Biquaternions in 3D geometric algebra (*Cl*3)

Miroslav Josipović, 2020

This text is another in a series of supplements to the book [1].

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Multivectors in *Cl*3

As in [1], e_1, e_2, and e_3 are the basis unit vectors, $B_i = je_i$ are the basis unit bivectors (and the quaternion units in *Cl*3), and $j = e_1 e_2 e_3$ is the unit pseudoscalar (and the commutative imaginary unit). A general multivector in *Cl*3 can be written as

$$M = \alpha + x_1 e_1 + x_2 e_2 + x_3 e_3 + n_1 B_1 + n_2 B_2 + n_3 B_3 + j\beta \,,\ \alpha, \beta, x_i, n_i \in \mathbb{R} \,,$$

or in more compact form

$$M = \alpha + \mathbf{x} + j\mathbf{n} + j\beta \,.$$

Note that we can see this as a complex *paravector*

$$M = \alpha + \mathbf{x} + j\left(\beta + \mathbf{n}\right),$$

(in *Cl*3, paravectors, like $\alpha + \mathbf{x}$, are real, see Sect. 2.7.5 in [1]), but also as a complex quaternion (complex *spinor*), i.e., a *biquaternion*.

$$M = \alpha + j\mathbf{n} + j\left(\beta - j\mathbf{x}\right).$$

This means that we can directly use such a multivector as a biquaternion. However, this definition is not in agreement with the standard conventions, because the definition of the unit quaternions in *Cl*3 (as given here) is in a way different from the standard definition. We can remove this “problem” easily by redefinition of the unit bivectors B_i; however, there are reasons to keep our definition (see [1]).

Biquaternions in the standard form

We define a biquaternion as

$$q = q_0 + q_1 \mathbf{I} + q_2 \mathbf{J} + q_3 \mathbf{K} \,,\ q_0, q_1, q_2, q_3 \in \mathbb{C} \,,$$

where $\mathbf{I}$, $\mathbf{J}$, and $\mathbf{K}$ are the unit quaternions. Then such a biquaternion has a faithful 2×2 complex matrix representation

$$\begin{bmatrix} q_0 + q_1 i & q_2 + q_3 i \\ -q_2 + q_3 i & q_0 - q_1 i \end{bmatrix}, \ i = \sqrt{-1}.$$

Biquaternions in *Cl*3

A simple procedure could be just to replace the imaginary unit by the pseudoscalar j and the quaternion units $\mathbf{I}$, $\mathbf{J}$, and $\mathbf{K}$ by the unit bivectors B_i. However, as we discussed above, the definitions of the unit bivectors B_i should be changed.

Anyhow, we know how to express this 2×2 complex matrix as a *Cl*3 multivector (see Sect. 4.2.2 in [1]). First, we can write the complex coefficients as

$$q_\mu = q_{\mu r} + i q_{\mu i}, \ q_{\mu r}, q_{\mu i} \in \mathbb{R}, \ \mu = 0,1,2,3,$$

and get

$$q = q_{0r} - q_{3i} e_1 - q_{2i} e_2 - q_{1i} e_3 + q_{3r} B_1 + q_{2r} B_2 + q_{1r} B_3 + j q_{0i}.$$

Defining the vectors $\mathbf{q}_i = q_{3i} e_1 + q_{2i} e_2 + q_{1i} e_3$ and $\mathbf{q}_r = q_{3r} e_1 + q_{2r} e_2 + q_{1r} e_3$, we have a more compact form

$$q = q_{0r} - \mathbf{q}_i + j\mathbf{q}_r + j q_{0i}.$$

Now we have a connection between the standard formulation and *Cl*3. All this can be calculated using the *Mathematica* package Cl3spectral.nb, which can be free-downloaded from [3].

Quaternion conjugations

Consider now the *complex conjugation* q^*, i.e., $q_\mu \to q_\mu^* = q_{\mu r} - i q_{\mu i}$, which gives the multivector

$$\hat{q} = q_{0r} + \mathbf{q}_i + j\mathbf{q}_r - j q_{0i}.$$

Thus, the complex conjugation of biquaternions is equivalent to the *grade involution* (see Sect. 1.6.3 in [1]).

Applying the *quaternion conjugation* ($q \to q^c = q_0 - q_1\mathbf{I} - q_2\mathbf{J} - q_3\mathbf{K}$), we get

$$\bar{q} = q_{0r} + \mathbf{q}_i - j\mathbf{q}_r + j q_{0i},$$

which is the *Clifford conjugation*. The *multivector amplitude* is

$$q\bar{q} = \left(q_{0r} + j q_{0i}\right)^2 + \mathbf{q}_r^2 - \mathbf{q}_i^2 + 2 j \mathbf{q}_i \cdot \mathbf{q}_r.$$

The convolution of these conjugations (*biconjugation*) gives

$$q^{\dagger} = q_{0r} - \mathbf{q}_i - j\mathbf{q}_r - jq_{0i} ,$$

which is the *reverse* involution, and we have ($\langle\ \rangle$ is for grade 0)

$$\left\langle qq^{\dagger} \right\rangle = \sum_{\mu} \left| q_{\mu} \right|^2 .$$

Multivectors as biquaternions

Starting with the multivector

$$M = \alpha + x_1 e_1 + x_2 e_2 + x_3 e_3 + n_1 B_1 + n_2 B_2 + n_3 B_3 + j\beta ,$$

we can now translate it to the standard biquaternion form

$$q = \alpha + i\beta + \left(n_3 - ix_3\right)\mathbf{I} + \left(n_2 - ix_2\right)\mathbf{J} + \left(n_1 - ix_1\right)\mathbf{K} .$$

The matrix representation is

$$q = \begin{pmatrix} \alpha + x_3 + i\left(\beta + n_3\right) & n_2 + x_1 + i\left(n_1 - x_2\right) \\ -n_2 + x_1 + i\left(n_1 + x_2\right) & \alpha - x_3 + i\left(\beta - n_3\right) \end{pmatrix} .$$

There are many reasons to use multivectors instead of the standard biquaternions, just to mention two of them. First, we have a clear geometric representation. Second, we can calculate using the power of geometric algebra, which means that we have the exponential form, the trigonometric forms, inverses, invariants, spectral decomposition and functions of multivectors, etc. Surprises are yet to come…

References

[1] Josipović, M.: *Geometric Multiplication of Vectors - An Introduction to Geometric Algebra in Physics*, Birkhäuser, 2019

[2] Tian, Y.: *Biquaternions and their complex matrix representations*, Beitrage zur Algebra und Geometrie, 2013

[3] Cl3spectral.nb, http://extras.springer.com/2019/978-3-030-01755-2