rangle = 0,$$

we get $De = 0$, where M_3 is closed. The second term in $\delta_v e_i^a - \delta e_i^a$ is $\epsilon^{abc}\omega_{jb}e_{ic}v^j$ we can relate it to local Lorentz transformation generated by $\tau_b = \omega_{jb}v^j$.

Proposition 5.2. $I_{CS} : \mathcal{A}_{ISO(2,1)}(P_3) \to \mathbb{R}$ *is invariant under the action of one-parameter family of diffeomorphisms.*

Proof. We need to verify that I_{CS} in (19) is invariant under arbitrary infinitesimal diffeomorphism generated by a vector field v on M_3. Let $\{\varphi_t\}_{t\in I}$ be the one-parameter family of diffeomorphisms $\varphi_t : M_3 \to M_3$ generated by a vector field $v \in \Gamma(TM_3)$, $\frac{d}{dt}\varphi_t = v \circ \varphi_t$. Then the derivative of Chern-Simons action is

$$\delta_v I_{CS} = \left.\frac{d}{dt}\right|_{t=0} \mathrm{I_{CS}}(\varphi_t^* A) = \int_{M_3} \mathcal{L}_v \left(e_a \wedge (d\omega^a + \frac{1}{2}\epsilon^{abc}\omega_b \wedge \omega_c)\right). \tag{24}$$

Note that $\theta = e_a \wedge (d\omega^a + \frac{1}{2}\epsilon^{abc}\omega_b \wedge \omega_c)$ is a 3-form in $3d$ manifold M_3, so that $d\theta = 0$, and its Lie derivative is

$$\mathcal{L}_v(\theta) = i_v(d\theta) + d(i_v(\theta)) = d(i_v(\theta)) = d(\text{2-form}),$$

the integration over d(2-form) term vanishes since M_3 has no boundary by assumption. So $\delta_v I_{CS} = 0$, and I_{CS} is a diffeomorphism invariant as required. $\square$

We can include a cosmological constant λ using gauge groups $SO(2,2)$ or $SO(3,1)$, but the universal covering space is not the Minkowski space since the condition $[P^a, P^b] \neq 0$ is necessary to include cosmological constant in the Lagrangian, whilst $[P^a, P^b] \neq 0$ is not true in the Minkowski space. This space is the de Sitter or the anti-de Sitter space, depending on the sign of λ. We include λ in the commutation relations of the generators $\{J^a, P^a\}$ as

$$[J_a, J_b] = \epsilon_{abc}J^c, \quad [J_a, P_b] = \epsilon_{abc}P^c \quad [P_a, P_b] = \lambda\epsilon_{abc}J^c, \tag{25}$$

these are the same commutation relations of generators of the groups $SO(2,2)$ and $SO(3,1)$, so the gravity with cosmological constant in $3d$ spacetime is Chern-Simons with gauge groups $SO(2,2)$ and $SO(3,1)$. We check the invariance of the bilinear form P_bJ^b of the Lie algebra (25),

$$[J_a, P_bJ^b] = P^b[J_a, J_b]+[J_a, P_b]J^b = P^b\epsilon_{abc}J^c+\epsilon_{abc}P^cJ^b = \epsilon_{abc}P^bJ^c-\epsilon_{acb}P^cJ^b = 0,$$

and

$$\begin{aligned}
[P_a, P_bJ^b] &= P^b[P_a, J_b] + [P_a, P_b]J^b = -P^b\epsilon_{bac}P^c + \lambda\epsilon_{abc}J^cJ^b \\
&= \epsilon_{abc}P^bP^c - \lambda\epsilon_{acb}J^cJ^b = \frac{1}{2}\epsilon_{abc}[P^b, P^c] - \frac{1}{2}\lambda\epsilon_{acb}[J^c, J^b] \\
&= \frac{1}{2}\epsilon_{abc}\lambda\epsilon^{bck}J_k - \frac{1}{2}\lambda\epsilon_{acb}\epsilon^{cbk}J_k = \frac{1}{2}\epsilon_{bca}\lambda\epsilon^{bck}J_k - \frac{1}{2}\lambda\epsilon_{cba}\epsilon^{cbk}J_k \\
&= \lambda\delta_a^kJ_k - \lambda\delta_a^kJ_k = 0.
\end{aligned}$$

Therefore bilinear form P_bJ^b is invariant under the adjoint action of $SO(3,1)$, and thus we have the invariant quadratic form on the Lie algebra of $SO(3,1)$,

$$\left\langle J^a, P^b \right\rangle = \eta^{ab}, \quad \left\langle J^a, J^b \right\rangle = \left\langle P^a, P^b \right\rangle = 0. \tag{26}$$

Let $P_4 \to M_4$ be a principal $SO(3,1)$-bundle over a connected oriented manifold M_4 with boundary M_3. The gauge field is a connection A on P_4, and for a local trivialization $P_4|_U \cong U \times G$ over a coordinate neighborhood (U, x), A is locally 1-form with values in the Lie-algebra $\mathfrak{g}$ of $SO(3,1)$ with generators $\{J_a, P_a\}$ (note that $\{J_a\}$ is closed subalgebra),

$$A = e + \omega = e_i^aP_adx^i + \omega_i^aJ_adx^i \in \Gamma(T^*U \otimes \mathfrak{g}).$$

Note that $SO(3,1)$ acts on hyperboloid $\mathcal{H} \subset \mathbb{M}^{3,1}$ transitively and the stabilizer $SO(3,1)_p$ at a point $p \in \mathcal{H}$ is isomorphic to $SO(2,1)$, so that

$$\mathcal{H} \cong \mathrm{SO}\,(3,1)\,/\mathrm{SO}\,(3,1)_p \cong \mathrm{SO}\,(3,1)\,/\mathrm{SO}\,(2,1)\,.$$

Then a $SO(3,1)$-connection A on the $SO(3,1)$-principal P_4 induces a connection $\underline{e}$ on the induced $\mathcal{H}$-bundle $P_4/SO(2,1) \to M_4$ and $\mathfrak{so}(2,1)$ valued 1-form $\omega = A - e \in \Omega^1(P_4; \mathfrak{so}(2,1))$, where $e := \pi^*_{SO(2,1)}\underline{e}$ for projection $\pi_{SO(2,1)} : P_4 \to P_4/SO(2,1)$.

Let $\mathcal{G} = Aut(P_4)$ be the group of all gauge transformations $g : P_4 \to P_4$. Then $\mathcal{G}$ acts on $\mathcal{A}_{SO(3,1)}$. An infinitesimal gauge transformation is $u = \rho^a P_a + \tau^a J_a \in \Omega^0(\mathfrak{g})$, with $\rho^a, \tau^a \in C^\infty(M_4)$. The infinitesimal gauge transformation of the connection A is

$$\delta A = -D_A u,$$

where D_A is covariant derivative with respect to the connection A defined by

$$D_A u = du + [A, u].$$

Then the infinitesimal gauge transformation of $A = e + \omega \in \Omega^1(U; \mathfrak{g})$ by $u = \rho + \tau \in \Omega^0(\mathfrak{g})$ ($\rho = \rho^a P_a$, $\tau = \tau^a J_a$) is

$$\delta A = \delta e + \delta\omega = -D_A(\rho + \tau).$$

regarding the commutation relations (25), we obtain

$$\delta e = -d\rho - [e, \tau] - [\omega, \rho]\,, \quad \delta\omega = -d\tau - [\omega, \tau] - \lambda\,[e, \rho]\,, \tag{27}$$

Now we calculate the curvature tensor of the connection A with respect to the Lie algebra (25),

$$\begin{aligned}
F(A) = dA + \frac{1}{2}\,[A, A] &= de + d\omega + \frac{\lambda}{2}\,[e, e] + [e, \omega] + \frac{1}{2}\,[\omega, \omega] \\
&= (de^a)\,P_a + (d\omega^a)\,J_a + \frac{1}{2} e^a \wedge e^b\,[P_a, P_b] + e^a \wedge \omega^b\,[P_a, J_b] + \frac{1}{2}\omega^a \wedge \omega^b\,[J_a, J_b] \\
&= (de^a)\,P_a + (d\omega^a)\,J_a + \frac{1}{2}\lambda e^a \wedge e^b \epsilon_{abc} J^c + \omega^a \wedge e^b \epsilon_{abc} P^c + \frac{1}{2}\omega^a \wedge \omega^b \epsilon_{abc} J^c \\
&= \left(de^a + \epsilon^a{}_{bc}\omega^b \wedge e^c\right) P_a + \left(d\omega^a + \frac{1}{2}\lambda \epsilon^a{}_{bc} e^b \wedge e^c + \frac{1}{2}\epsilon^a{}_{bc}\omega^b \wedge \omega^c\right) J_a \\
&= (de + [\omega, e]) + (d\omega + [\omega, \omega]/2 + \lambda[e, e]/2) \in \Omega^2(\mathfrak{g}_{P_4}),
\end{aligned}$$

$$(28)$$

its components on M_4 are

$$F_{ij} = [D_i, D_j] = (\partial_i e^a_j - \partial_j e^a_i + \epsilon^{abc}(\omega_{ib} e_{jc} + e_{ib}\omega_{jc}))P_a \\ + (\partial_i \omega^a_j - \partial_j \omega^a_i + \epsilon^{abc}(\omega_{ib}\omega_{jc} + \lambda e_{ib} e_{jc}))J_a.$$

The topological invariant Pontryagin form $Tr\,(F(A) \wedge F(A))$, with using quadratic form (26), is

$$\begin{aligned} U = \langle F(A) \wedge F(A) \rangle &= \left\langle (de + [\omega \wedge e]) \wedge \left(d\omega + \frac{1}{2}[\omega \wedge \omega] + \frac{\lambda}{2}[e \wedge e] \right) \right\rangle \\ &= \left(de^a + \epsilon^a{}_{bc}\omega^b \wedge e^c \right) \wedge \left(d\omega^e + \frac{1}{2}\epsilon^e{}_{fk}\omega^f \wedge \omega^k + \frac{\lambda}{2}\epsilon^e{}_{fk} e^f \wedge e^k \right) \eta_{ae}. \end{aligned} \tag{29}$$

As we did before, in $3d$ spacetime, we need to regard the connection e^a as gravitational field (tensor), not connection. Therefore we relate the transformation which generated by $\{P^a\}$ to diffeomorphism, and produce a covariant derivative with respect to local Lorentz symmetry $SO(2,1)$ with generators $\{J^a\}$, the corresponding connection is ω, and the covariant derivative is $D = de^a + \omega$. The second part of the curvature (28) is a curvature and so satisfies the Bianchi identity

$$DF(\omega) = dF(\omega) + [\omega, F(\omega)] = 0.$$

We write (29) as

$$\begin{aligned} U &= \left\langle (de + [\omega \wedge e]) \wedge \left(d\omega + \frac{1}{2}[\omega \wedge \omega] + \frac{1}{2}[e \wedge e] \right) \right\rangle \\ &= \left\langle (De) \wedge \left(F(\omega) + \frac{1}{2}[e \wedge e] \right) \right\rangle = \langle De \wedge F(\omega) \rangle + \frac{1}{2} \langle De \wedge [e \wedge e] \rangle . \end{aligned}$$

Note that

$$\langle (De \wedge [e \wedge e]) \rangle = \frac{1}{3} d \, \langle (e \wedge [e \wedge e]) \rangle \,, \text{ and } DF(\omega) = 0,$$

we obtain

$$U = d\, \langle e \wedge F(\omega) \rangle + \frac{1}{6} d\, \langle e \wedge [e \wedge e] \rangle \, .$$

Using the quadratic form (26), we get

$$
\begin{aligned}
U &= d\left(e^a \wedge F^b(\omega)\right)\langle P_a, J_b\rangle + \frac{1}{6} d\left(e^a \wedge e^b \wedge e^c\right)\langle P_a, [P_b, P_c]\rangle \\
&= d\left(e^a \wedge F^b(\omega)\right)\eta_{ab} + \frac{1}{6}\lambda\varepsilon^d_{bc} d\left(e^a \wedge e^b \wedge e^c\right)\langle P_a, P_d\rangle . \\
&= d\left(e^a \wedge F^b(\omega)\right)\eta_{ab} + \frac{1}{6}\lambda\varepsilon^d_{bc} d\left(e^a \wedge e^b \wedge e^c\right)\eta_{ad} \\
&= d\left(e^a \wedge F^b(\omega)\right)\eta_{ab} + \frac{1}{6}\lambda\varepsilon_{bca} d\left(e^a \wedge e^b \wedge e^c\right). \\
&= d\left(e^a \wedge F^b(\omega)\right)\eta_{ab} + \frac{1}{6}\lambda\varepsilon_{abc} d\left(e^a \wedge e^b \wedge e^c\right).
\end{aligned}
$$

By integration on the $4d$ manifold M_4 with boundary M_3, we obtain

$$
\begin{aligned}
\int_{M_4} U &= \int_{M_4} d\left(e^a \wedge F^b(\omega)\eta_{ab} + \frac{1}{6}\lambda\epsilon_{abc} e^a \wedge e^b \wedge e^c\right) \\
&= \int_{M_3} \left(e^a \wedge F^b(\omega)\eta_{ab} + \frac{1}{6}\lambda\epsilon_{abc} e^a \wedge e^b \wedge e^c\right),
\end{aligned}
$$

so

$$
\int_{M_4} U = \int_{M_3} e_a \wedge \left(d\omega^a + \frac{1}{2}\epsilon^a{}_{bc}\omega^b \wedge \omega^c + \frac{1}{6}\lambda\epsilon^a{}_{bc} e^b \wedge e^c\right).
$$

Definition 5.2. Let $\mathcal{A}_{SO(3,1)}(P_3)$ be the space of all $SO(3,1)$-connections on the trivial principal $SO(3,1)$-bundle $P_3 \to M_3$. Let $\lambda > 0$ be a constant. Then the Chern-Simons action functional $I_{CS} : \mathcal{A}_{SO(3,1)}(P_3) \to \mathbb{R}$ is defined to be

$$
\begin{aligned}
I_{CS}(A) &= \int_{M_3} \left\langle A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right\rangle \\
&= I_{CS}(e + \omega) = \int_{M_3} \left\langle e \wedge F(\omega) + \frac{1}{6} e \wedge [e \wedge e] \right\rangle \\
&= \int_{M_3} \epsilon^{ijk} e_{ia} \left(\partial_j \omega^a_k - \partial_k \omega^a_j + \epsilon^{abc}\omega_{jb}\omega_{kc} + \frac{1}{3}\lambda\epsilon^{abc} e_{jb} e_{kc}\right) dV.
\end{aligned}
$$

For $\lambda < 0$, we define the corresponding Chern-Simons action functional I_{CS} by replacing $SO(3,1)$ with $SO(2,2)$. This is gravity Lagrangian in $3d$ spacetime manifold when we consider only the Lorentz $SO(2,1)$ symmetry. But also by regarding e^a and ω^a as connections of the gauge group $SO(3,1)$, this Lagrangian is Chern-Simons.

Thus we obtain the following.

Theorem 5.2. *Let $I_{CS} : \mathcal{A}_G(P_3) \to \mathbb{R}$ be the Chern-Simons action functional defined in Definition 5.1 for $G = ISO(2,1), \lambda = 0$ or Definition 5.2 for $G = SO(3,1), \lambda > 0$, and $G = SO(2,2)$, $\lambda < 0$.*

(1) The critical points of the action I_{CS} are flat G-connections, and the space $\mathcal{F}/\mathcal{G}$ of all gauge equivalent classes of G-connections can be identified with the space of all conjugacy classes of G-representations of the fundamental group $\pi_1(M_3)$, $\mathcal{F}/\mathcal{G} \cong Hom(\pi_1(M_3), G)/\sim$, where $G = ISO(2,1)$ for $\lambda = 0$, $G = SO(3,1)$ for $\lambda > 0$, and $G = SO(2,2)$ for $\lambda < 0$.

(2) The equations of motion $F(A) = 0$ for $A = \omega + e$ of the action I_{CS} give the constraints $De = 0$ and $F(\omega) + \lambda[e,e]/2 = 0$.

In quantum theory, e is regarded as the conjugate momentum of ω in the spatial part Σ of the spacetime $M_3 = \mathbb{R} \times \Sigma$ and the state $\hat{\psi}(\omega) \in \mathcal{H}(\mathcal{F}/\mathcal{G})$ which solves $\hat{F}\psi(\omega) = 0$ depends only on the topology of M_3.

For the quantization of the system we need to find the phase space of the system and the constraints which generate symmetry transformations of a three manifold $M_3 = \mathbb{R} \times \Sigma$. Since $M_3 = \mathbb{R} \times \Sigma$ is contractible to Σ, the solutions will depend on the topology of Σ. On $M_3 = \mathbb{R} \times \Sigma$, the action becomes

$$\begin{aligned} I = \int_{\mathbb{R}} dt \int_{\Sigma} & \epsilon^{0jk} e_{0a} \left(\partial_j \omega_k^a - \partial_k \omega_j^a + \epsilon^{abc} \omega_{jb} \omega_{kc} + \frac{1}{3} \lambda \epsilon^{abc} e_{jb} e_{kc} \right) \\ & + \epsilon^{i0k} e_{ia} \left(\partial_0 \omega_k^a - \partial_k \omega_0^a + \epsilon^{abc} \omega_{0b} \omega_{kc} + \frac{1}{3} \lambda \epsilon^{abc} e_{0b} e_{kc} \right) \\ & + \epsilon^{ij0} e_{ia} \left(\partial_j \omega_0^a - \partial_0 \omega_j^a + \epsilon^{abc} \omega_{jb} \omega_{0c} + \frac{1}{3} \lambda \epsilon^{abc} e_{jb} e_{0c} \right). \end{aligned}$$

Now we introduce $\epsilon^{0jk} \equiv \epsilon^{jk}$ the invariant anti-symmetric Levi-Civita tensor on

$2d$ surface Σ to obtain

$$I = \int_{\mathbb{R}} dt \int_{\Sigma} e_{0a}\epsilon^{ij}\left(\partial_i \omega_j^a - \partial_j \omega_i^a + \epsilon^{abc}\omega_{ib}\omega_{jc} + \frac{1}{3}\lambda\epsilon^{abc}e_{ib}e_{jc}\right)$$
$$- \epsilon^{ij}e_{ia}\left(\partial_0 \omega_j^a - \partial_j \omega_0^a + \epsilon^{abc}\omega_{0b}\omega_{jc} + \frac{1}{3}\lambda\epsilon^{abc}e_{0b}e_{jc}\right)$$
$$+ \epsilon^{ij}e_{ia}\left(\partial_j \omega_0^a - \partial_0 \omega_j^a + \epsilon^{abc}\omega_{jb}\omega_{0c} + \frac{1}{3}\lambda\epsilon^{abc}e_{jb}e_{0c}\right).$$

Note that sum over the three λ terms give $\lambda\epsilon^{ij}\epsilon^{abc}e_{0a}e_{ib}e_{jc}$, and since Σ is closed, we have

$$\int_{\Sigma} \epsilon^{ij}e_{ia}\partial_j\omega_0^a = -\int_{\Sigma} \epsilon^{ij}\omega_0^a\partial_j e_{ia} = \int_{\Sigma} \epsilon^{ij}\omega_0^a\partial_i e_{ja},$$

using it in I and and reordering, we get

$$I = -2\int_{\mathbb{R}} dt \int_{\Sigma} \epsilon^{ij}e_{ia}\partial_0\omega_j^a$$
$$+ \int_{\mathbb{R}} dt \int_{\Sigma} e_{0a}\epsilon^{ij}\left(\partial_i \omega_j^a - \partial_j \omega_i^a + \epsilon^{abc}\omega_{ib}\omega_{jc} + \lambda\epsilon^{abc}e_{ib}e_{jc}\right)$$
$$+ \int_{\mathbb{R}} dt \int_{\Sigma} \omega_{0a}\epsilon^{ij}\left(\partial_i e_j^a - \partial_j e_i^a + \epsilon^{abc}(\omega_{ib}e_{jc} + e_{ib}\omega_{jc})\right).$$

The term $-2dt\epsilon^{ij}e_{ia}\partial_0\omega_j^a$ does not depends on time parametrization for some diffeomorphism $\phi : \mathbb{R} \to \mathbb{R}$, $t \mapsto \phi(t)$. Therefore the fields e_i^a and ω_i^a can be regarded as a canonical coordinate and momenta of the phase space on a slice of constant time $\{t\} \times \Sigma$. To obtain the Poisson brackets, we recall that the term

$$dt p_i \partial_0 q^i$$

in the Lagrangian gives the Poisson brackets $\{q^i, p_j\} = \delta^i_j$. Now our first term in the Lagrangian

$$-2dt\epsilon^{ij}e_{ia}(x)\partial_0\omega_j^a(x)$$

gives the Poisson brackets

$$\{\omega_j^a(x), -2\epsilon^{ik}e_{ib}(y)\} = \delta_j^k\delta_b^a\delta^2(x-y).$$

Then multiply $\epsilon_{\ell k}$ to both sides and using $\epsilon_{\ell k}\epsilon^{ik} = \delta^i_\ell$, it becomes

$$\{\omega^a_j(x), e_{\ell b}(y)\} = \frac{1}{2}\epsilon_{j\ell}\delta^a_b\delta^2(x-y).$$

Thus we obtain the Poisson brackets

$$\{\omega^a_i(x), e^b_j(y)\} = \frac{1}{2}\epsilon_{ij}\eta^{ab}\delta^2(x-y), \quad \{e^a_i(x), e^b_j(y)\} = 0 \quad \{\omega^a_i(x), \omega^b_j(y)\} = 0. \tag{30}$$

Since the fields e^a_0 and ω^a_0 do not have the time derivative, they are not dynamical, and their equation of motion are constraints,

$$\begin{aligned} \frac{\delta I}{\delta \omega_{0a}} &= \epsilon^{ij}\left(\partial_i e^a_j - \partial_j e^a_i + \epsilon^{abc}(\omega_{ib}e_{jc} + e_{ib}\omega_{jc})\right) = 0, \\ \frac{\delta I}{\delta e_{0a}} &= \epsilon^{ij}\left(\partial_i \omega^a_j - \partial_j \omega^a_i + \epsilon^{abc}\omega_{ib}\omega_{jc} + \lambda\epsilon^{abc}e_{ib}e_{jc}\right) = 0. \end{aligned} \tag{31}$$

The first equation of (31) is the standard torsion-free condition that determines ω in terms of e. If $\lambda = 0$ then the second equation implies that the connection ω is flat, $F(\omega) = d\omega + [\omega \wedge \omega]/2 = 0$.

If we regard e^I as a component of a connection $A = e + \omega$, these constraints are the vanishing of the curvature $F_A = 0$, gauge connections which locally are gauge $A = g^{-1}dg$, for $g \in \mathcal{G}$. The only gauge-invariant observables that do not vanish when the constraints are imposed are global observables, such as holonomies around possible non-contractible loops in M. The Poisson brackets (30) suggest either ω or e are the canonical coordinates, so we need to choose one of them. Then the quantum state is either $\psi(\omega) \in L^2(\mathcal{F}/\mathcal{G})$ or $\psi(e) \in L^2(\mathcal{F}/\mathcal{G})$. If we choose $\psi(e^a_i)$ then the infinitesimal gauge transformation of $A = e + \omega$ for $\lambda = 0$ are

$$\begin{aligned} \delta e^a_i &= -\partial_i \rho^a - \epsilon^{abc}e_{ia}\tau_c - \epsilon^{abc}\omega_{ib}\rho_c, \\ \delta \omega^a_i &= -\partial_i \tau^a - \epsilon^{abc}\omega_{ib}\tau_c, \end{aligned} \tag{32}$$

which implies that δe^a_i contains not only e^a_i but also ω^a_i, so we can not choose e^a_i as a coordinates. But ω^a_i transforms only to ω^a_i, thus we can choose ω^a_i as a canonical coordinates. The physical state is a state that annihilates the operator corresponds to the curvature of $A = e + \omega$,

$$\hat{F}^a_{ij}(e) = \partial_i \hat{e}^a_j - \partial_j \hat{e}^a_i + \epsilon^{abc}(\hat{\omega}_{ib}\hat{e}_{jc} + \hat{e}_{ib}\hat{\omega}_{jc}),$$

and

$$\hat{F}^a_{ij}(\omega) = \partial_i \hat{\omega}^a_j - \partial_j \hat{\omega}^a_i + \epsilon^{abc}\hat{\omega}_{ib}\hat{\omega}_{jc} + \lambda \epsilon^{abc}\hat{e}_{ib}\hat{e}_{jc}.$$

To do this we replace the Poisson brackets (30) by commutators such

$$[\hat{\omega}^a_i(x), \hat{e}^b_j(y)] = \frac{1}{2}\epsilon_{ij}\eta^{ab}\delta^2(x-y), \quad [\hat{e}^a_i(x), \hat{e}^b_j(y)] = 0 \quad [\hat{\omega}^a_i(x), \hat{\omega}^b_j(y)] = 0,$$

by this we define $\hat{e}^a_i$ by

$$\hat{e}^a_i = \frac{1}{2}\varepsilon_{ij}\eta^{ab}\frac{\delta}{\delta\omega^b_j}.$$

The physical state is solution of the quantization of equations of the motion, $\hat{F}^a_{ij}(e)\psi(\omega) = 0$ and $\hat{F}^a_{ij}(\omega)\psi(\omega) = 0$.

The constraint $\hat{F}^a_{ij}\psi(\omega) = 0$ generates diffeomophism invariance of $\psi(\omega)$, the infinitesimal transformation of the connection ω under the infinitesimal diffeomophism generated by a vector field v is

$$\delta_v \omega^a_i = \mathcal{L}_v \omega^a_i = v^j F^a_{ij}(\omega).$$

Then the derivative of the state function $\psi(\omega)$ with respect to this infinitesimal diffeomophism is

$$\psi(\omega) \mapsto \psi(\omega + \delta_v\omega^a_i) = \psi(\omega) + \delta_v\omega^a_i \frac{\delta}{\delta\omega^a_i}\psi(\omega) = \psi(\omega) + v^j F^a_{ij}(\omega)\frac{\delta}{\delta\omega^a_i}\psi(\omega).$$

Therefore if $F^a_{ij} = 0$, the state $\psi(\omega)$ is invariant under this transformation. Therefore the state $\psi(\omega)$ depends only on the homotopy class $[\gamma] \in \pi_1(\Sigma)$. Let $\Omega(\Sigma, x_0)$ be the space of all smooth loops $\gamma : I \to \Sigma$ based on x_0, then we have the holonomy map

$$\mathrm{hol}(\omega) : \Omega(\Sigma, x_0) \to SO(2,1)$$

with respect to a connection ω, and ([6])

$$\big(\gamma \simeq \gamma' \Rightarrow hol_\gamma(\omega) = hol_{\gamma'}(\omega)\big) \Leftrightarrow F = 0.$$

Note that $F(A) = 0$ ($A = e + \omega$) if and only if $F(\omega) = 0$ and $De = 0$, and the gauge equivalence classes of flat connections $A = e + \omega$ are completely characterized by conjugacy classes of holonomy representation $hol(A) : \pi_1(\Sigma) \to ISO(2, 1)$.

Now we define the state $\psi(\omega)$ by $\psi_\gamma(\omega) = Tr\mathrm{hol}_\gamma(\omega)$. If the connection ω is flat, the holonomy of ω is determined by the holonomy of $A = e + \omega$, and ω is determined by e by $De = 0$, then we have

$$\psi : Hom\,(\pi_1(\Sigma), SO(2, 1)) \to \mathbb{C}.$$

This state satisfies $\hat{F}^a_{ij}(\omega)\psi(\omega) = 0$ as required for the quantization. We give the states $\psi(\omega)$ physical aspects by making them eigenstates of hermitan operator that measure some physical quantities, such as area, volume, etc. Therefore for two distinct eigenvalues correspond to orthonormal eigenstates with respect to an inner product which keeps the eigenvalues invariant. We can define the states $\psi(\omega)$ using a spin network basis ([14]), in which the eigenvalues of area and volume are invariant under gauge transformations and diffeomorphisms, and hence both of them are unitary operators that keep the inner product invariant. Therefore the states $\psi(\omega)$ on a loop γ depends on the homotopy classes $[\gamma]$ of that loop.

6 Summary

We have seen that the Chern-Simons theory is a gauge theory that measures the topological invariants, like the abelian Chern-Simons theory measures a of linking and self-linking numbers of a knot. The non-Abelian Chern-Simons measures link invariants, like Jones polynomial which associates with spin 1/2 representation in an $SU(2)$ Chern-Simons theory. Also we have seen that the solution of equation of motion of Chern-Simons action give a theory, it is WZW theory. We have given two examples in which the WZW action induces Minkowski and Taub-NUT metrics on the group manifolds. We saw that by using the Chern-Simons theory in describing the gravity in $3 + 1$ dimension gives a solutions depends only on global measurements of the manifold, like the holonomy of flat connections.

7 Appendix A

A principal fibre bundle consists of the following data: - a manifold P, called the total space; - a Lie group G acting freely on P on the right:

$$P \times G \to P, \quad (p, g) \mapsto pg.$$

The free action means that the stabilizer of every point is trivial, that every element of G (except the identity) moves every point in P. We assume that the space of orbits $\Sigma = P/G$ is a manifold (the base space). With projection $\pi : P \to \Sigma$ and for every $p \in \Sigma$, the submanifold $\pi^{-1}(p) \subset P$ is fibre over Σ. Let $\{U_\alpha\}$ be open cover of Σ, the local trivialization is G-equivariant diffeomorphisms

$$\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G,$$

given by $\psi_\alpha(p) = (\pi(p), g_\alpha(p))$ for some G-equivariant map $g_\alpha : \pi^{-1}(U_\alpha) \to G$. Equivariance means that $g_\alpha(pg) = g_\alpha(p)g$. We say that the bundle is trivial if there exists a diffeomorphism $\psi : P \to \Sigma \times G$ such that $\psi(p) = (\pi(p), \psi(p))$ and such that $\psi(pg) = \psi(p)g$. This last condition is simply the G-equivariance of ψ.

We separate T_pP to vertical and horizontal vector spaces at each point $p \in P$, we get the vertical vector fields by acting of group G on P by

$$\sigma_p(X) = \frac{d}{dt} \left(pe^{tX}\right)\Big|_{t=0},$$

for every vector $X \in \mathfrak{g}$, this satisfies

$$\pi_* \sigma_p(X) = \frac{d}{dt} \left(\pi \left(pe^{tX}\right)\right)\Big|_{t=0} = \frac{d}{dt} \left(\pi\left(p\right)\right)\big|_{t=0} = 0,$$

thus $\sigma_p(X)$ is vertical vector field at $p \in P$.

In this bundle the connection is defined as a map

$$A : \Gamma(TP) \to \Gamma(T_eG),$$

and since P is locally product, then $T\ P|_U = TU \otimes T_eG$, so the connection becomes

$$A : \Gamma(T\Sigma) \to \Gamma(T_eG).$$

We can get this map by letting A be in $T^*\Sigma \otimes T_eG$ and so this map is pairing TM with T^*M. Since A is flat, it can be given in

$$A = g^{-1}dg \in \Gamma(T^*\Sigma \otimes T_eG).$$

Then

$$d(g^{-1}dg) = d(g^{-1})dg = -g^{-1}dg \wedge g^{-1}dg \to d(g^{-1}dg) + g^{-1}dg \wedge g^{-1}dg = 0,$$

so

$$F(A) = dA + A \wedge A = 0.$$

8 Appendix B

We calculate the change of Chern-Simons action under gauge transformations, we use $A \to g^{-1}Ag + g^{-1}dg$ in $L(A) = Tr\left(AdA + \frac{2}{3}A^3\right)$, this gives

$$\begin{aligned} L(A^g) = &- Tr\left(g^{-1}Agg^{-1}(dg)g^{-1}Ag)\right) + Tr\left(g^{-1}Agg^{-1}(dA)g\right) - Tr\left(g^{-1}Agg^{-1}Adg\right) \\ &+ Tr\left(g^{-1}Agd\alpha\right) + Tr\left(\alpha d(g^{-1}Ag)\right) + Tr(\alpha d\alpha) + \frac{2}{3}Tr\left(g^{-1}Ag\right)^3 \\ &+ \frac{2}{3}Tr(\alpha^3) + 2Tr\left((g^{-1}Ag)^2g^{-1}dg\right) + 2Tr\left((g^{-1}Ag)g^{-1}(dg)g^{-1}dg\right) \\ &- Tr\left(A(dg)g^{-1}A)\right) + Tr\left(A(dA)\right) - Tr\left(g^{-1}A^2dg\right) + Tr\left(g^{-1}Agd\alpha\right) \\ &+ Tr\left(\alpha d(g^{-1}Ag)\right) \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} L(A^g) = &- Tr\left(A(dg)g^{-1}A)\right) + Tr\left(A(dA)\right) - Tr\left(g^{-1}A^2dg\right) + Tr\left(g^{-1}Agd\alpha\right) \\ &+ Tr\left(\alpha d(g^{-1}Ag)\right) + Tr(\alpha d\alpha) + \frac{2}{3}Tr\left(A\right)^3 + \frac{2}{3}Tr(\alpha^3) \\ &+ 2Tr\left(g^{-1}Agg^{-1}Agg^{-1}dg\right) + 2Tr\left(g^{-1}Agg^{-1}(dg)g^{-1}dg\right) \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} L(A^g) = &- Tr\left(A^2(dg)g^{-1}\right) + Tr\left(AdA\right) - Tr\left(g^{-1}A^2dg\right) + Tr\left(g^{-1}Agd\alpha\right) \\ &+ Tr\left(\alpha d(g^{-1}Ag)\right) + Tr(\alpha d\alpha) + \frac{2}{3}Tr\left(A\right)^3 + \frac{2}{3}Tr(\alpha^3) \\ &+ 2Tr\left(g^{-1}AAdg\right) + 2Tr\left(g^{-1}A(dg)g^{-1}dg\right) \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} L(A^g) =& Tr\,(AdA) + \frac{2}{3}Tr\,(A)^3 + Tr(\alpha d\alpha) + \frac{2}{3}Tr(\alpha^3) - Tr\left(A^2(dg)g^{-1}\right) \\ & - Tr\left(g^{-1}A^2dg\right) + Tr\left(g^{-1}Agd\alpha\right) + Tr\left(\alpha d(g^{-1}Ag)\right) \\ & + 2Tr\left(g^{-1}AAdg\right) + 2Tr\left(g^{-1}A(dg)g^{-1}dg\right) \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} L(A^g) =& L(A) + L(\alpha) - Tr\left(g^{-1}A^2dg\right) - Tr\left(g^{-1}A^2dg\right) + Tr\left(g^{-1}Agd\alpha\right) \\ & + Tr\left(\alpha d(g^{-1}Ag)\right) + 2Tr\left(g^{-1}A^2dg\right) + 2Tr\left(g^{-1}A(dg)g^{-1}dg\right) \end{aligned}$$

$\Rightarrow$

$$L(A^g) = L(A)+L(\alpha)+Tr\left(g^{-1}Agd\alpha\right)+Tr\left(\alpha d(g^{-1}Ag)\right)+2Tr\left((g^{-1}Ag)g^{-1}(dg)g^{-1}dg\right).$$

By using $g^{-1}(dg)g^{-1}dg = -d\left(g^{-1}dg\right)$, this becomes

$$L(A^g) = L(A)+L(\alpha)+Tr\left(g^{-1}Agd\alpha\right)+Tr\left(\alpha d(g^{-1}Ag)\right)-2Tr\left((g^{-1}Ag)d(g^{-1}dg)\right)$$

$\Rightarrow$

$$\begin{aligned} L(A^g) &= L(A) + L(\alpha) + Tr\left(g^{-1}Agd\alpha\right) + Tr\left(\alpha d(g^{-1}Ag)\right) - 2Tr\left((g^{-1}Ag)d\alpha\right) \\ &= L(A) + L(\alpha) + Tr\left(\alpha d(g^{-1}Ag)\right) - Tr\left((g^{-1}Ag)d\alpha\right). \end{aligned}$$

And using

$$d\left(\alpha g^{-1}Ag\right) = (d\alpha)g^{-1}Ag - \alpha d(g^{-1}Ag),$$

we obtain

$$L(A^g) = L(A) + L(\alpha) - dtr\left(\alpha g^{-1}Ag\right),$$

therefore

$$L(A^g) - L(A) = L(\alpha) - dtr\left(\alpha g^{-1}Ag\right).$$

9 Appendix C, WZW model

As we saw, WZW model has been obtained from required of gauge invariance of wave function of solution of ($F_A\psi(A) = 0$) in Chern-Simons theory with a group G on $2d$ space-like surface Σ. In this section we discuss the symmetries in WZW model, the $G_R \times G_R$ symmetry and conformal symmetry, the conserved currents satisfy the Kac-Moody algebra ([15, 16]). Let X be compact oriented smooth $(2n+1)$-manifold with boundary $\partial M = \Sigma$. We construct trivial principle bundle $P = \Sigma \times G \to \Sigma$.

The WZW actions for $g \in Map(M, G)$ are

$$S^{\pm}(g) = \frac{c}{4\pi}\int_{\Sigma} Tr(g^{-1}\partial g g^{-1}\bar{\partial} g) \pm \frac{c}{12\pi}\int_X Tr(g^{-1}dg)^3. \tag{33}$$

We write the WZ term in $2n+1$ dimensional space M as

$$S_{WZ}(g) = \frac{c}{12\pi}\int_M Tr(g^{-1}dg)^{2n+1},$$

the variation of this term depends only on fields on the $2n$ dimensional boundary $\partial M = \Sigma$, we see this by using the fact that Maurer-Cartan form $Tr(g^{-1}dg)^{2n+1}$ is closed on contractible space like the cylindrical space $B = I \times M$ (with $I = [0,1]$). Let d_B, δ and d_M be the exterior derivatives on B, I and M, we have

$$d_B Tr(g^{-1}d_B g)^{2n+1} = 0,$$

and let $d_B = d_M + \delta$, so

$$(d_M + \delta)Tr(g^{-1}(d_M + \delta)g)^{2n+1} = 0,$$

its component of type (1, 2n + 1) is

$$\delta Tr(g^{-1}d_M g)^{2n+1} + (2n+1)d_M Tr(g^{-1}\delta g)(g^{-1}d_M g)^{2n} = 0,$$

by using $\int_I \delta \int_M d_M = -\int_M d_M \int_I \delta$, we get

$$\delta S_{WZ}(g) = (2n+1)\frac{c}{12\pi}\int_M d_M Tr(g^{-1}\delta g)(g^{-1}d_M g)^{2n},$$

so

$$\delta S_{WZ}(g) = (2n+1)\frac{c}{12\pi}\int_\Sigma Tr(g^{-1}\delta g)(g^{-1}dg)^{2n}. \tag{34}$$

Therefore the varying of $S_{WZ}(g)$ depends only on $Map(\Sigma \to G)$.

We use the Killing form on $Lie(G)$ and metric on Σ to construct the first term in (33):

$$\int_\Sigma Tr(g^{-1}\partial g g^{-1}\bar{\partial} g),$$

but in order to write it in arbitrary n dimensional surface Σ, we write it using Hodge star as

$$\frac{1}{2}\int_\Sigma Tr(g^{-1}dg \wedge *g^{-1}dg),$$

and so

$$S^{\pm}_{WZW}(g) = \frac{c}{4\pi}\frac{1}{2}\int_\Sigma Tr(g^{-1}dg \wedge *g^{-1}dg) \pm \frac{c}{12\pi}\int_X Tr(g^{-1}dg)^{n+1}.$$

In two dimensions, the first term reads

$$\frac{1}{2}\int_\Sigma Tr(g^{-1}dg \wedge *g^{-1}dg) = \frac{1}{2}\int_\Sigma Tr(g^{ij}\theta_i\theta_j)d^2x, \tag{35}$$

we write $\theta = g^{-1}dg$, and g^{ij} is metric on Σ.

The WZW actions can be written as

$$S^{\pm}_{WZW}(g) = \frac{c}{4\pi}\frac{1}{2}\int_\Sigma Tr(g^{ij}\theta_i\theta_j)d^2x \pm \frac{c}{12\pi}\int_X Tr(g^{-1}dg)^3.$$

The variation of first term under $\delta\theta$ is

$$\frac{1}{2}\delta\int_\Sigma Tr(g^{ij}\theta_i\theta_j)d^2x = \int_\Sigma Tr(g^{ij}(\delta\theta_i)\theta_j)d^2x = \int_\Sigma Tr(\delta\theta * \theta),$$

and

$$\delta\theta = \delta(g^{-1}dg) = d(g^{-1}\delta g) + [g^{-1}dg, g^{-1}\delta g],$$

the second term vanish in tracing, so

$$\frac{1}{2}\delta\int_\Sigma Tr(g^{ij}\theta_i\theta_j)d^2x = \int_\Sigma Tr\left(d(g^{-1}\delta g) * \theta\right) = -\int_\Sigma Tr((g^{-1}\delta g)d(*\theta)). \quad (36)$$

Using (34) and (36), the variation of WZW actions under δg is

$$\delta S^{\pm}_{WZW}(g) = \frac{c}{4\pi}\frac{1}{2}\delta\int_\Sigma Tr(g^{-1}dg\wedge *g^{-1}dg) \pm \frac{c}{12\pi}\delta\int_X Tr(g^{-1}dg)^3.$$

$$= -\frac{c}{4\pi}\int_\Sigma Tr((g^{-1}\delta g)d(*\theta)) \pm \frac{3c}{12\pi}\int_\Sigma Tr(g^{-1}\delta g)(g^{-1}dg)^2.$$

The equations of motion are

$$-\frac{c}{4\pi}d(*g^{-1}dg) \pm \frac{3c}{12\pi}(g^{-1}dg)^2 = 0,$$

or

$$-d(*g^{-1}dg) \pm (g^{-1}dg)^2 = 0.$$

Using $d(g^{-1}dg) = -(g^{-1}dg)(g^{-1}dg)$, these equations becomes

$$d(*g^{-1}dg) \pm d(g^{-1}dg) = 0,$$

so

$$d(*g^{-1}dg \pm g^{-1}dg) = 0.$$

In the coordinates $(\bar{z}, z)$, it becomes

$$dz\partial_z\left(g^{-1}\partial_{\bar{z}}g(*d\bar{z}) \pm g^{-1}\partial_{\bar{z}}gd\bar{z}\right) = 0, \text{ and } d\bar{z}\partial_{\bar{z}}\left(g^{-1}\partial_z g(*dz) \pm g^{-1}\partial_z gdz\right) = 0. \quad (37)$$

We can derive the Hodge operator for complex coordinates $(\bar{z}, z)$ from the corresponding Euclidean coordinates (x, y) with Euclidean metric. $*dx = dy$ and $*dy = -dx$, so by linearity of Hodge operator we obtain

$$*dz = *(dx + idy) = dy - idx = -i(dx + idy) = -idz,$$

and

$$*d\bar{z} = *(dx - idy) = dy + idx = i(dx - idy) = id\bar{z}.$$

Using this in (37), we get

$$dz\partial_z \left(ig^{-1}\partial_{\bar{z}}g \pm g^{-1}\partial_{\bar{z}}g\right) d\bar{z} = 0, \text{ and } d\bar{z}\partial_{\bar{z}} \left(-ig^{-1}\partial_z g \pm g^{-1}\partial_z g\right) dz = 0,$$

or

$$\partial_z \left(g^{-1}\partial_{\bar{z}}g\right) = 0, \text{ and } \partial_{\bar{z}} \left(g^{-1}\partial_z g\right) = 0. \tag{38}$$

Therefore we have two conserved currents $J_{\bar{z}} = g^{-1}\partial_{\bar{z}}g$ and $J_z = g^{-1}\partial_z g$ on Σ, so there is two symmetries corresponding to these currents, to find that symmetries we test the WZW Lagrangian under a global transformations like $z \to fz$ and $\bar{z} \to g\bar{z}$, where f and g are constants. The group elements transform as $g(z, \bar{z}) \to g(fz, \bar{z})$ and $g(z, \bar{z}) \to g(z, g\bar{z})$ which can be written as

$$g(z, \bar{z}) \to e^{\epsilon_+(f)}g(z, \bar{z}), \text{ and } g(z, \bar{z}) \to e^{\epsilon_-(g)}g(z, \bar{z}),$$

the functions $\epsilon_+(f)$ and $\epsilon_-(g)$ have to satisfy $\epsilon_+(0) = 0$ and $\epsilon_-(0) = 0$. The WZW actions

$$S^{\pm}(g) = \frac{c}{4\pi}\int_\Sigma Tr(g^{-1}\partial g g^{-1}\bar{\partial}g) \pm \frac{c}{12\pi}\int_X Tr(g^{-1}dg)^3$$

are invariant under $g(z, \bar{z}) \to e^{\epsilon_+(f)}g(z, , \bar{z})$ and $g(z, \bar{z}) \to e^{\epsilon_-(g)}g(z, \bar{z})$ separately, that is, they neither require a relation between the functions ϵ_- and ϵ_+ nor between the constants f and g, this changes the metric $d\bar{z}dz$ by scaling it in addition to rotation. Therefore we have two global conformal symmetries, left and right. To get local symmetry, we let the functions f and g and so ϵ_+ and ϵ_- depend on the coordinates $(\bar{z}, z)$ with requiring the action be invariant up to boundary terms, like

$$\delta S_{WZW}(g) = \int_\Sigma Tr(\partial_\mu B^\mu).$$

For infinitesimal transformation $g(z,\bar{z}) \to e^{\epsilon_+(z)} g(z,\bar{z})$, $e^{\epsilon_+(z)} \approx 1 + \epsilon_+(z)$, we use the property

$$S^{\pm}(g_1 g_2) = S^{\pm}(g_1) + S^{\pm}(g_2) + \frac{1}{\pi} \int_{\Sigma} Tr(g_1^{-1} \partial g_1)(g_2^{-1} \bar{\partial} g_2),$$

by setting $g_1 = e^{\epsilon_+(z)}$ and $g_2 = g(z,\bar{z})$, and ignore $S^{\pm}(e^{\epsilon_+}) = O(\epsilon_+^2)$, we obtain

$$\delta S^{\pm}(g) = S^{\pm}((1+\epsilon_+)g) - S^{\pm}(g) = \frac{1}{\pi} \int_{\Sigma} Tr(e^{-\epsilon_+(z)} \partial e^{\epsilon_+(z)} g^{-1} \bar{\partial} g)$$

$$\approx \frac{1}{\pi} \int_{\Sigma} Tr(\partial \epsilon_+)(g^{-1} \bar{\partial} g) = -\frac{1}{\pi} \int_{\Sigma} Tr(\epsilon_+ \partial (g^{-1} \bar{\partial} g)).$$

As we saw before, the term $\partial(g^{-1}\bar{\partial} g)$ vanishes by the equation of motion (38), therefore $\delta S^{\pm}(g) = 0$, and same thing we find for $g(z,\bar{z}) \to e^{\epsilon_-(\bar{z})} g(z,\bar{z})$. Thus the WZW action is invariant under conformal transformation.

We call the current $J_{\bar{z}}(\bar{z},z) = g^{-1}\partial_{\bar{z}} g$ left current, since it is relates to left conformal transformation $g(\bar{z},z) \to e^{\epsilon_+(z)} g(\bar{z},z)$. We find the right current from the relation

$$\partial_z (g^{-1} \partial_{\bar{z}} g) = g^{-1} \left(\partial_{\bar{z}} (\partial_z g g^{-1}) \right) g,$$

and so $\partial_z(g^{-1}\partial_{\bar{z}} g) = 0$ implies $\partial_{\bar{z}}(\partial_z g g^{-1}) = 0$. We call the current $J_z(\bar{z},z) = (\partial_z g) g^{-1}$ right current, since it is invariant under right conformal transformation $g(\bar{z},z) \to g(\bar{z},z) e^{\epsilon_-(\bar{z})}$:

$$J_z(\bar{z},z) = (\partial_z g) g^{-1} \to \partial_z (g e^{\epsilon_-}) e^{-\epsilon_-} g^{-1} = \partial_z g g^{-1} = J_z(\bar{z},z).$$

Therefore the WZW actions has the symmetry $G_R(\bar{z}) \times G_L(z)$ groups:

$$G_R = Map(\Sigma \to G) \text{ such } \bar{z} \mapsto h_r(\bar{z}),$$

$$G_L = Map(\Sigma \to G) \text{ such } z \mapsto h_l(z).$$

These act on $g(\bar{z},z) \in G$ by

$$g(\bar{z},z) \to h_l(z) g(\bar{z},z) h_r(\bar{z}).$$

When $\Sigma = S^1 \times \mathbb{R}$, we can change the metric $d\bar{z}dz$ to Minkowski metric $dx^+dx^- = -dt^2 + dx^2$ so we change the coordinates like $z \to x^+ = x + t$ and $\bar{z} \to x^- = x - t$. The conformal transformations become

$$g(x^+, x^-) \to h_l(x^+)g(x^+, x^-)h_r(x^-), \tag{39}$$

with currents

$$J_L(x^+, x^-) = J_+(x^+, x^-) = (\partial_+ g)g^{-1} \text{ and } J_R(x^+, x^-) = J_-(x^+, x^-) = g^{-1}\partial_- g. \tag{40}$$

The left and right currents defines Kac-Moody algebra.

10 Appendix D, Virasoro and Affine Lie Algebra

We have seen that the WZW action on Σ is invariant under local conformal transformations with acting $G_L \times G_R$ of the group G, where $G_L = Map(\Sigma \to G)$, $z \mapsto h_l(z)$. And $G_R = Map(\Sigma \to G)$, $\bar{z} \mapsto h_r(\bar{z})$. The corresponding currents are $\bar{\partial}J(z, \bar{z}) = 0$(holomorphic) and $\partial\bar{J}(\bar{z}, z) = 0$(anti-holomorphic). We find that the algebra of these currents is the affine Lie algebra of the group G on Σ, but when we use Sugawara construction for that algebra, we obtain Virasoro algebra, where we use the fact that in quantum theory the currents are regarded as operators that generate the corresponding transformations (conformal transformations) ([10]). Since the current $J(\bar{z}, z) \in \mathfrak{g}$ is holomorphic, we can consider its Laurent expansion around the origin ($z \neq 0$):

$$J(z) = \sum_{n \in \mathbb{Z}} J_n z^{-n-1},$$

and write $\bar{J}(\bar{z})$ as

$$\bar{J}(\bar{z}) = \sum_{n \in \mathbb{Z}} \bar{J}_n \bar{z}^{-n-1}.$$

We study the commutation relations of these currents on a circle $S^1 \subset \Sigma$ in equal time with respect to the coordinates $(x^+, x^-) = (x + i\tau, x - i\tau)$, with transformation

$$z = e^{2\pi i n x^+}, \quad \bar{z} = e^{2\pi i n x^-}.$$

The corresponding metric is

$$x^+ x^- = x^2 + \tau^2, \quad g_{+-} = g_{-+} = 1, \; g_{++} = g_{--} = 0 \tag{41}$$

on Σ. Different values of τ associate with different circles radiuses in Σ. We define loop group of G as set of all maps $S^1 \to G$:

$$LG = maps(S^1, G).$$

Thus the WZW actions have the symmetry $(LG)_L \times (LG)_R$ on $S^1 \subset \Sigma$.
The group $(LG)_L \times (LG)_R$ is given by the maps

$$(LG)_L = Map(S^1 \to G), \quad e^{2\pi i n x^+} \mapsto g(e^{2\pi i n x^+}),$$

$$(LG)_R = Map(S^1 \to G), \quad e^{2\pi i n x^-} \mapsto g(e^{2\pi i n x^-}),$$

at $\tau = constant$. The Lie algebra $Lie(LG) = L\mathfrak{g}$, is vector space of maps $S^1 \to \mathfrak{g}$. If $\{T^a\}$, $a = 1, ..., dim(G)$ is basis of $Lie(G)$, with bilinear skew-symmetric map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ defined by $[T^a, T^b] = f^{ab}{}_c T^c$, and Killing form $Tr(T^a T^b) = \delta^{ab}$. We can describe the maps $S^1 \to \mathfrak{g}$ using Fourier expansion

$$S^1 \ni e^{2\pi i n x^\pm} \mapsto \sum\nolimits_a J^a(x^\pm) T^a = \frac{1}{2} \sum\nolimits_a \sum_{n \in \mathbb{Z}} J^a_n T^a e^{2\pi i n x^\pm} \in L\mathfrak{g}, \quad \tau = constant,$$

so

$$J^a(x^\pm) = \frac{1}{2} \sum_{n \in \mathbb{Z}} J^a_n e^{2\pi i n x^\pm}. \tag{42}$$

We regard $\{J^a(x^\pm)\}$ as a basis for Lie algebra of the loop group LG with the same commutation relations of $\{T^a\}$. The fact that J^a is anti-hermitian operator asserts that $(J^a_n)^\dagger = -J^a_{-n}$. Therefore

$$[J^a(x^\pm), J^b(x^\pm)] = f^{ab}{}_c J^c(x^\pm), \tag{43}$$

and so

$$\frac{1}{4}\sum_{n,m\in\mathbb{Z}}[J_n^a,J_m^b]e^{2\pi i(n+m)x^\pm}=\frac{1}{2}f_c^{ab}\sum_{\ell\in\mathbb{Z}}J_\ell^c e^{2\pi i\ell x^\pm}=\frac{1}{4}f_c^{ab}\sum_{\ell_1,\ell_2\in\mathbb{Z}}J_{\ell_1+\ell_2}^c e^{2\pi i(\ell_1+\ell_2)x^\pm},$$

this gives

$$[J_n^a,J_m^b]=f_c^{ab}J_{n+m}^c.$$

This is bilinear map $L\mathfrak{g}\times L\mathfrak{g}\to L\mathfrak{g}$.

The construction $(J_n^a)^\dagger=-J_{-n}^a$ defines an antilinear involution on $L\mathfrak{g}$, this is the map $\omega:L\mathfrak{g}\to L\mathfrak{g}$ defined by $\omega(J_n^a)=-J_{-n}^a$, it satisfies

$$\omega\circ\omega=1, \tag{44}$$

$$\omega(ax)=a^*\omega(x),\quad x\in L\mathfrak{g},\quad a\in\mathbb{C},$$

$$[\omega(x),\omega(y)]=\omega[x,y].$$

There is a nontrivial central extension of $L\mathfrak{g}$ by $\mathbb{C}$, generated by the map $\Phi:L\mathfrak{g}\times L\mathfrak{g}\to\mathbb{R}$,

$$\Phi(J_n^a,J_m^b)=ic\delta^{ab}n\delta_{n,-m},\quad c\in\mathbb{R}.$$

By this the Lie algebra of $L\mathfrak{g}$ extends to affine Lie algebra $\hat{\mathfrak{g}}$ with bilinear map $\hat{\mathfrak{g}}\times\hat{\mathfrak{g}}\to\hat{\mathfrak{g}}$

$$[J_n^a,J_m^b]=f^{ab}{}_cJ_{n+m}^c+ic\delta^{ab}n\delta_{n,-m}. \tag{45}$$

We show that WZW theory has conformal symmetry with affine Lie algebra (45), that is the currents

$$J_L(x^+,x^-)=J_+(x^+,x^-)=(\partial_+g)g^{-1}\text{ and }J_R(x^+,x^-)=J_-(x^+,x^-)=g^{-1}\partial_-g$$

have the algebra (45) when we regard these currents as generators of the conformal transformations. The left and right conformal transformations are (39)

$$g(x^+,x^-)\mapsto h_l(x^+)g(x^+,x^-)h_r(x^-).$$

If we write $h_l(x^+) = e^{\epsilon(x^+)}$, so $g(x^+, x^-) \mapsto e^{\epsilon(x^+)} g(x^+, x^-)$, the current $J_+(x^+, x^-) = (\partial_+ g) g^{-1}$ transforms as

$$\begin{aligned} J_+ = (\partial_+ g) g^{-1} \mapsto J'_+ &= (\partial_+ e^{\epsilon(x^+)} g) g^{-1} e^{-\epsilon(x^+)} \\ &= e^{\epsilon(x^+)} J_+ e^{-\epsilon(x^+)} + \epsilon'(x^+), \quad \epsilon'(x^+) = \frac{d}{dx^+}\epsilon(x^+). \end{aligned} \tag{46}$$

For small conformal transformation parameter $\epsilon(x^+) \in \mathfrak{g}$, this transformation becomes

$$\delta_\epsilon J_+ = J'_+ - J_+ = [\epsilon(x^+), J_+(x^+)] + \epsilon'(x^+),$$

expanding it in the basis $\{T^a\}$ of $Lie(G)$, we obtain

$$\delta_\epsilon J_+^a = [\epsilon(x^+), J_+^a(x^+)] + (\epsilon^a)'(x^+). \tag{47}$$

This agree with regarding the current as anti-hermitian operator since the generators $\{T^a\}$ of the group G are regarded as anti-hermitian operators. The current J_+ corresponds to the conformal transformation by $h_r(x^+)$, so according to the quantum theory, $\sum_a \int \epsilon^a J_+^a$ is a generator of this conformal transformation. Thus, we obtain the following equation for any operator $\mathcal{O}$,

$$\delta_\epsilon \mathcal{O} = \sum\nolimits_a [\int \epsilon^a(x^+) J_+^a(x^+), \mathcal{O}].$$

Like this, the transformation (47) can be expressed by

$$\delta_\epsilon J_+^b = \sum\nolimits_a [\int \epsilon^a(x^+) J_+^a(x^+), J_+^b].$$

Combining this equation with (47), we have the equality

$$\sum\nolimits_a [\int \epsilon^a(y^+) J^a(y^+), J_+^b(x^+)] = [\epsilon(x^+), J_+^b(x^+)] + (\epsilon^b)'(x^+),$$

We solve this by expanding $\epsilon(x^+)$ in the Lie algebra of $(LG)_L$, this is $\epsilon(x^+) = \sum_a \epsilon^a(x^+) J^a(x^+)$, therefore

$$\sum\nolimits_a \int dy^+ [\epsilon^a(y^+) J^a(y^+), J^b(x^+)] = \sum\nolimits_a [\epsilon^a(x^+) J^a(x^+), J^b(x^+)] + (\epsilon^b)'(x^+),$$

or

$$\sum_a \int dy^+ \epsilon^a(y^+)[J^a(y^+), J^b(x^+)] = \sum_a \epsilon^a(x^+)[J^a(x^+), J^b(x^+)] + (\epsilon^b)'(x^+),$$

which can be solved by

$$[J^a(y^+), J^b(x^+)] = f^{ab}{}_c J^c(x^+)\delta(y^+ - x^+) - \delta^{ab}\frac{d}{dy^+}\delta(y^+ - x^+), \tag{48}$$

with the commutation relation (43). We have to note that when $a = b$ the term $\epsilon^a(x^+)[J^a(x^+), J^b(x^+)]$ does not exist because it comes from $e^{\epsilon^a J^a} J^b e^{-\epsilon^a J^a}$ in (46). We obtain the affine algebra (45) by expanding the generators $\{J^a(x^+)\}$ on circle S^1, like

$$J^a(x^+) = \sum_{n\in\mathbb{Z}} J^a_n e^{2\pi i n x^+}, \quad \delta(y^+ - x^+) = \sum_{k\in\mathbb{Z}} e^{i2\pi k(y^+ - x^+)}, \quad \tau_1 = \tau_2.$$

Using them in (48), we get

$$\begin{aligned}&\sum_{n,m\in\mathbb{Z}} [J^a_n, J^b_m] e^{2\pi i(ny^+ + mx^+)} \\ &= f^{ab}{}_c \sum_{n_1,m_1\in\mathbb{Z}} J^c_{n_1} e^{2\pi i n_1 x^+} e^{2\pi i m_1(y^+ - x^+)} - \delta^{ab}\frac{d}{dy^+}\sum_{n_2\in\mathbb{Z}} e^{2\pi i n_2(y^+ - x^+)},\end{aligned}$$

or

$$\begin{aligned}&\sum_{n,m\in\mathbb{Z}} [J^a_n, J^b_m] e^{2\pi i(ny^+ + mx^+)} \\ &= f^{ab}{}_c \sum_{n_1,m_1\in\mathbb{Z}} J^c_{n_1} e^{2\pi i m_1 y^+} e^{2\pi i(n_1 - m_1)x^+} - 2\pi i\delta^{ab}\sum_{n_2\in\mathbb{Z}} n_2 e^{2\pi i n_2(y^+ - x^+)}.\end{aligned}$$

Identifying with respect to $e^{2\pi i n y^+}$ and $e^{2\pi i m x^+}$ separately, we obtain

$$[J^a_n, J^b_m] = f^{ab}{}_c J^c_{n+m} - 2\pi i\delta^{ab} n\delta_{n,-m},$$

this is same (45) for $c = -2\pi$. Thus we have seen that the Lie algebra of conformal transformations in WZW theory is affine lie algebra (45) of G. Now we see that we need Sugawara construction for this algebra in order to get finite energy spectrum in WZW theory.

As we saw we can use the metric $dx^+dx^- = d\tau^2 + dx^2$ in WZW theory on Σ, by Wick rotation $\tau \mapsto it$ we obtain Lorentz metric $-dt^2 + dx^2$, that is the changing $(x^+, x^-) = (x + i\tau, x - i\tau) \mapsto (x + t, x - t) = (y^+, y^-)$, and so

$$z^{\pm} = e^{2\pi i x^{\pm}} \mapsto z'^{\pm} = e^{2\pi i y^{\pm}},$$

where $z^{\pm}$ are complex coordinates on Σ, by this we have one circle ($|z'| = 1$) with two opposite motions (left and right) instead of circles with different values of radiuses in different times. There is a metric in the first term of

$$S^{\pm}(g) = \frac{k}{4\pi} \int_{\Sigma} Tr(U^{-1}\partial_+ U \wedge *U^{-1}\partial_- U) \pm \frac{k}{12\pi} \int_X Tr(U^{-1}dU)^3,$$

this allows us to calculate the stress-energy tensor using the formula ([17])

$$T_{\mu\nu} = -2\frac{\delta}{\delta g^{\mu\nu}} S.$$

We write $Tr(U^{-1}\partial_i U U^{-1}\partial_j U) g^{ij} d^2x$ with respect to an arbitrary metric tensor g_{ij}, so

$$T_{ij} = -2\frac{\delta}{\delta g^{ij}} S = -2\frac{k}{4\pi} Tr(U^{-1}\partial_i U U^{-1}\partial_j U).$$

Then substituting Lorentain metric $g = g_{ij}dx^i dx^j = -dt^2 + dx^2$, with $x^0 = t$, this gives

$$T_{ij} = -2\frac{\delta}{\delta g^{ij}} S = -2\frac{k}{4\pi} Tr(U^{-1}\partial_i U U^{-1}\partial_j U).$$

Its components are

$$T_{00} = -\frac{k}{2\pi} Tr(U^{-1}\partial_0 U U^{-1}\partial_0 U) = T^{00}, \quad T_{11} = -\frac{k}{2\pi} Tr(U^{-1}\partial_1 U U^{-1}\partial_1 U),$$
$$\text{and } T_{01} = -\frac{k}{2\pi} Tr(U^{-1}\partial_0 U U^{-1}\partial_1 U) = T_{10},$$

The total energy is

$$H \sim \oint dx T^{00}, \quad t = constant,$$

in the new coordinates $y^+ = x + t$, $y^- = x - t$. As we have seen that WZW has affine Lie algebra (45), therefore we write the currents using Fourier expansion on the circle $S^1 \subset \Sigma$ (with $t = constant$) like

$$J_0(y^+) = U^{-1}\partial_0 U = \sum\nolimits_a \sum_{n\in\mathbb{Z}} J_n^a T^a e^{2\pi i n y^+},$$

and

$$J_1^a(y^-) = U^{-1}\partial_1 U = \sum\nolimits_a \sum_{n\in\mathbb{Z}} J'^a_n T^a e^{2\pi i n y^-}.$$

therefore

$$T_{00} = \frac{-k}{2\pi} \sum_{n,m\in\mathbb{Z}} J_n^a J_m^b Tr(T^a T^b) e^{2\pi i(n+m)y^+}, \quad T_{11} = \frac{-k}{2\pi} \sum_{n,m\in\mathbb{Z}} J'^a_n J'^b_m Tr(T^a T^b) e^{2\pi i(n+m)y^-},$$

and using $Tr(T^a T^b) = \delta^{ab}$, we get

$$T^{00} = \frac{-k}{2\pi} \sum\nolimits_a \sum_{n,m\in\mathbb{Z}} J_n^a J_m^a e^{2\pi i(n+m)y^+}, \quad T_{11} = \frac{-k}{2\pi} \sum\nolimits_a \sum_{n,m\in\mathbb{Z}} J'^a_n J'^a_m e^{2\pi i(n+m)y^-}.$$

The energy density is

$$T_{00} = \frac{-k}{2\pi} Tr \sum_{n\in\mathbb{Z}} \left(\sum_{m\in\mathbb{Z}} \sum_{a} J_{n-m}^a J_m^a \right) e^{2\pi i n y^+} = \frac{-k}{2\pi} Tr \sum_{n\in\mathbb{Z}} L_n e^{2\pi i n y^-},$$

with $L_n = \sum\limits_{a,m} J_{n-m}^a J_m^a$.

We use the normalization $L_n = \frac{1}{2c} \sum\limits_{a,m} J_{n-m}^a J_m^a$, with c is central extension real constant in the affine Lie algebra (45)

$$[J_n^a, J_m^b] = i f^{ab}{}_c J_{n+m}^c + c\delta^{ab} n \delta_{n,-m}, \tag{49}$$

using this algebra, we find that generators $\{L_n\}$ satisfies $[L_n, L_m] = (n-m)L_{n+m}$ when $n + m \neq 0$. The hermitian conjugate is

$$(L_n)^\dagger = \sum_{a,m} \left(J_{n-m}^a J_m^a \right)^\dagger = \sum_{a,m} (J_m^a)^\dagger (J_{n-m}^a)^\dagger,$$

using $(J^a_n)^\dagger = J^a_{-n}$, it becomes $(L_n)^\dagger = L_{-n}$, which defines a an antilinear involution on $\{L_n\}$, this is the map $\omega : \{L_n\} \to \{L_n\}$ defined by $\omega(L_n) = L_{-n}$, it satisfies

$$\omega \circ \omega = 1,$$

$$\omega(ax) = a^*\omega(x), x \in \{L_n\}, a \in \mathbb{C},$$

$$[\omega(x), \omega(y)] = \omega[y, x].$$

But for $n \neq 0$, we have $\omega(L_n) \neq L_n$, thus the element L_n, $n \neq 0$, is not self-adjont operator, it does not have real eigenvalues under any representation. But L_0 is self-adjont operator, let $|h\rangle$ be eigenstate with eigenvalue h, so $L_0 |h\rangle = h |h\rangle$. By using $[L_0, L_n] = -nL_n$, we find that $L_n |h\rangle \to |h - n\rangle$, and $L_{-n} |h\rangle \to |h + n\rangle$, therefore L_n and L_{-n} are lowering and raising operators with step n. But there is underline algebra of $\{L_n\}$, it is the algebra of $\{J^a_n\}$.

There is a divergence in $L_0 = \sum\limits_{n\in\mathbb{Z}} J^a_{-n} J^a_n$ (this includes the sum over a), we see this divergence when we calculate $L_0(J^b_m |h\rangle)$, we find $L_0(J^b_m |h\rangle) = \infty |h\rangle$. By using the affine algebra (45)

$$[J^a_n, J^b_m] = if^{ab}{}_c J^c_{n+m} + c\delta^{ab} n \delta_{n,-m},$$

and using the involution map $\omega(J^a_n) = J^a_{-n}$ (44), we obtain

$$[J^b_m, L_0] = \frac{1}{2c}[J^b_m, \sum_{n\in\mathbb{Z}} J^a_{-n} J^a_n]$$

$$= \frac{1}{2c}\sum_{n\in\mathbb{Z}} J^a_{-n}[J^b_m, J^a_n] + \frac{1}{2c}\sum_{n\in\mathbb{Z}}[J^b_m, J^a_{-n}]J^a_n + \frac{1}{2c}\sum_{n\in\mathbb{Z}} J^a_{-n} c\delta^{ab} m \delta_{m,-n} + \frac{1}{2c}\sum_{n\in\mathbb{Z}} c\delta^{ab} m \delta_{m,n} J^a_n$$

$$= \frac{1}{2c}\sum_{n\in\mathbb{Z}} J^a_{-n} i f^{ba}_c J^c_{m+n} + \frac{1}{2c}\sum_{n\in\mathbb{Z}} i f^{ba}_c J^c_{m-n} J^a_n + \frac{1}{2} J^b_m m + \frac{1}{2} m J^b_m$$

$$= \frac{1}{2c}\sum_{n\in\mathbb{Z}} i f^{ba}_c J^a_{-n} J^c_{m+n} + \frac{1}{2c}\sum_{n\in\mathbb{Z}} i f^{ba}_c J^c_{m-n} J^a_n + m J^b_m$$

$$
\begin{aligned}
&= \frac{1}{4c}\sum_{n\in\mathbb{Z}} if_c^{ba}[J_{-n}^a, J_{m+n}^c] + \frac{1}{4c}\sum_{n\in\mathbb{Z}} if_c^{ba}[J_{m-n}^c, J_n^a] + mJ_m^b \\
&= \frac{1}{4c}\sum_{n\in\mathbb{Z}} if_c^{ba} if_d^{ac} J_m^d + \frac{1}{4c}\sum_{n\in\mathbb{Z}} if_c^{ba} if_d^{ca} J_m^d + mJ_m^b \\
&= \frac{1}{4c}\sum_{n\in\mathbb{Z}} fJ_m^b + \frac{1}{4c}\sum_{n\in\mathbb{Z}} fJ_m^b + mJ_m^b,
\end{aligned}
$$

where we used $f^{abc}f_{abd} = f\delta_d^c$ is given by Killing form on Lie algebra $\left\langle J^a \,\middle|\, J^b \right\rangle = f\delta^{ab}$. Therefore

$$
[J_m^b, L_0] = \frac{1}{2c} f \sum_{n\in\mathbb{Z}} J_m^b + mJ_m^b.
$$

Let it act on the eigenstate $|h\rangle$:

$$
J_m^b L_0 |h\rangle - L_0 J_m^b |h\rangle = \frac{1}{2c} f \sum_{n\in\mathbb{Z}} J_m^b |h\rangle + mJ_m^b |h\rangle ,
$$

so

$$
hJ_m^b |h\rangle - L_0 J_m^b |h\rangle = \frac{1}{2c} f \sum_{n\in\mathbb{Z}} J_m^b |h\rangle + mJ_m^b |h\rangle ,
$$

then

$$
\begin{aligned}
L_0(J_m^b |h\rangle) &= hJ_m^b |h\rangle - \frac{1}{2c} f \sum_{n\in\mathbb{Z}} J_m^b |h\rangle - mJ_m^b |h\rangle \\
&= \left(h - \frac{1}{2c}\sum_{n\in\mathbb{Z}} f - m \right) J_m^b |h\rangle .
\end{aligned}
\tag{50}
$$

Therefore J_m^b is lowering operator when $m, f, c > 0$, but the summing in $\sum\limits_{n\in\mathbb{Z}} f$ is infinity, this is undefined, but if there is a highest number $n_0 > 0$ satisfying $J_{n_0}^a |h\rangle = 0$, and since $L_{-n}(n > 0)$ is raising operator and L_n is lowering, the only non-zero of $\sum\limits_{n\in\mathbb{Z}} J_{-n}^a J_n^a$ when acting on $|h\rangle$ is the ordering $J_{-n}^a J_n^a$, for $0 < n < n_0$, and zero otherwise, so

$$
\sum_{n\in\mathbb{Z}} J_{-n}^a J_n^a |h\rangle = 2 \sum_{0<n<n_0} J_{-n}^a J_n^a |h\rangle .
$$

We then get $\sum_n J^b_m |h\rangle = 2n_0 J^b_m |h\rangle$ which is finite. Therefore we rewrite L_n using normal ordering $L_n = \frac{1}{2c}\sum_{m\in\mathbb{Z}} : J^a_{n-m}J^a_m :$ (Sugawara construction) defined in

$$: J^a_{n-m}J^a_m := \begin{cases} J^a_{n-m}J^a_m, & n-m \leq m \\ J^a_m J^a_{n-m}, & m \leq n-m. \end{cases}$$

This gives

$$L_0 = \frac{1}{2c}\sum_{n\in\mathbb{Z}} J^a_{-n}J^a_n = \frac{2}{2c}\sum_{n=0}^{\infty} J^a_{-n}J^a_n.$$

Therefore we obtain

$$\frac{1}{2c}[: J^a_{-n}J^a_n :, J^b_m] = -\frac{1}{2c}(2fn_0 + 2cm)J^b_m,$$

or

$$[: J^a_{-n}J^a_n :, J^b_m] = -2(fn_0 + cm)J^b_m,$$

when $m = n_0$, it becomes

$$[: J^a_{-n}J^a_n :, J^b_{n_0}] = -2n_0(f+c)J^b_{n_0}.$$

The generators $L_n = \frac{1}{2c}\sum_{m\in\mathbb{Z}} : J^a_{n-m}J^a_m :$ satisfies Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{h}{12}(n^3 - n)\delta_{n,-m}, \quad h = \frac{c}{c+f}dim\mathfrak{g},$$

these infinity number of non-commutative generators of Virasoro algebra, the algebra of conformal symmetry in two dimensions. This Virasoro algebra is non-trivial central extensions of Witt algebra. The solution of this algebra gives the stress-energy tensor $T_{00} + T_{11}$. The operator L_n, $n > 0$ is lowering, while L_{-n}, $n > 0$ is raising, and the involution relation between them is $(L_n)^\dagger = L_{-n}$, therefore the only adjoint operator of them is L_0 which has eigenstates, like $|h\rangle$, with eigenvalues h as mentioned above. The spectrum of determines the energy spectrum. As we saw, the energy must be bounded below in order to get finite results, so if $|\psi_0\rangle$ is lowest energy state, then $L_n |\psi_0\rangle = 0, n > 0$.

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