Analysis of a purely nonlinear generalized isotonic oscillator equation

M. D. Monsia*[1]

1-Department of Physics, University of Abomey-Calavi, Abomey-Calavi, 01.B P.526, Cotonou, BENIN

Abstract

We perform in this paper a mathematical analysis of a supposed purely nonlinear isotonic oscillator designed to be a generalization of the Ermakov-Pinney differential equation. We calculate its exact and general solution. This allows the determination of new periodic solutions to the Ermakov-Pinney equation as well as non-periodic solutions as complex-valued function. In this context all motions corresponding to this nonlinear isotonic oscillator are not periodic so it is not consistent to consider such differential equations with real coefficients as conservative oscillators which can only have real and periodic solutions like the harmonic oscillator equation.

Keywords: Nonlinear isotonic oscillator, harmonic oscillator, Ermakov-Pinney equation, periodic solution, complex-valued function.

Introduction

During the last decades many autonomous second-order nonlinear differential equations with real coefficients are said to be conservative oscillators and have been mathematically treated as such. For instance, expressions of the time period or angular frequency, and analytical or numerical approximate solutions have been carried out while no exact and explicit general solution is known.

According to [1, 2] an autonomous conservative system with a single degree of freedom can only have periodic motions characterized by the angular frequency, amplitude and initial phase of oscillations. In other words, an autonomous second-order differential equation with real coefficients representing a conservative oscillator can have only real and periodic solutions like the harmonic oscillator equation from which we know the fundamental notion of the so-called conservative oscillator. In this respect, Gadella and Lara [3] investigated the differential equation

* Corresponding author : E-mail: monsiadelphin@yahoo.fr

$$\ddot{x} + \frac{1}{x} = 0 \tag{1}$$

and proved the existence of non-periodic solutions while the equation (1) has been known in the literature as a truly non-linear oscillator [2-4]. In a paper Apedo et al. [5] calculated exactly and explicitly using the generalized Sundman transformation [6-8] the two non-periodic solutions of (1) mentioned in [3]. Motivated by these results, we recently investigated the so-called truly nonlinear oscillator equation

$$\ddot{x} + x + x^{\frac{1}{3}} = 0 \tag{2}$$

We calculated for the first time its exact and explicit general solution as a complex-valued function [9]. Also we examined analytically the supposed truly nonlinear oscillator with power nonlinearity

$$\ddot{x} + c_\alpha^2\, x|x|^{\alpha-1} = 0 \tag{3}$$

where $\alpha > 0$. It is remarkable to notice that the equation (3) has no known exact solutions. For some values of $\alpha > 0$, the exact and explicit general solutions of the equation (3), have been computed for the first time [10]. It is found that the equation (3) may exhibit non-periodic solutions [10]. The identification of these complex-valued solutions shows that the indirect or approximate methods used to treat mathematically these equations are not sufficient or adequate. They are only valid when all the solutions of the oscillator equations are real and periodic. Now consider the autonomous second-order nonlinear differential equation

$$\ddot{x} + \frac{(m+1)c}{8} x^m = \frac{(m+1)c}{8} \frac{1}{x^{m+2}} \tag{4}$$

where $m > 0$, and $c > 0$, proposed by Ghose-Choudhury et al. [11] as a purely nonlinear isotonic oscillator which includes the famous Ermakov-Pinney equation introduced in the literature over a century ago

$$\ddot{x} + \mu x = \frac{\mu}{x^3} \tag{5}$$

where $\mu > 0$. The authors [11] in this way claimed to calculate the time period of (4) using the so-called symmetrization procedure for $0 < x < \infty$. Such a result means that the equation (4) has only real and periodic solutions for any $m > 0$. This observation applied then to the Ermakov-Pinney equation. However, the objective of the present paper is to prove that the equation (4) may exhibit of course, periodic solutions, but also non-periodic solutions by showing the existence for the first time of non-periodic solutions to the celebrated Ermakov-Pinney equation, as well as, of new periodic solutions. To do so we first show that the equation (4) is a particular case of the second-order differential equation equivalent to the first-order equation we have introduced in 2016 in the literature [12] for solving second-order differential equations, and give its exact and general solution (section 2). Secondly we exhibit non-periodic and periodic solutions for the Ermakov-Pinney equation (section 3). Finally a conclusion is given for the work.

2- Statement of the equation (4)

In 2016, Monsia et al. [12] proposed the first-order differential equation

$$b(x,\dot{x}) = \lambda\, \dot{x}^{\ell} g^{q}(x)\, e^{\gamma\int h(x)dx} + a\, \dot{x}^{s} f^{\alpha}(x)\, e^{\sigma\int \varphi(x)dx} \tag{6}$$

for solving second-order nonlinear differential equations. Performing the differentiation of (6) with respect to time one may, after a few mathematical treatments, secure the general second-order differential equation

$$\ddot{x} + \left\{\lambda\, g^{q}(x)\, e^{\gamma\int h(x)dx}\left[q\frac{g'(x)}{g(x)} + \gamma\, h(x)\right] + a\, f^{\alpha}(x)\, e^{\sigma\int \varphi(x)dx}\left[\alpha\frac{f'(x)}{f(x)} + \sigma\varphi(x)\right]\right\}\times \frac{b^{2/\ell}}{\ell\left[\lambda\, g^{q}(x)\, e^{\gamma\int h(x)dx} + a\, f^{\alpha}(x) e^{\sigma\int \varphi(x)dx}\right]^{\frac{2}{\ell}+1}} = 0 \tag{7}$$

where the prime means differentiation with respect to x, the quadratic term $\dot{x}^2$ is eliminated using (6) and $b(x,\dot{x})$ is taken as a constant. The parameters $\lambda, \ell, q, \gamma, s, \alpha.\sigma,$ and a, are arbitrary constants and $f(x),\ g(x),\ h(x)$, and $\varphi(x)$ are arbitrary functions of x. By application of $\lambda = 1, \gamma = s = \sigma = 0$, and $g(x) = f(x) = x$, the first-order differential equation (6) reduces to [9]

$$b = \dot{x}^{\ell} x^{q} + a\, x^{\alpha} \tag{8}$$

and the equation (7) turns into

$$\ddot{x} + \frac{q}{\ell\, x}\left(b\, x^{-q} - a\, x^{\alpha-q}\right)^{\frac{2}{\ell}} + \frac{a\alpha}{\ell} x^{\alpha-q-1}\left(b\, x^{-q} - a\, x^{\alpha-q}\right)^{\frac{2-\ell}{\ell}} = 0 \tag{9}$$

where $\ell \neq 0$. Setting $\ell = 2$, the equation (8) becomes

$$b = \dot{x}^2 x^q + a\,x^\alpha \tag{10}$$

and the equation (9) reads

$$\ddot{x} + \frac{1}{2}(\alpha - q)a\,x^{\alpha-q-1} + \frac{qb}{2}x^{-q-1} = 0 \tag{11}$$

Choosing $q = m+1$, and $\alpha = 2q$, one may obtain the nonlinear differential equation

$$\ddot{x} + \frac{(m+1)a}{2}x^m + \frac{(m+1)b}{2}x^{-m-2} = 0 \tag{12}$$

which is exactly identical to the equation (4) proposed by Ghose-Choudhury et al. [11] by putting $b = -a < 0$, and $a = \frac{c}{4} > 0$. In this way, one may solve the equation (12) using the first-order differential equation (10). Therefore for $q = m+1$, the equation (10) may be written

$$b = \dot{x}^2 x^{m+1} + a\,x^{2m+2} \tag{13}$$

From (13) one may obtain

$$\dot{x}^2 = b\,x^{-m-1} - a\,x^{m+1}, \quad x \neq 0 \tag{14}$$

such that the exact and general solution of (4) is given by the quadrature

$$\pm(t+K) = \int\left(\frac{x^{m+1}}{b - a\,x^{2m+2}}\right)^{\frac{1}{2}} dx \tag{15}$$

where K is an integration constant.

That being so it becomes easy to compute the exact and explicit general periodic and non-periodic solutions for the well-known Ermakov-Pinney equation.

3- Periodic and Non-periodic solutions to the Ermakov-Pinney equations

The Ermakov-Pinney equation [13] has been the object of a rich and various study in the literature since it arises in mathematical modeling in classical and quantum mechanics. The Ermakov-Pinney equation has found also applications in the field of cosmology. The physical importance of this equation results from the fact that an exact periodic solution is known [13]. On the basis of this solution given by Pinney [13] the equation (5) is studied in the literature as a conservative nonlinear isotonic oscillator. However, the objective of this section is of course, to secure new periodic solutions for the Emarkov-Pinney equation as well as non-periodic solutions for this equation for the first time. Now, notice that the Ermakov-Pinney equation (5) is obtained from (12) by putting $m=1$, and $\mu=a$. Hence, its exact and explicit general solution may be derived from (15) as

$$\int \frac{xdx}{\sqrt{-a-ax^4}} = \pm(t+K) \tag{16}$$

which may read

$$\int \frac{xdx}{\sqrt{1+x^4}} = \pm i\sqrt{a}\,(t+K) \tag{17}$$

The integration of the left hand side is immediate and gives

$$\frac{1}{2}\sinh^{-1}\left(x^2\right) = \pm i\sqrt{a}\,(t+K) \tag{18}$$

from which the exact and general solution of the Ermakov-Pinney equation takes the form

$$x(t) = \pm\sqrt{\sinh\left[\pm 2i\sqrt{a}\,(t+K)\right]} \tag{19}$$

which becomes definitively

$$x(t) = \pm\sqrt{i\,\sin\left[\pm 2\sqrt{a}\,(t+K)\right]} \tag{20}$$

where i is the purely imaginary number. As one can see, the solution (20) is a complex-valued function.

It is worth to note that the present nonlinear differential equation theory may handle also the Ermakov-Pinney equation of the form

$$\ddot{x} + a\,x + \frac{a}{x^3} = 0 \tag{21}$$

where $a>0$, to give a new exact and explicit general periodic solution. The equation (21) is obtained from (12) by setting $m=1$, and $b=a$. The equation (21) has been also widely investigated in the literature. An interesting solution has been given by Chandrasekar and coworkers in [14, 15]. Here in the context of preceding values of parameters m and b, the relation (15) turns into

$$\int \frac{xdx}{\sqrt{a - a x^4}} = \pm (t + K_1) \tag{22}$$

where K_1 is a constant of integration. The equation (22) may be written as

$$\int \frac{xdx}{\sqrt{1 - x^4}} = \pm \sqrt{a}\,(t + K_1) \tag{23}$$

to give

$$\frac{1}{2}\sin^{-1}(x^2) = \pm \sqrt{a}\,(t + K_1) \tag{24}$$

from which the new exact and explicit general periodic solution to the Ermakov-Pinney equation (21) is secured in the form

$$x(t) = \pm \sqrt{\sin\left[2\sqrt{a}\,(t + K_2)\right]} \tag{25}$$

where K_2 is arbitrary constant. As one may notice, the period of the solution (25) is $T = \frac{2\pi}{\left(2\sqrt{a}\right)}$ as it is known for the solution of the harmonic oscillator equation.

Now our intention is to compute a new real-periodic solution for the Ermakov-Pinney equation using another approach. To that end consider the reduced form of the first-order differential equation (6) as

$$b = \dot{x}^2 g(x) + a\, f(x) \tag{26}$$

obtained for $\lambda = 1$, $\ell = 2$, $q = 1$, $\gamma = s = \sigma = 0$, and $\alpha = 1$. In this context one may secure by differentiation with respect to time the equivalent second-order differential equation

$$\ddot{x} + \frac{g'(x)}{g(x)}\frac{\dot{x}^2}{2} + \frac{a}{2}\frac{f'(x)}{g(x)} = 0 \tag{27}$$

For $g(x) = 1$, and $f(x) = \frac{2a_1}{n+1}x^{n+1} + \frac{2a_2}{n+1}x^{-n-1}$, the equation (27) becomes

$$\ddot{x} + a\, a_1 x^n = \frac{a\, a_2}{x^{n+2}} \tag{28}$$

Now for $n = 1$, (28) is equivalent to

$$\ddot{x} + a\,a_1 x = \frac{a\,a_2}{x^3} \tag{29}$$

which is identical to (5) for $\mu = a\,a_1$, and $a_1 = a_2$.

In this respect the equation (26) turns into

$$b = \dot{x}^2 + a\,a_1 x^2 + \frac{a\,a_1}{x^2}$$

that is

$$b = \dot{x}^2 + \mu\,x^2 + \frac{\mu}{x^2} \tag{30}$$

The relation (30) gives the general solution of (5) in the quadrature form

$$\int \frac{x\,dx}{\sqrt{bx^2 - \mu\,x^4 - \mu}} = \pm(t + K_3) \tag{31}$$

where K_3 is an integration constant. Making $X = x^2$, where $dx = \pm\frac{dX}{2X^{\frac{1}{2}}}$, the relationship (31) leads to

$$\int \frac{dX}{\sqrt{bX - \mu\,X^2 - \mu}} = \pm 2(t + K_3) \tag{32}$$

which may read

$$\int \frac{dX}{\sqrt{X^2 - \frac{b}{\mu}X + 1}} = \pm 2i\sqrt{\mu}(t + K_3) \tag{33}$$

The equation (33) may be rewritten as

$$\int \frac{dX}{\sqrt{\left(X - \frac{b}{2\mu}\right)^2 - \frac{b^2}{4\mu^2} + 1}} = \pm 2i\sqrt{\mu}\,(t + K_3) \tag{34}$$

Putting $\beta^2 = \frac{b^2}{4\mu^2} - 1 > 0$, and $\beta v = X - \frac{b}{2\mu}$, one may arrive at the relation

$$\int \frac{dv}{\sqrt{v^2 - 1}} = \pm 2i\sqrt{\mu}(t + K_3) \tag{35}$$

that is

$$ch^{-1}(v) = \pm 2i\sqrt{\mu}\,(t + K_3) \tag{36}$$

from which one may secure the new periodic solution to (5) in the form

$$x(t) = \pm\left[\frac{b}{2\mu} + \frac{\sqrt{b^2 - 4\mu^2}}{2\mu}\cos\left(2\sqrt{\mu}(t + K_3)\right)\right]^{\frac{1}{2}} \tag{37}$$

The equation (37) is calculated for $\frac{b^2}{4\mu^2} > 1$. Consequently, it is reasonable to examine the solution of (5) when $\frac{b^2}{4\mu^2} < 1$. In this situation, substituting $p^2 = 1 - \frac{b^2}{4\mu^2} > 0$, and $pv = X - \frac{b}{2\mu}$, where $dX = pdv$, into (34) leads to

$$\int \frac{dv}{\sqrt{1 + v^2}} = \pm 2i\sqrt{\mu}(t + K_4) \tag{38}$$

where K_4 is an arbitrary constant, which yields

$$sh^{-1}(v) = \pm 2i\sqrt{\mu}\,(t + K_4) \tag{39}$$

from which the exact and general non-periodic solution for (5) may be secured in the form

$$x(t) = \pm\left[\frac{b}{2\mu} + i\frac{\sqrt{4\mu^2 - b^2}}{2\mu}\sin\left(2\sqrt{\mu}(t + K_4)\right)\right]^{\frac{1}{2}} \tag{40}$$

One may consider also the case $\frac{b^2}{4\mu^2} = 1$. This situation leads, using (34), to obtain

$$\int \frac{dX}{\sqrt{\left(X - \frac{b}{2\mu}\right)^2}} = \pm 2i\sqrt{\mu}\,(t + K_5) \tag{41}$$

where K_5 is an arbitrary constant. The evaluation of the indefinite integral of the left hand side allows one to get immediately the solution to (5) in the form

$$x(t) = \pm\left[\frac{b}{2\mu} + e^{\pm 2i\sqrt{\mu}(t+K_5)}\right]^{\frac{1}{2}} \tag{42}$$

Now consider the Ermakov-Pinney equation (21). In the context of the generalized equation (28), the equation (21) is established for $a_2 = -a_1 = -1$, and $n = 1$. In this regard, the equation (30) becomes

$$b = \dot{x}^2 + a\,x^2 - \frac{a}{x^2} \tag{43}$$

which implies

$$\int \frac{x\,dx}{\sqrt{bx^2 - a\,x^4 + a}} = \pm(t + K_6) \tag{44}$$

where K_6 is an arbitrary constant. The relation (44) may be rewritten as

$$\int \frac{x\,dx}{\sqrt{x^4 - \frac{b}{a}x^2 - 1}} = \pm 2i\sqrt{a}\,(t + K_6) \tag{45}$$

The change of variable $X = x^2$, leads to

$$\int \frac{dX}{\sqrt{X^2 - \frac{b}{a}X - 1}} = \pm 2i\sqrt{a}\,(t + K_6) \tag{46}$$

as $a > 0$. The equation (46) may also read

$$\int \frac{dX}{\sqrt{\left(X - \frac{b}{2a}\right)^2 - \left(\frac{b^2}{4a^2} + 1\right)}} = \pm 2i\sqrt{a}\,(t + K_6) \tag{47}$$

In this perspective the change of variable $r^2 = \frac{b^2}{4\mu^2} + 1$, and $rv = X - \frac{b}{2a}$, allows us to obtain

$$\int \frac{dv}{\sqrt{v^2 - 1}} = \pm 2i\sqrt{a}\,(t + K_6) \tag{48}$$

from which we may get

$$ch^{-1}(v) = \pm 2i\sqrt{a}\,(t + K_6) \tag{49}$$

The formula (49) gives the new exact and general periodic solution of the Ermakov-Pinney equation (21) in the form

$$x(t) = \pm\left[\frac{b}{2a} + \frac{\sqrt{b^2 + 4a^2}}{2a}\cos\left(2\sqrt{a}(t + K_6)\right)\right]^{\frac{1}{2}} \tag{50}$$

Conclusion

The identification of nonlinear differential equations which are able to represent conservative mechanical and electrical systems has become an attractive problem during the last decades. As we know from the conservative harmonic oscillator, an autonomous second-order ordinary differential equation representing a conservative oscillator can only have real and periodic solutions to assure a reliable and efficient implementation in design. In this context we have investigated the so-called purely nonlinear generalized isotonic oscillator including the celebrated Ermakov-Pinney equation as special case. In contrast to the result following which the Ermakov-Pinney equation has only periodic solutions, we have shown that such a differential equation with singularity at the origin may of course exhibit non-periodic solutions, that is, complex-valued solutions. As such, the designation of the concerned equations as conservative oscillators is not consistent.

References

[1] A. A. Andronov, A. A. Vitt and S. E. Khaikin (1966). Theory of Oscillators, Pergamon Press, Oxford.

[2] R. E. Mickens (2010). Truly Nonlinear Oscillators, World Scientific, Singapore,

[3] M. Gadella and L. P. Lara, On the solutions of a nonlinear pseudo-oscillator equation, Phys. Scr. 89 105205 (2014).

[4] A. Beléndez, E. Gimeno, E. Fernandez, D. I. Méndez and M. L. Alavez, Accurate approximate solution to nonlinear oscillators in which the restoring force is inversely proportional to the dependent variable, Phys. Scr. 77 065004 (2008).

[5] E. A. Doutetien, A. R. Yehossou, P. Mallick, B. Rath, M. D. Monsia, On the General Solutions of a Nonlinear Pseudo-Oscillator Equation and Related Quadratic Liénard Systems, PINSA(2020), Doi:10.16943/ptinsa/2020/154987

[6] J. Akande, D. K. K Adjaï, M. D. Monsia, Theory of exact trigonometric periodic solutions to quadratic Liénard type equations, Journal of Mathematics and Statistics 14 (1) (2018) 232-240, doi:10.3844jmssp.2018.232.240.

[7] D. K. K. Adjaï, L. H. Koudahoun, J. Akande, Y. J. F. Kpomahou, M. D. Monsia, Solutions of the Duffing and Painlevé-Gambier equations by generalized Sundman transformation, Journal of Mathematics and Statistics 14 (1) (2018) 241-252, doi:10.3844jmssp.2018.241.252.

[8] L. H. Koudahoun, Y. J. F. Kpomahou, J. Akande, K. K. D. Adjaï and M. D. Monsia, Integrability Analysis of a Generalized Truly Nonlinear Oscillator Equation, Math.Phys.,viXra.org/1906.0212v2.pdf (2019).

[9] M. D. Monsia, On the Exact Periodic Solution of a Truly Nonlinear oscillator Equation, Math.Phys.,viXra.org/2009.0057v2.pdf (2020).

[10] M. D. Monsia, The non-periodic solution of a truly nonlinear oscillator with power nonlinearity, Math.Phys.,viXra.org/2009.0174v1.pdf (2020).

[11] A. Ghose-Choudhury, Aritra Ghosh, Partha Guha and A. Pandey, On purely nonlinear oscillators generalizing an isotonic potential. International Journal of Non-Linear Mechanics 106 (2018) 55–59.

[12] M. D. Monsia, J. Akande, D. K. K. Adjaï, L. H. Koudahoun, Y. J. F. Kpomahou, A theory of exactly integrable quadratic Liénard type equations, Int. J. Adv. Appl. Math. And Mech. 4 (1) 2016, 21-25.

[13] E. Pinney, The nonlinear differential equation $y''+p(x)y+c\,y^{-3}=0$, Proc. Amer. Math. Soc. 1, 681 (1950).

[14] V. K. Chandrasekar, M. Senthilvelan, A. Kundu and M. Lakshmanan, A nonlocal connection between certain linear and nonlinear ordinary differential equations/oscillators, J. phys. A. Math. Gen. 39 (2006) 9743-9754.

[15] R. Gladwin Pradeep, V. K. Chandrasekar, M. Senthilvelan and M. Lakshmanan, Nonlocal symmetries of a class of scalar and coupled nonlinear ordinary differential equations of any order, Journal of Physics A Mathematical and Theoretical 44(44) (2011), Doi: 10.1088/1751-8113/44/44/445201.