THE WARING RANK OF THE 3×3 DETERMINANT

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Abstract. Let f be a homogeneous polynomial of degree d with coefficients in $\mathbb{C}$. The *Waring rank* of f is the smallest integer r such that f is a sum of r powers of linear forms. We show that the Waring rank of the polynomial

$$x_1y_2z_3 - x_1y_3z_2 + x_2y_3z_1 - x_2y_1z_3 + x_3y_1z_2 - x_3y_2z_1$$

is at least 18, which matches the known upper bound.

1. Introduction

In this paper, we work over the field $\mathbb{C}$ of complex numbers. A polynomial is called *homogeneous* if its non-zero terms have equal degrees, and it is called a *linear form* if its degree is 1. The *Waring rank* of a homogeneous polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ of degree d is the smallest integer r for which one can write

$$f = c_1 \ell_1^d + \ldots + c_r \ell_r^d \tag{1.1}$$

for some linear forms $\ell_1, \ldots, \ell_r \in \mathbb{C}[x_1, \ldots, x_n]$ and scalars $c_1, \ldots, c_r \in \mathbb{C}$. This definition transfers to arbitrary fields of characteristic greater than d, which is a sufficient condition to the existence of the decomposition (1.1) for some r [3]. We remark that, if the ground field is $\mathbb{C}$, then every number admits a root of degree d, and the coefficients $c_1, \ldots, c_r$ can be omitted without loss of generality.

Waring rank is an active topic of modern research [1, 3, 6, 20, 22, 26, 30], and it appears in the study of secant varieties and other questions in algebraic geometry [2, 5, 11, 24, 30, 32]. Its practical applications may include the study of matrix multiplication [7], parametrized algorithms [29], independent component analysis [4, 8, 9]. Waring rank is NP-hard to compute [33], and its value remains unknown for many relevant instances. This paper is devoted to the polynomial $\det_d \in \mathbb{F}[x_{11}, \ldots, x_{dd}]$ defined as the determinant of a generic $d \times d$ matrix. The Waring rank of the determinant grows exponentially with d, but the ratios between the known lower and upper bounds are still exponential [22]. No exact value was known for the Waring rank of the $d \times d$ determinant except the trivial cases

$$\mathrm{WR}(\det_1) = 1, \ \ \mathrm{WR}(\det_2) = 4,$$

where $\mathrm{WR}(f)$ stands for the Waring rank of f. Our aim is to prove the following.

Theorem 1.1. *The Waring rank of $\det_3$ equals* 18.

This solves the problem with the Waring rank of the 3×3 determinant, previously discussed in several research papers [3, 10, 11, 12, 14, 15, 21, 22, 23, 26, 32] and popular media such as the *Open Problem Garden* [38] and *MathOverflow* [39].

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A general lower bound

$$\mathrm{WR}(\det_d) \geqslant \frac{1}{2}\binom{2d}{d} \tag{1.2}$$

was proved by Shafiei [32] using the result of Ranestad, Schreyer [30], and it implies $\mathrm{WR}(\det_3) \geqslant 10$. Landsberg, Teitler [26] used a lower bound based on the singularities of the hypersurface of a given polynomial and proved that $\mathrm{WR}(\det_3) \geqslant 14$. Farnsworth [15] improved the inequality (1.2), and his results allowed to conclude that the border Waring rank of $\det_3$ is at least 14. Since the border Waring rank is less than or equal to the Waring rank, the result of Farnsworth gives a different proof of the inequality $\mathrm{WR}(\det_3) \geqslant 14$. Derksen, Teitler [14] proved an analogue of (1.2) in terms of the so-called *cactus rank*, and they showed that the cactus rank of $\det_3$ is at least 14. Again, since the cactus rank is a lower bound for the Waring rank, this result gives another proof of the inequality $\mathrm{WR}(\det_3) \geqslant 14$. A further improvement was made by Boij, Teitler [3], who used the syzygies of the corresponding apolar ideal and showed that $\mathrm{WR}(\det_3) \geqslant 15$. Finally, the current lower bound was given by Conner, Harper, Landsberg [11] with a computer calculation based on a technique from algebraic geometry. In fact, they proved that the border rank of the tensor corresponding to $\det_3$ equals 17, which implies $\mathrm{WR}(\det_3) \geqslant 17$.

An early upper bound on $\mathrm{WR}(\det_d)$ is given by Gurvits [26] using the inequality

$$\mathrm{WR}(x_1 \cdot x_2 \cdot \ldots \cdot x_d) \leqslant 2^{d-1} \tag{1.3}$$

and the standard expansion of the determinant, which lead to

$$\mathrm{WR}(\det_d) \leqslant 2^{d-1} d! \tag{1.4}$$

and give $\mathrm{WR}(\det_3) \leqslant 24$. We refer the reader to [22] for further improvements on the general bound (1.4), and we note that the inequality in (1.3) can be replaced by the equality [6, 30]. Derksen [13] and Krishna, Makam [23] expressed $\det_3$ as the sum of five products of linear forms, which implied $\mathrm{WR}(\det_3) \leqslant 20$. This bound cannot be further improved by this approach because, as shown by Ilten, Teitler [21], one cannot write $\det_3$ as the sum of less than five such products. Ilten, Süss [20] reiterated this result and gave a proof free of computer calculations. Nevertheless, Conner, Gesmundo, Landsberg, Ventura [3, 10] gave an explicit decomposition of $\det_3$ into the sum of 18 third powers of linear forms. So we have

$$17 \leqslant \mathrm{WR}(\det_3) \leqslant 18,$$

but whether the Waring rank of $\det_3$ is 17 or 18 remained open until now. The aim of this paper is to show that $\mathrm{WR}(\det_3) \geqslant 18$ and complete the proof of Theorem 1.1.

2. Partially symmetric tensors

Similarly to the approach taken in a sister paper [36], we employ the natural correspondence between the Waring rank and symmetric tensor decompositions [1]. Since the polynomial $\det_3$ has degree three, we switch to three-dimensional tensors, and, as in the introduction, all tensors, matrices, vectors, and scalars considered below are taken over $\mathbb{C}$. All linear spaces are assumed to be $\mathbb{C}$-linear and finite dimensional, and $\operatorname{span} \Phi$ denotes the $\mathbb{C}$-linear span of a family Φ of vectors in some $\mathbb{C}$-linear space. It should be noted, however, that the proof of our lower bound remains valid for the Waring rank computed with respect to any other field.

A *symmetric tensor* T is an $n \times n \times n$ array of scalars such that the value of $T(i|j|k)$ remains invariant under a permutation of elements i, j, k in an indexing set of cardinality n. The *symmetric rank* of T is the smallest integer r for which there exist length-n vectors $u_1, \ldots, u_r$ and scalars $c_1, \ldots, c_r$ such that

$$T = c_1 u_1^{\otimes 3} + \ldots + c_r u_r^{\otimes 3} \tag{2.1}$$

with $v^{\otimes 3}$ being the tensor whose $(i|j|k)$ coordinate equals $v_i v_j v_k$.

The Waring rank of a polynomial f equals the symmetric rank of f viewed as a symmetric tensor [1, 26, 30]. The paper [36] contains a more detailed comment on this correspondence and an example of a situation similar to our current setting. We recall that, for any fixed index k, the k-th *slice* of an $n \times n \times n$ tensor T is defined as the $n \times n$ matrix whose (i, j) entry equals $T(i|j|k)$. The linear space spanned by the slices of the $9 \times 9 \times 9$ tensor corresponding to $\det_3$ is

$$\mathcal{L} = \left(\begin{array}{c|ccc|ccc|ccc}
 & x_{11} & x_{12} & x_{13} & x_{21} & x_{22} & x_{23} & x_{31} & x_{32} & x_{33} \\ \hline
x_{11} & 0 & 0 & 0 & 0 & k & -h & 0 & -f & e \\
x_{12} & 0 & 0 & 0 & -k & 0 & g & f & 0 & -d \\
x_{13} & 0 & 0 & 0 & h & -g & 0 & -e & d & 0 \\ \hline
x_{21} & 0 & -k & h & 0 & 0 & 0 & 0 & c & -b \\
x_{22} & k & 0 & -g & 0 & 0 & 0 & -c & 0 & a \\
x_{23} & -h & g & 0 & 0 & 0 & 0 & b & -a & 0 \\ \hline
x_{31} & 0 & f & -e & 0 & -c & b & 0 & 0 & 0 \\
x_{32} & -f & 0 & d & c & 0 & -a & 0 & 0 & 0 \\
x_{33} & e & -d & 0 & -b & a & 0 & 0 & 0 & 0
\end{array}\right) \tag{2.2}$$

with the first row and first column indicating the labels in the indexing set. The x_{11} slice of $\det_3$ is obtained by taking the variable a equal to $1/6$ and all other variables equal to 0 in (2.2). Similarly, the variable b corresponds to the x_{12} slice, the variable c indicates the x_{13} slice, and so on.

Definition 2.1. (See [5].) Let L be a linear space spanned by a family of symmetric $n \times n$ matrices. The *partially symmetric rank* of L is the smallest cardinality of a family Φ of symmetric rank-one matrices such that $L \subseteq \operatorname{span} \Phi$.

Every slice of a tensor T satisfying (2.1) belongs to $\operatorname{span}\{u_1 \otimes u_1, \ldots, u_r \otimes u_r\}$, so the symmetric rank of a tensor is greater than or equal to the corresponding partially symmetric rank. We are going to prove the following result.

Theorem 2.2. *We have* $\operatorname{psr} \mathcal{L} \geqslant 18$, *where* $\mathcal{L}$ *is the linear space in* (2.2).

Here, the notation $\operatorname{psr} \mathcal{L}$ stands for the partially symmetric rank of $\mathcal{L}$. The rest of this paper is devoted to the proof of Theorem 2.2, which implies the desired lower bound in Theorem 1.1 by the above discussion. In the forthcoming Section 3, we proceed with some further notations and basic results. In Section 4, we compare this study with a recent investigation of the Waring rank of the 3×3 permanent [36], and we explain the relevance of the symmetry of $\det_3$ to the current study. In Section 5, we analyze the possible representations of $\mathcal{L}$ which might imply $\operatorname{psr} \mathcal{L} \leqslant 17$, and we use the symmetry to reduce them to several particular situations as described in Theorem 5.6. In Sections 6 and 7, we analyze the possible partially symmetric decompositions of the restrictions of $\mathcal{L}$ to the two relevant submatrices, and Sections 8 and 9 are devoted to several basic properties of small families of symmetric 6×6 rank-one matrices. In Section 10, we put all technical results together, and we prove Theorem 2.2 by showing that none of the situations described in Theorem 5.6 can actually arise. Finally, Section 11 contains several further remarks and relations to several other open problems on tensor decompositions.

3. Our notation and general observations

We begin with some notation related to matroid theory [28].

Definition 3.1. Let H be a finite family of elements in a linear space V. An element $v \in H$ is called a *coloop* of H if v does not belong to the linear span of $H \setminus \{v\}$. A *collinear pair* in H is a linearly dependent subset with two elements.

We are going to prove Theorem 2.2 by contradiction, and, to this end, we adopt one convention in the manner similar to Assumption 3.1 in [36]. More precisely, the following statement is the negation of Theorem 2.2.

Assumption 3.2. There exists a family $\alpha = (\alpha_1, \ldots, \alpha_{17})$ of vectors in $\mathbb{C}^9$ such that α has no collinear pairs, and the linear space

$$\Lambda = \operatorname{span}\{\alpha_1 \otimes \alpha_1, \ldots, \alpha_{17} \otimes \alpha_{17}\} \tag{3.1}$$

contains the space $\mathcal{L}$ as in (2.2).

Remark 3.3. The assumption that α has 17 elements does not cause a loss of generality because, otherwise, we can add a generic vector to α without breaking any relevant property. Similarly, we can assume without loss of generality that no collinear pair appears in α, because otherwise we can replace one of the collinear elements by a generic vector again without breaking any relevant property.

We proceed with some further notation similar to [36].

Notation 3.4. We denote by M the six-dimensional subspace of $\mathbb{C}^9$ cut by the $(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23})$ coordinates, which corresponds to taking the six leftmost columns with respect to the block partition of (2.2).

Notation 3.5. We define m_i as the projection of α_i onto M.

The following statement is similar to Claim 3.4 in [36].

Lemma 3.6. *Let P be a subspace of the space Λ as in* (3.1)*. If $\dim P \geqslant 9$, then*

$$\dim \mathcal{L} \cap P \geqslant \dim P - 8,$$

where $\mathcal{L}$ is the space (2.2)*.*

Proof. We have $\dim(\mathcal{L} + P) \leqslant 17$ because every matrix in $\mathcal{L} \cup P$ belongs to the linear span of the 17 matrices $\alpha_i \otimes \alpha_i$ as in Assumption 3.2. We get

$$17 \geqslant \dim(\mathcal{L} + P) = \dim \mathcal{L} + \dim P - \dim \mathcal{L} \cap P,$$

and the result follows because $\dim \mathcal{L} = 9$. □

The following statement is almost trivial but important for our technique; we used it in the consideration of the 3×3 permanent as well [36].

Observation 3.7. *Let $c_1, \ldots, c_n$ be a family of non-zero scalars, and let $v_1, \ldots, v_n$ be a family of vectors taken in some linear space. If*

$$w = c_1(v_1 \otimes v_1) + \ldots + c_k(v_n \otimes v_n), \tag{3.2}$$

then $\operatorname{rank} w \geqslant 2 \dim \operatorname{span}\{v_1, \ldots, v_n\} - n$*.*

Proof. If vectors $v_{i_1}, \ldots, v_{i_t}$ are linearly independent, then the total of the corresponding t summands in (3.2) has rank t. Since there are $n-t$ summands remaining, their total has rank at most $n-t$, and hence $\operatorname{rank} w \geqslant t-(n-t)$. $\square$

4. Using the symmetry

One may want to compare the current problem of computing $\mathrm{WR}(\det_3)$ with the analogous question for the 3×3 permanent, which is the polynomial

$$\mathrm{per}_3 = x_1y_2z_3 + x_1y_3z_2 + x_2y_3z_1 + x_2y_1z_3 + x_3y_1z_2 + x_3y_2z_1.$$

The upper bound $\mathrm{WR}(\mathrm{per}_3) \leqslant 16$ was known for a while [18, 31] before the matching lower bound was confirmed [12, 36]. As explained above, the current approach may look similar to [36], but the problem of computing $\mathrm{WR}(\det_3)$ has two substantial differences. In particular, the desired lower bound of 18 is larger than the corresponding bound for per_3, so the current paper may require a more deeper combinatorial analysis as compared to [36]. However, the complexity of the problem can be reduced by the use of the symmetries of $\det_3$ as in [11]. In fact, for any pair of $n \times n$ matrices A and B with $\det(AB) = 1$, the mappings

$$X \to AXB \quad \text{and} \quad X \to AX^\top B \tag{4.1}$$

preserve[1] the determinant of an unknown $n \times n$ matrix X. The linear transformations preserving the permanent of X have the form (4.1) as well, but the matrices A and B should be restricted to have only one non-zero entry in every row and in every column [27]. As we can see, the symmetry group of $\det_3$ is much richer, and we are going to exploit it in the result of this auxiliary section. Before we proceed, we need to recall the tensor analogue of a linear substitution of variables.

Definition 4.1. Let t be a positive integer, let U and V be linear spaces. Any linear mapping $\mathcal{A} : U \to V$ induces the linear mapping

$$\underbrace{U \otimes \ldots \otimes U}_{t \text{ times}} \to \underbrace{V \otimes \ldots \otimes V}_{t \text{ times}}$$

defined by the formula

$$\sum_{i=1}^{k} a_{i1} \otimes \ldots \otimes a_{it} \quad \to \quad \sum_{i=1}^{k} (\mathcal{A}a_{i1}) \otimes \ldots \otimes (\mathcal{A}a_{it})$$

for any positive integer k and arbitrary vectors (a_{ij}) in U. We write $T_{\mathcal{A}}$ to denote the image of a tensor $T \in U \otimes \ldots \otimes U$ under this mapping.

Remark 4.2. Let α_i be a vector in the family α as in Assumption 3.2. In what follows, we think of α_i as the 3×3 matrix which has the (p, q) position equal to the corresponding x_{pq} entry in the vector α_i. In particular, a pair of 3×3 non-singular matrices A, B induce the invertible linear mapping on $\mathbb{C}^9$ defined as $X \to AXB$.

The following is the main result of this section.

Lemma 4.3. *Let $\mathcal{A}$ be an invertible mapping of the form* (4.1)*. The condition* (3.1) *in Assumption 3.2 remains true if every α_i is replaced by $\mathcal{A}\alpha_i$.*

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[1]By an old result of Frobenius [16], no other linear transformation of X preserves $\det X$.

Proof. The validity of (3.1) means that there exist $\beta_1, \dots, \beta_{17}$ such that

$$\Delta = \alpha_1 \otimes \alpha_1 \otimes \beta_1 + \ldots + \alpha_{17} \otimes \alpha_{17} \otimes \beta_{17},$$

where Δ is the tensor corresponding to $\det_3$. As explained above, the tensor Δ is invariant under the corresponding mapping as in Definition 4.1, so we get

$$\Delta = (\mathcal{A}\alpha_1) \otimes (\mathcal{A}\alpha_1) \otimes (\mathcal{A}\beta_1) + \ldots + (\mathcal{A}\alpha_{17}) \otimes (\mathcal{A}\alpha_{17}) \otimes (\mathcal{A}\beta_{17}),$$

which gives a desired conclusion. □

A matrix-theoretic description of the determinant preservers is as follows.

Lemma 4.4. *If $\mathcal{L}$ is the space as in* (2.2)*, then $\mathcal{A}\mathcal{L}\mathcal{A}^\top = \mathcal{L}$, whenever*

$$\mathcal{A} = \left(\begin{array}{c|c|c} A & O & O \\ \hline O & A & O \\ \hline O & O & A \end{array}\right)$$

with all blocks being 3×3, and A is a non-singular 3×3 matrix.

Proof. The mapping $X \to XA^\top$ corresponds to the matrix $\mathcal{A}$. □

Lemma 4.5. *If $\mathcal{L}$ is the space as in* (2.2)*, then $\mathcal{B}\mathcal{L}\mathcal{B}^\top = \mathcal{L}$, whenever*

$$\mathcal{B} = \left(\begin{array}{c|c|c} b_{11}I & b_{12}I & b_{13}I \\ \hline b_{21}I & b_{22}I & b_{23}I \\ \hline b_{31}I & b_{32}I & b_{33}I \end{array}\right)$$

with all blocks being 3×3, and $B = (b_{ij})$ is a non-singular 3×3 matrix.

Proof. The mapping $X \to BX$ corresponds to the matrix $\mathcal{B}$. □

5. A stronger version of Assumption 3.2

In this section, we employ Lemma 4.3 to give several conditions on the set α which can be added to Assumption 3.2 without loss of generality.

Lemma 5.1. *Let V be a k-dimensional vector space. Let $u = (u_1, \dots, u_k)$ and $v = (v_1, \dots, v_k)$ be two families of vectors in V. Then the vectors*

$$\pi_1 u_1 + \ldots + \pi_k u_k \quad \text{and} \quad \pi_1 v_1 + \ldots + \pi_k v_k$$

are collinear for some $(\pi_1, \dots, \pi_k) \neq (0, \dots, 0)$.

Proof. If v is linearly dependent, then we complete the proof immediately by choosing $(\pi_1, \dots, \pi_k) \neq (0, \dots, 0)$ such that $\pi_1 v_1 + \ldots + \pi_k v_k = 0$. Otherwise, v is a basis of V, and we write the vectors of the coordinates of $u_1, \dots, u_k$, with respect to the basis v, as the columns of the $k \times k$ matrix which we call U. Since the ground field is algebraically closed, we can find t such that $\det(tI - U) = 0$, and then there exist $(\pi_1 \dots, \pi_k) \neq (0, \dots, 0)$ such that

$$\pi_1(tv_1 - u_1) + \ldots + \pi_k(tv_k - u_k) = 0,$$

which implies the desired conclusion. □

Definition 5.2. We write $\operatorname{row} a$ to denote the row space of a matrix a.

Lemma 5.3. *Let a and b be non-collinear 3×3 matrices such that*

$$\dim(\operatorname{row} a + \operatorname{row} b) \geqslant 2.$$

Then, for some 1×3 matrix ℓ, the matrices ℓa and ℓb are not collinear.

Proof. If x and y are non-singular 3×3 matrices, then the current statement is true if and only if it is true for (xay, xby) instead of (a, b). This allows us to focus on the cases when a has one of the forms

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and their analysis is straightforward. □

Remark 5.4. The set of all appropriate matrices ℓ in Lemma 5.3 is generic.

The following is a key step towards the main result of this section.

Lemma 5.5. *Let α be a family as in Assumption 3.2. Then there exists a matrix*

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & s_1 & s_2 \\ 0 & s_3 & s_4 \end{pmatrix}$$

with $s_1 s_4 \neq s_2 s_3$ such that the linear span of

$$(\sigma\alpha_1) \otimes (\sigma\alpha_1), \, \ldots, \, (\sigma\alpha_{11}) \otimes (\sigma\alpha_{11})$$

contains a non-zero element with the zero projection onto $M \otimes M$, where M is the subspace specified in Notation 3.4, and σ is the mapping $A \to SA$.

Proof. We define the subspace H as

$$H = \mathcal{L} \cap \operatorname{span}\{\alpha_1 \otimes \alpha_1, \, \ldots, \, \alpha_{11} \otimes \alpha_{11}\},$$

where $\mathcal{L}$ is the space in (2.2), and we note that $\dim H \geqslant 3$ by Lemma 3.6. In particular, we can take three linearly independent matrices $h_1, h_2, h_3 \in H$.

Further, we note that the top middle and top right blocks of any matrix in $\mathcal{L}$ are skew-symmetric, where the partition into the blocks is taken with respect to (2.2). Therefore, the space of all possible blocks of this type has dimension three, and hence we can apply Lemma 5.1 with $k = 3$, where u_1, u_2, u_3 are the top middle blocks of h_1, h_2, h_3, and v_1, v_2, v_3 are the top right blocks of h_1, h_2, h_3, respectively. Lemma 5.1 tells that, for some non-zero $h \in H$, the top middle block of h is collinear to the top right block of h. Since, for any $\ell \in \mathcal{L}$, the top middle block of the matrix ℓ_σ as in Definition 4.1 equals s_1 times the top middle block of ℓ plus s_2 times the top right block of ℓ, we can find $(s_1, s_2) \neq 0$ such that the top middle block of h_σ is zero. It remains to find arbitrary (s_3, s_4) such that $s_1 s_4 \neq s_2 s_3$. □

We proceed with the main result of this section. We recall that m_i is the projection of α_i onto the first six coordinates as explained in Notation 3.5.

Theorem 5.6. *If Theorem 2.2 is false, then there exists a family as in Assumption 3.2 which satisfies, additionally, one of the following conditions:*

(1) $m_{16} = m_{17} = 0$,
(2) *the family*

$$(m_1 \otimes m_1, \ldots, m_{17} \otimes m_{17})$$

has no collinear pairs and has a dependent subset with 11 *elements.*

Proof. As explained in Section 3, the negation of Theorem 2.2 implies the existence of a family α as in Assumption 3.2. We analyze two possible cases separately.

Case 1: There exist $i, j \in \{1, \ldots, 17\}$ such that $i \neq j$ and

$$\dim R \leqslant 1 \quad \text{with} \quad R = \operatorname{row} \alpha_i + \operatorname{row} \alpha_j. \tag{5.1}$$

In view of Assumption 3.2, we have $\dim R = 1$, so we obtain

$$AR^\top = \left\{ \begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix} \quad \text{with} \quad r \in \mathbb{C} \right\}$$

for some non-singular 3×3 matrix A. Now we apply Lemma 4.3 to the mapping $\mathcal{A}$ defined as $X \to AX^\top$ and get a family α' satisfying Assumption 3.2 and, additionally, containing $(\mathcal{A}\alpha_i) \otimes (\mathcal{A}\alpha_i)$ and $(\mathcal{A}\alpha_j) \otimes (\mathcal{A}\alpha_j)$; this family α' satisfies the assertion (1) of the current theorem up to the relabeling $i \to 16$, $j \to 17$.

Case 2: The condition (5.1) is invalid, for all distinct $i, j \in \{1, \ldots, 17\}$.

We consider the mappings

$$\mathcal{A} : X \to GX \quad \text{and} \quad \mathcal{B} : X \to SGX$$

where G is a generic 3×3 matrix, and S is the matrix obtained from the application of Lemma 5.5 to the family

$$(\mathcal{A}\alpha_1) \otimes (\mathcal{A}\alpha_1), \ldots, (\mathcal{A}\alpha_{11}) \otimes (\mathcal{A}\alpha_{11}).$$

Now let $\mathcal{M}$ be the family of the projections of the members in

$$(\mathcal{B}\alpha_1) \otimes (\mathcal{B}\alpha_1), \ldots, (\mathcal{B}\alpha_{17}) \otimes (\mathcal{B}\alpha_{17}) \tag{5.2}$$

onto $M \otimes M$, where M is the subspace introduced in Notation 3.4. In order to see that the family (5.2) satisfies the assertion (2) in the current theorem, we need to check that the family $\mathcal{M}$ has a dependent subset with 11 elements and has no collinear pairs. The first of these conditions follows from the relation of our construction to Lemma 5.5, and the second condition is implied by Remark 5.4 because the mapping $X \to SX$ does not change the first row of X.

Cases 1 and 2 cover all possibilities, so the proof is complete. □

6. The upper right 6×3 block of $\mathcal{L}$

We proceed with some relevant information on the following matrix space.

Definition 6.1. We define $\mathcal{L}_*$ as the set of all restrictions of the matrices in $\mathcal{L}$ to

$$\{x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}\} \times \{x_{31}, x_{32}, x_{33}\},$$

that is, to the upper right 6×3 blocks in (2.2).

We need to explore the behavior of $\mathcal{L}_*$ under the multiplication from the left.

Lemma 6.2. *If A is a non-zero 1×6 matrix, then* $\dim A\mathcal{L}_* \geqslant 2$.

Proof. Let B be an arbitrary non-singular 3×3 matrix. A straightforward checking (or, alternatively, an application of Lemma 4.4) shows that

$$\mathcal{L}_* = \left(\begin{array}{c|c} B & O \\ \hline O & B \end{array} \right) \cdot \mathcal{L}_* \cdot B^\top$$

which allows us to reduce the situation to either

$$A = (x\ 0\ 0 \,|\, y\ 0\ 0) \ \text{ or } \ A = (1\ 0\ 0 \,|\, 0\ 0\ 1).$$

We have $\dim A\mathcal{L}_* = 2$ in the first case and $\dim A\mathcal{L}_* = 3$ in the second case. $\square$

Lemma 6.3. *If A is a rank-two 2×6 matrix, then $\dim A\mathcal{L}_* \geqslant 3$. Moreover, the equality $\dim A\mathcal{L}_* = 3$ holds if and only if*

$$A = (c_1 A' \,|\, c_2 A') \tag{6.1}$$

for some rank-two 2×3 matrix A' and for some $c_1, c_2 \in \mathbb{C}$.

Proof. We begin with the consideration of one special case.

Special case: The row space of A contains a non-zero vector whose restriction to the first three coordinates is collinear to its restriction to the last three coordinates. In other words, this means that s_1 times the former restriction equals s_2 times the latter restriction with some $(s_1, s_2) \neq 0$. Using Lemma 4.5 with

$$b_{13} = b_{23} = b_{31} = b_{32} = 0,\ b_{33} = 1,\ b_{12} = s_1,\ b_{22} = -s_2,$$

we can assume that some non-zero vector of the form

$$a = \left(\begin{array}{ccc|ccc} x_1 & y_1 & z_1 & 0 & 0 & 0 \end{array}\right)$$

belongs to the row space of A. If every row of A has all zeros at the last three positions, then $\dim A\mathcal{L}_* = 3$, and also we fall into (6.1). Otherwise, we get $\dim A\mathcal{L}_* \geqslant 4$ by the application of Lemma 6.2 because the upper 3×3 block of $\mathcal{L}_*$ and the bottom 3×3 block of $\mathcal{L}_*$ depend on the non-intersecting families of variables.

We proceed with the general situation. If the condition of the special case does not realize, then we can use an argument similar to Lemma 6.2 and assume that

$$a = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

is a row of A without loss of generality, and also either

$$A = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ x_1 & 0 & y_1 & u_1 & 0 & v_1 \end{array}\right) \text{ or } A = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & x & y & z \end{array}\right) \tag{6.2}$$

with $(x, y) \neq (0, 0)$. The second option in (6.2) implies $\dim A\mathcal{L}_* \geqslant 4$ with a straightforward checking. In the first case, according to Lemma 5.1, the row space of A contains a non-zero vector with the left 1×3 part collinear to its right 1×3 part, and hence the situation reduces to the special case above. $\square$

Lemma 6.4. *Let A be a rank-three 3×6 matrix. Then either $\dim A\mathcal{L}_* = 3$ or $\dim A\mathcal{L}_* \geqslant 5$, and the former condition applies if and only if*

$$A = (c_1 A' \,|\, c_2 A')$$

for some non-singular 3×3 matrix A' and for some $c_1, c_2 \in \mathbb{C}$.

Proof. Using Lemma 5.1, we conclude that the row space of A contains a non-zero vector whose restriction to the first three coordinates is collinear to the restriction to the last three coordinates. An argument as in the special case of Lemma 6.3 allows us to assume that some non-zero vector of the form

$$a = \left(\begin{array}{ccc|ccc} x_1 & y_1 & z_1 & 0 & 0 & 0 \end{array}\right)$$

belongs to the row space of A. If every row of A has all zeros at the last three coordinates, then we have $\dim A\mathcal{L}_* = 3$ by Lemma 6.3; otherwise, some vector

$$b = \left(\begin{array}{ccc|ccc} x_2 & y_2 & z_2 & x_3 & y_3 & z_3 \end{array}\right)$$

with $(x_3, y_3, z_3) \neq 0$ appears as a row of A, and Lemma 6.2 implies $\dim A\mathcal{L}_* \geqslant 4$. In this case, according to Lemma 6.3, we can have $\dim A\mathcal{L}_* \leqslant 4$ only if the restriction of any row of A to the last three coordinates is collinear to (x_3, y_3, z_3), and the

$$\left(\begin{array}{ccc|ccc} * & * & * & 0 & 0 & 0 \end{array}\right)$$

part of the row space of A is spanned by a. But then $\operatorname{rank} A = 2$, which contradicts the initial assumption and proves that $\dim A\mathcal{L}_* \geqslant 5$. □

Lemma 6.5. *If A is a rank-four 4×6 matrix, then $\dim A\mathcal{L}_* \geqslant 5$.*

Proof. Since $\operatorname{rank} A + 3 > 6$, the row space of A has a non-zero intersection with any three-dimensional subspace of the full six-dimensional space. Therefore, the row space of A contains non-zero vectors of both of the forms

$$\left(\begin{array}{ccc|ccc} * & * & * & 0 & 0 & 0 \end{array}\right) \quad \text{and} \quad \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & * & * & * \end{array}\right),$$

and the rest follows from Lemma 6.4. □

7. The upper left 6×6 block of $\mathcal{L}$

Now we need to estimate the partially symmetric rank of the restriction of $\mathcal{L}$ to the upper left 6×6 block, where the partition into the blocks corresponds to (2.2).

Theorem 7.1. *The partially symmetric rank of the linear space*

$$\mathcal{G} = \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & k & -h \\ 0 & 0 & 0 & -k & 0 & g \\ 0 & 0 & 0 & h & -g & 0 \\ \hline 0 & -k & h & 0 & 0 & 0 \\ k & 0 & -g & 0 & 0 & 0 \\ -h & g & 0 & 0 & 0 & 0 \end{array}\right) \tag{7.1}$$

is at least nine.

Proof. We argue by contradiction, and we assume that

$$\mathcal{G} \subseteq \operatorname{span}\{v_1 \otimes v_1, \ldots, v_8 \otimes v_8\}$$

with some vectors $v_1, \ldots, v_8$. We consider two possible cases separately.

Case 1: There is a linearly dependent family $F \subset \{v_1, \ldots, v_8\}$ with $|F| = 6$.

We get $F = \{v_1, \ldots, v_6\}$ after an appropriate relabeling, and we take a maximal linearly independent subfamily F' of F. Also, we complete F' to a basis B of the full six-dimensional space by adding one or two of the vectors v_7, v_8. We assume without loss of generality that v_7 appears in B and take the subspace

$$H = \mathcal{G} \cap \operatorname{span}\{v_1 \otimes v_1, \ \ldots, \ v_7 \otimes v_7\},$$

and we note that $\dim H \geqslant 3 + 7 - 8 = 2$. Written with respect to the new basis, every matrix $h \in H$ has the form

$$h = \left(\begin{array}{c|c} x_h & O_{1\times 5} \\ \hline O_{5\times 1} & h' \end{array}\right) \tag{7.2}$$

with $x_h \in \mathbb{C}$; we denote by H' the set of all possible choices of h' in (7.2). It can be noted directly from the formulation of the theorem that

$$\operatorname{rank} h = 4, \quad \text{for any non-zero} \quad h \in \mathcal{G}, \tag{7.3}$$

so the rank of a generic matrix in H' should equal 3. Therefore, every matrix in H' has rank at most three, but since $\dim H \geqslant 2$, we can find a non-zero matrix $g \in H$ with $x_g = 0$. This implies $\operatorname{rank} g \leqslant 3$ and contradicts to (7.3).

Case 2: Every family of six vectors in $(v_1, \ldots, v_8)$ is linearly independent.

By Observation 3.7 and the condition (7.3), every non-zero matrix in $\mathcal{G}$ equals

$$\lambda_1(v_1 \otimes v_1) + \ldots + \lambda_8(v_8 \otimes v_8)$$

with either exactly four or exactly eight numbers among $(\lambda_1, \ldots, \lambda_8)$ being non-zero. This is a contradiction because $\mathbb{C}^8$ has no three-dimensional subspace in which every non-zero vector has either exactly four non-zeros or exactly eight non-zeros.

Cases 1 and 2 cover all possibilities, so the proof is complete. □

Remark 7.2. A similar and more thorough reasoning leads to $\operatorname{psr} \mathcal{G} = 10$, but we decided to stay with a weaker bound which is sufficient for our purposes.

8. Split subspaces of small matrices

Our approach requires a description of certain subspaces of $m \times m$ matrices with relatively small dimensions. We begin with formulations of two useful properties.

Definition 8.1. Let V be a linear space and $S \subseteq V \otimes V$. The set S *splits* if there exist $V_1 \subsetneq V$ and $V_2 \subsetneq V$ such that $V = V_1 \oplus V_2$, and every $s \in S$ represents as

$$s = s_1 \oplus s_2 \quad \text{with} \quad s_1 \in V_1 \otimes V_1 \quad \text{and} \quad s_2 \in V_2 \otimes V_2.$$

We say that the pair (V_1, V_2) is a *witness* of the splitting of S.

Definition 8.2. Let V be a linear space and $S \subseteq V \otimes V$. A *full splitting* of S is a sequence $(V_1, \ldots, V_k)$ with $V = V_1 \oplus \ldots \oplus V_k$ such that every $s \in S$ represents as

$$s = s_1 \oplus \ldots \oplus s_k \quad \text{with} \quad s_j \in V_j \otimes V_j \quad \text{for any} \quad j \in \{1, \ldots, k\},$$

every V_j is non-zero, and the restriction of S to any $V_j \otimes V_j$ does not split.

Remark 8.3. If Definition 8.2 applies with some family S and some integer k, and if q is an integer with $q \leqslant k$, then S said to *split into at least q parts.*

Remark 8.4. If $S \subseteq V \otimes V$ does not split, then Definition 8.2 applies with $k = 1$.

One further auxiliary definition is needed to proceed.

Definition 8.5. Let S be a family of symmetric rank-one $t \times t$ matrices. The *type* of S is (a, b), where a is the dimension of the linear span of the set of all columns of the matrices in S, and b is the dimension of the linear span of S itself.

We proceed with the main result of this section. It gives a sufficient condition under which a set of the type (t, b) splits, for $t \in \{2, 3, 4, 5, 6\}$ and $b \leqslant 2t - 2$.

Lemma 8.6. *Let t be an integer with $2 \leqslant t \leqslant 6$. Let S be a family of symmetric rank-one $t \times t$ matrices with $\dim \operatorname{span} S \leqslant 2t - 2$. If S has no coloops and no collinear pairs, then S splits.*

Proof. We can assume that the type of S is indeed (t, x), for some x, because, otherwise, there is a basis in which all the matrices in S have all their entries collected in the top left 5×5 submatrix, which would witness the splitting of S.

Now let us fix a subset $B \subset S$ that is a basis of $\operatorname{span} S$. Since S contains no coloops and no collinear pairs, any matrix $\beta \in B$ admits a linear combination

$$\psi = \sum_{b\in B} \lambda_b b \quad \text{with} \quad \operatorname{rk}\psi = 1 \tag{8.1}$$

such that $\lambda_\beta \neq 0$ and ψ is not collinear to β. According to Observation 3.7, the set of all b satisfying $\lambda_b \neq 0$ in (8.1) is a type (τ, y) family with $\tau \geqslant 2$ and $y \geqslant 2\tau - 1$. Therefore, for any β, there is a type $(\tau, 2\tau - 1)$ subfamily $\varphi_\beta \subset B$ containing β.

We proceed with the analysis of two appropriate special cases.

Special case 1: S possesses a type $(t-1, 2t-3)$ subfamily F.

Let B' be a basis of the linear span of all the columns of the matrices in F, and let v be a non-zero column of some matrix that lies in S but outside $\operatorname{span} F$. Then the matrices of S, if written with respect to the basis $B' \cup \{v\}$, are block-diagonal with a 5×5 block corresponding to B' and a 1×1 block corresponding to v.

Special case 2: S possesses a type $(t-2, 2t-5)$ subfamily F'.

Similarly to the previous special case, we can represent S with respect to an appropriate basis so that $\operatorname{span} S$ admits a basis B consisting of the matrices

$$\left(\begin{array}{cc|c} 0 & 0 & O \\ 0 & 0 & O \\ \hline O & O & B' \end{array}\right), \quad \left(\begin{array}{cc|c} 1 & 0 & O \\ 0 & 0 & O \\ \hline O & O & O \end{array}\right), \quad \left(\begin{array}{cc|c} 0 & 0 & O \\ 0 & 1 & O \\ \hline O & O & O \end{array}\right),$$

and one additional matrix s. Here, the size of the B' block is $(t-2) \times (t-2)$, and B' is a concise version of a basis of $\operatorname{span} F'$. If one of the rows of the top right block of s is zero, then we end up with the conclusion of the special case 1. This should happen, indeed, because, otherwise, any linear combination of B that is rank-one and involves s with a non-zero coefficient should be collinear to s, which is a contradiction as noted in the second paragraph of this proof.

We proceed the argument. In view of the discussion of the second paragraph, the special cases 1 and 2 solve the cases $t = 2$, $t = 3$, $t = 4$ immediately. If $t = 5$, then we can assume without loss of generality that any subfamily φ_β has type $(2, 3)$ again because of the special cases 1 and 2. We take one such subfamily φ_β, and also we take one arbitrary matrix γ in B outside $\operatorname{span} \varphi_\beta$. Further, we take a matrix δ in B outside $\operatorname{span} \varphi_\beta + \operatorname{span} \varphi_\gamma$, and we note that one of the families

$$\varphi_\beta \cup \varphi_\gamma, \ \varphi_\beta \cup \varphi_\delta, \ \varphi_\gamma \cup \varphi_\delta$$

does not split, and then the special case 2 applies, because, if they all split, then the dimension of $\operatorname{span} \varphi_\beta + \operatorname{span} \varphi_\gamma + \operatorname{span} \varphi_\delta$ should equal $3 + 3 + 3 = 9 > 2t - 2$.

Now we focus on the case $t = 6$. If the family S contains a type $(\tau, 2\tau - 1)$ subfamily with $\tau \geqslant 4$, then we appeal to the special cases 1 and 2 and complete the proof. If S contains two subfamilies φ_1 and φ_2 of the type $(3, 5)$, and the corresponding sums of the row spaces V_1 and V_2 are not equal, then

- (V_1, V_2) witnesses the splitting of S if $V_1 \cap V_2 = 0$,
- one of the special cases 1 and 2 applies to the family $\varphi_1 \cup \varphi_2$ otherwise.

Therefore, if we take an arbitrary matrix γ in B outside $\operatorname{span} \varphi_1$, then the family φ_γ can be assumed to have the type $(2, 3)$. Further, we take a matrix δ in B outside $\operatorname{span} \varphi_1 + \operatorname{span} \varphi_\gamma$, and, similarly to the case $t = 5$, we note that one of the families

$$\varphi_1 \cup \varphi_\gamma, \ \varphi_1 \cup \varphi_\delta, \ \varphi_\gamma \cup \varphi_\delta$$

does not split (and then one of the above special cases applies) because otherwise

$$\dim(\operatorname{span}\varphi_\beta + \operatorname{span}\varphi_\gamma + \operatorname{span}\varphi_\delta) = 5 + 3 + 3 = 11 > 2t - 2.$$

So we can assume that every family φ_β is of the type $(2, 3)$. An argument similar to the case $t = 5$ reduces the situation to the case when the basis B consists of

$$\left(\begin{array}{c|c|c} \Phi_1 & O & O \\ \hline O & O & O \\ \hline O & O & O \end{array}\right), \quad \left(\begin{array}{c|c|c} O & O & O \\ \hline O & \Phi_2 & O \\ \hline O & O & O \end{array}\right), \quad \left(\begin{array}{c|c|c} O & O & O \\ \hline O & O & O \\ \hline O & O & \Phi_3 \end{array}\right)$$

and one additional matrix s, where Φ_1, Φ_2, Φ_3 are bases of the space of the symmetric 2×2 matrices. If one of the diagonal blocks of s is zero, then B splits immediately, and the proof is complete. Otherwise, every off-diagonal block of s is non-zero, and then any linear combination of B that is rank-one and involves s with a non-zero coefficient should be collinear to s, which is impossible, as explained above, because S has no coloops and no collinear pairs. □

The following remark is not used in our further considerations.

Remark 8.7. The statement of Lemma 8.6 does not generalize to arbitrary large t. In fact, a counterexample to the potential generalization of Lemma 8.6 to any $t \geqslant 9$ can be constructed as follows. We find an even integer u such that

$$t + 1 \leqslant u \leqslant \frac{4t - 4}{3}$$

and take a generic family $(v_1, \ldots, v_u)$ of vectors of length t. If

$$S = \{v_1 \otimes v_1, \ldots, v_u \otimes v_u\} \cup W_1 \cup \ldots \cup W_{u/2}$$

with W_i being the set of all $(v_{2i-1} + \varepsilon v_{2i}) \otimes (v_{2i-1} + \varepsilon v_{2i})$ with $\varepsilon \neq 0$, then S does not split, and S has neither a coloop nor a collinear pair. Also, we have $\dim S \leqslant 2t - 2$.

We finalize the section with a corollary of Lemma 8.6.

Corollary 8.8. *Let F be a type $(6, 9)$ family of symmetric rank-one matrices. If F has no coloops and no collinear pairs, then F splits into at least three parts.*

Proof. This family admits a splitting (U, V) by Lemma 8.6, and the same lemma applies again either to the $U \otimes U$ or $V \otimes V$ part of the initial splitting. □

9. Almost split subspaces of small matrices

In this section, we discuss another notion similar to that of splitting. Our goal is to get a partial generalization of Lemma 8.6 to the case $t = 6$, $\dim \operatorname{span} S = 11$.

Definition 9.1. Let V be a linear space, and let S be a family of symmetric rank-one matrices in $V \otimes V$. The set S is *almost split* if there exist $V_1 \subsetneq V$ and $V_2 \subsetneq V$ such that $V = V_1 + V_2$, $\dim V_1 \cap V_2 \leqslant 1$, and the columns of any matrix $s \in S$ belong to either V_1 or V_2. The pair (V_1, V_2) is called a *witness* of the almost splitting of S.

One easy relation between Definitions 8.1 and 9.1 is as follows.

Lemma 9.2. *Let S be an almost split family of symmetric rank-one matrices as witnessed by the pair (V_1, V_2). Let β be a non-zero vector in $V_1 \cap V_2$. Let S_2 be the set of all restrictions of the matrices in S to $V_2 \otimes V_2$. If S_2 splits and, additionally, S contains a non-zero matrix collinear to $\beta \otimes \beta$, then S splits as well.*

Proof. Let (U_1, U_2) be a witness of the splitting of S_2. Then a non-zero matrix collinear to $\beta \otimes \beta$ belongs to either

$$(U_1 \otimes U_1) \oplus 0 \quad \text{or} \quad 0 \oplus (U_2 \otimes U_2)\,,$$

which corresponds to the splitting of S as either $(U_1 + V_1, U_2)$ or $(U_1, V_1 + U_2)$. $\square$

We proceed with the desired generalization of Lemma 8.6.

Lemma 9.3. *Let S be a set of rank-one symmetric 6×6 matrices. If*

(1) *the linear span of S has dimension 11,*
(2) *S has a dependent set of 11 elements,*
(3) *S has no coloops and no collinear pairs,*

then S is almost split.

Proof. Similarly to the argument in Lemma 8.6, we fix a subset $B \subset S$ that is a basis of $\operatorname{span} S$, and we conclude that any matrix $\beta \in B$ admits a linear combination

$$\psi = \sum_{b \in B} \lambda_b b \quad \text{in} \quad S \tag{9.1}$$

such that $\lambda_\beta \neq 0$ and ψ is not collinear to β, and, since $\psi \in S$, we get $\operatorname{rk} \psi = 1$. We define φ_β as the set collecting all b satisfying $\lambda_b \neq 0$ in (9.1), and Observation 3.7 shows that φ_β has the type (τ, y) with $\tau \geqslant 2$ and $y \geqslant 2\tau - 1$.

Now we consider a dependent subset $F \subset S$ with 11 elements, which exists because of the assumption (2) in the formulation. Then we take an inclusion minimal linearly dependent subfamily $F' \subseteq F$, and an application of Observation 3.7 shows that any subfamily of F' with cardinality $|F'| - 1$ has the type (ρ, δ) with $\delta \geqslant 2\rho - 1$. We remark that $\rho \geqslant 2$ by the assumption (3) in the formulation, and we also get $\rho \leqslant 5$ because $|F'| - 1 \leqslant |F| - 1 = 10$. So we can take a family

$$\Phi \text{ of the type } (r, 2r-1) \text{ with maximal possible } r \in \{2, 3, 4, 5\}. \tag{9.2}$$

From now on, we write $\mathcal{V}$ to denote the linear span of the columns of the matrices in Φ. We have $\dim \mathcal{V} = r$ by the definition of the type, and we write V to denote an arbitrary fixed basis of $\mathcal{V}$. Also, we can assume that there exist matrices[2]

$$(u_1 \otimes u_1)\,,\ \ldots,\ (u_{6-r} \otimes u_{6-r}) \text{ in } S \tag{9.3}$$

such that $U \cup V$ is a basis of the full six-dimensional space, where

$$U = (u_1, \ldots, u_{6-r}).$$

Now we can find a family $B \subset S$ which is a basis of $\operatorname{span} S$ consisting of

(1B) $2r - 1$ matrices in $0_{\mathcal{U}} \oplus (\mathcal{V} \otimes \mathcal{V})$,
(2B) $6 - r$ matrices in $(\mathcal{U} \otimes \mathcal{U}) \oplus 0_{\mathcal{V}}$,
(3B) $6 - r$ other matrices,

with $\mathcal{U} = \operatorname{span} U$. Here, (1B) can be an arbitrary maximal linearly independent subfamily of Φ, and (2B) are the matrices in (9.3). Also, we define the integer α as the quantity of all those matrices in B which do not lie in

$$(\mathcal{U} \otimes \mathcal{U}) \oplus (\mathcal{V} \otimes \mathcal{V})$$

or, equivalently, which have a non-zero $U \times V$ block when written with respect to the basis $U \cup V$. Also, we define A as the multiset of all the non-zero restrictions of the matrices in B to the $U \times V$ block; a non-zero matrix a appears in A with the

[2]Otherwise, there would be a common kernel vector for S, and hence S would be split.

multiplicity m if any only if there are precisely m matrices in B whose restrictions to the $U \times V$ block coincide with a. We have $|A| = \alpha$, and the matrices in (1B) and (2B) cannot contribute to A; we get

$$0 \leqslant \alpha \leqslant 6 - r. \tag{9.4}$$

We assume that α is minimal possible over all choices of the matrices (9.3). We are going to complete the proof with a separate consideration of every possible value of α, but we need several additional special cases in the course of this argument. The rest of the proof gives a list of several conditions each of which is sufficient to deduce the desired assertion, which is the almost splitting of S, and we will see it later that the presented conditions cover all possible values of α.

Sufficient condition 1: The rows of the matrices in A are all collinear.

In this case, the pair $(\mathcal{U} + \operatorname{row} a, \mathcal{V})$ is the desired almost splitting of S, where a is any matrix in B which contributes to A, that is, a matrix in B which has a non-zero $U \times V$ block. If there are no such matrices a, then $(\mathcal{U}, \mathcal{V})$ is the splitting of S, and, since S is split, it has to be almost split as well.

Sufficient condition 2: Either $\alpha = 0$ or $\alpha = 1$.

This reduces to the sufficient condition 1.

Sufficient condition 3: S contains two matrices s_1, s_2 such that the restrictions of s_1, s_2 to the $U \times V$ block are non-zero and have collinear columns.

Since s_1, s_2 are not collinear by the assumption (3) in the formulation of the lemma, the rows of s_1, s_2 are not collinear as well. Therefore, the family $\Phi \cup \{s_1, s_2\}$ has the type $(r+1, 2r+1)$, and this implies $r = 5$ by the maximality of r. We get $\alpha \leqslant 1$ by the inequality (9.4), and it remains to apply the sufficient condition 2.

Sufficient condition 4: A contains two matrices with collinear columns.

Follows immediately from the sufficient condition 3 above.

Sufficient condition 5: There exists $a \in A$ such that every rank-one linear combination of A that involves a with a non-zero coefficient is collinear to a.

Let $b \in B$ be the matrix that corresponds to a, that is, the matrix a appears to be the restriction of b to the $U \times V$ block. Since b is not a coloop of S by the assumption (3) in the formulation of the lemma, there should exist a matrix $s \in S$ which is a linear combination of B that involves b with a non-zero coefficient. According to the current condition 5, the $U \times V$ block of s has to be collinear to a, but s itself is not collinear to b again because of the assumption (3) in the formulation. Therefore, we get the family $\Phi \cup \{b, s\}$ of the type $(r+1, 2r+1)$ and complete the proof similarly to the sufficient condition 3.

Sufficient condition 6: One has $\alpha = 2$.

Follows immediately from the sufficient conditions 1, 4, 5 above.

Sufficient condition 7: One has $r = 2$, and there exist families $f_1, f_2, f_3 \subset S$ each of which has the type $(2, 3)$ such that $\dim \operatorname{span} f_1 \cup f_2 \cup f_3 = 9$.

If, for $j = 1, 2, 3$, the notation V_j stands for the linear span of the rows of f_j, then, by the maximality of r in (9.2), we should have the direct sum $V_1 \oplus V_2 \oplus V_3$ equal to the full six-dimensional space. Since $\dim \operatorname{span} S = 11$, there exist two matrices $b, c \in S$ such that $f_1 \cup f_2 \cup f_3 \cup \{b, c\}$ is a basis of $\operatorname{span} S$. Now, if the rows of the restrictions of b, c to the $(V_1 + V_2) \otimes V_3$ are collinear, then the pair

$$(V_1 + V_2 + \operatorname{row} b + \operatorname{row} c, V_3)$$

is the desired witness of the almost splitting of S. Alternatively, if the columns of the restrictions of b, c to $(V_1 + V_2) \otimes V_3$ are collinear, then the family $f_3 \cup \{b, c\}$ has the type $(3, 5)$, which contradicts to the maximality of r in (9.2). It remains to consider the case when the restrictions of b, c to the $(V_1 + V_2) \otimes V_3$ have neither collinear rows nor collinear columns, but then, similarly to the sufficient condition 5, we can find a matrix $s \in S$ which is not collinear to b but for which the restriction of s to $(V_1+V_2)\otimes V_3$ is collinear to such a restriction of b. This shows that $f_3 \cup \{b, s\}$ has the type $(3, 5)$ and contradicts to the maximality of r again.

Sufficient condition 8: One has $\alpha = 3$.

In this case, we have three matrices $b, c, d \in B$ whose restrictions to the $U \times V$ blocks are non-zero. Further, we can assume that the restrictions of b, c, d to the $U \times V$ blocks are linearly independent, and the sum of the column spaces of these restrictions has dimension two, because otherwise we cannot avoid all the sufficient conditions 1, 4, 5 simultaneously. Therefore, there is a dimension two subspace $\mathcal{U}' \subset \mathcal{U}$ such that the rows of b, c, d belong to $\mathcal{U}' + \mathcal{V}$. Now we take the set φ_b as in the first paragraph of this proof, and the application of (9.1) allows us to express a matrix $\psi \in S$ as a linear combination of the matrices in φ_b taken with non-zero coefficients. Since the restrictions of b, c, d to the $U \times V$ block are linearly independent, there exists at least one non-zero column of ψ with an index in V. As explained above, this column belongs to $\mathcal{U}' + \mathcal{V}$, and, since ψ is rank-one, we get

$$\operatorname{row} \psi \subset \mathcal{U}' + \mathcal{V}.$$

If we have $\psi \notin \operatorname{span} \Phi \cup \{b, c, d\}$, then the family $\Phi \cup \{b, c, d, \psi\}$ has the type $(r+2, 2r+3)$, which implies $r = 4$ by the maximality of r in (9.2) and contradicts to the inequality (9.4). Therefore, the matrix ψ is a linear combination of $\Phi \cup \{b, c, d\}$, and hence φ_b is a subset of $\Phi \cup \{b, c, d\}$. If $\varphi_b \cap \Phi \neq \varnothing$, then the family $\varphi_b \cup \Phi$ witnesses a contradiction via the maximality of r again. Since φ_b has the type (ρ, x) with some $\rho \geqslant 2$ and $x \geqslant 2\rho - 1$, the only remaining possibility is that the family φ_b equals $\{b, c, d\}$ and has the type $(2, 3)$, which means that

$$\dim(\operatorname{row} b + \operatorname{row} c + \operatorname{row} d) = 2. \tag{9.5}$$

In the case $r = 3$, we get a contradiction to the minimality of α as follows. We proceed with b, c, g in the role of the matrices (9.3), where g is an element of S such that $\mathcal{U}' + \mathcal{V} + \operatorname{row} g$ is the full six-dimensional space. In view of (9.5), the full six-dimensional space can be expressed as

$$\mathcal{U}'' \oplus \mathcal{V} \;\text{ with }\; \mathcal{U}'' = \operatorname{row} b + \operatorname{row} c + \operatorname{row} d + \operatorname{row} g,$$

and we immediately have

$$b, c, d, g \in (\mathcal{U}'' \otimes \mathcal{U}'') \oplus 0_{\mathcal{V}} \;\text{ and }\; \Phi \subset 0_{\mathcal{U}''} \oplus (\mathcal{V} \otimes \mathcal{V})\,.$$

So we see that the corresponding partition (1B)–(3B) has at most $11 - 4 - 5 = 2$ matrices with the non-zero $\mathcal{U}'' \otimes \mathcal{V}$ block, which gives the desired contradiction with the minimality of α because the initial assumption is $\alpha = 3$.

Now we can focus on the case $r = 2$. Similarly to the previous paragraph, we take matrices g, h in S such that the full six-dimensional space expresses as

$$\mathcal{U}'' \oplus \mathcal{V} \;\text{ with }\; \mathcal{U}'' = \operatorname{row} b + \operatorname{row} c + \operatorname{row} d + \operatorname{row} g + \operatorname{row} h.$$

We proceed by replacing the matrices (9.3) with b, c, g, h, and we get

$$b, c, d, g, h \in (\mathcal{U}'' \otimes \mathcal{U}'') \oplus 0_{\mathcal{V}} \;\text{ and }\; \Phi \subset 0_{\mathcal{U}''} \oplus (\mathcal{V} \otimes \mathcal{V})\,,$$

which means that at most $11 - 5 - 3 = 3$ matrices in the corresponding partition (1B)–(3B) can have their $\mathcal{U}'' \otimes \mathcal{V}$ blocks non-zero. In order to avoid the sufficient conditions 2 and 6, we can assume that there are exactly three such matrices. These matrices must form a $(2,3)$ type subfamily by repeating the current argument, so the sufficient condition 7 concludes the proof.

Sufficient condition 9: One has $\alpha = 4$.

We have exactly four matrices $b, c, d, e \in B$ with the non-zero restrictions to the $U \times V$ blocks. If H be the sum of the row spaces of b, c, d, e, then the rows of the matrices in $\Phi \cup \{b, c, d, e\}$ belong to $H + \mathcal{V}$. We remark that $r = 2$ by the inequality (9.4), so the maximality of r in (9.2) implies that either

$$\dim(H + \mathcal{V}) = 5 \quad \text{or} \quad \dim(H + \mathcal{V}) = 6. \tag{9.6}$$

Similarly to the special case 8, we note that the restrictions of b, c, d, e to the $U \times V$ blocks are non-zero. We consider the two possibilities in (9.6) separately.

Special case 9.1: Assume $\dim(H + \mathcal{V}) = 6$. Then the columns of the restrictions of b, c, d, e to the $U \times V$ blocks span the full four-dimensional subspace corresponding to the indexes in U. The only way to avoid the sufficient conditions 1 and 5, up to the relabeling of b, c, d, e, is that the rows of b, c are collinear modulo U, and the rows of d, e are collinear modulo U. This does not allow any of the sets φ_b and φ_d as in the first paragraph of the proof to be of the type $(6, 11)$, and hence they should be of the type $(2, 3)$ by the maximality of r in (9.2). Therefore, the families φ_b, φ_d, Φ allow an application of the sufficient condition 7.

Special case 9.2: Assume $\dim(H + \mathcal{V}) = 5$. Then we complete V, which is a basis of $\mathcal{V}$, to a basis $W \cup V$ of $\operatorname{span}(H + \mathcal{V})$ by adding a family W with $|W| = 3$. Also, we complete this new basis by adding some vector γ to get a basis

$$\Gamma = \{\gamma\} \cup W \cup V \tag{9.7}$$

of the full six-dimensional space. Finally, we define L as the linear span of all possible γ-th rows of the matrices in S written with respect to the basis Γ.

We write $\dim L = \delta$, and if $\delta = 0$, then the matrices in S have a common zero row, which implies that S is split and completes the proof. So we can focus on the case $\delta \geqslant 1$, and then we take matrices $\ell_1, \ldots, \ell_\delta \in S$ with linearly independent γ-th rows. In fact, there should exist one more matrix $\ell_0 \in S$ with γ-th row non-zero, because the set S has no coloops by the assumption (3) of this lemma. We have

$$\dim(\mathbb{C}\ell_0 + \ldots + \mathbb{C}\ell_\delta) = \delta + 1$$

because S has no collinear pairs again by the assumption (3). We have $L \cap \mathcal{V} = 0$ from our construction, so there exist matrices $g_1, \ldots, g_{4-\delta} \in S$ such that the full six-dimensional space represents as

$$\mathcal{L} \oplus \mathcal{V} \quad \text{with} \quad \mathcal{L} = \operatorname{row} g_1 + \ldots + \operatorname{row} g_{4-\delta} + L.$$

We take the matrices $g_1, \ldots, g_{4-\delta}, \ell_1, \ldots, \ell_\delta$ in the role of (9.3), and we have

$$\ell_0, \ldots, \ell_\delta,\, g_1, \ldots, g_{4-\delta} \in (\mathcal{L} \otimes \mathcal{L}) \oplus 0_{\mathcal{V}} \quad \text{and} \quad \Phi \subset 0_{\mathcal{L}} \oplus (\mathcal{V} \otimes \mathcal{V}).$$

So we see that the corresponding partition (1B)–(3B) has at most $11 - 5 - 3 = 3$ matrices with the non-zero $\mathcal{L} \otimes \mathcal{V}$ block, which contradicts to the minimality of α because the initial assumption is $\alpha = 4$. Therefore, the case 9.2 does not realize, and this completes the consideration of the final sufficient condition 9.

Since we have $r \geqslant 2$ from (9.2), the inequalities (9.4) imply $0 \leqslant \alpha \leqslant 4$. Therefore, one of the sufficient conditions 2, 6, 8, 9 satisfies, and the result follows. $\square$

10. The proof of Theorem 5.6

Now we are prepared to complete the proof of Theorem 2.2, which is done by showing that neither the case 1 nor case 2 of Theorem 5.6 can realize. We need several further notations to be used in this section.

Definition 10.1. We use the letter α to denote the family that satisfies Assumption 3.2. Also, we write μ to denote the set of $m_i \otimes m_i$ over all $i \in \{1, \ldots, 17\}$, where m_i is defined in Notation 3.5.

Definition 10.2. We write μ' to denote the family obtained by removing all zeros and all coloops from the family μ as in the previous definition.

Observation 10.3. *The family μ' has no coloops.*

Definition 10.4. A collinear pair is called *non-zero* if both parts of it are non-zero.

We need to specify one subspace of the space $\mathcal{L}$ as in (2.2).

Definition 10.5. We define $\mathcal{L}_\circ$ as the linear space consisting the matrices obtained from (2.2) by taking $g = h = k = 0$.

We still use the notation Λ for the space defined in Assumption 3.2.

Lemma 10.6. *Let P be a subspace of Λ. Then* $\dim \mathcal{L}_\circ \cap P \geqslant \dim P - 11$.

Proof. We have $\dim(\mathcal{L}_\circ + P) \leqslant 17$ because every matrix in $\mathcal{L}_\circ \cup P$ belongs to the linear span of the 17 matrices $\alpha_i \otimes \alpha_i$ as in Assumption 3.2. We get

$$17 \geqslant \dim(\mathcal{L}_\circ + P) = \dim \mathcal{L}_\circ + \dim P - \dim \mathcal{L}_\circ \cap P,$$

and the result follows because $\dim \mathcal{L}_\circ = 6$. $\square$

We proceed with one further technical result.

Lemma 10.7. *If μ has no non-zero collinear pairs, then μ' does not split.*

Proof. We argue by contradiction. Assuming that μ' splits and μ has no non-zero collinear pairs, we aim to check that $\mathcal{L}_\circ$ is not a subset of the space Λ as in (3.1). Up to the relabeling of the vectors in α, we can assume that

$$\mu' = (m_1 \otimes m_1, \ldots, m_c \otimes m_c)$$

for some $c \leqslant 17$. Then a linear combination

$$\lambda_1(\alpha_1 \otimes \alpha_1) + \ldots + \lambda_{17}(\alpha_{17} \otimes \alpha_{17})$$

can have all zeros at the upper left 6×6 block only if $\lambda_j = 0$ for all $j > c$, and hence it suffices to prove that $\mathcal{L}_\circ$ is not contained in

$$\Lambda' = \operatorname{span}\{\alpha_1 \otimes \alpha_1, \ldots, \alpha_c \otimes \alpha_c\}.$$

Now let $(V_1, \ldots, V_k)$ be a full splitting of μ' as in Definition 8.2. We have $k \geqslant 2$ by the initial assumption, and also, since μ' has no coloops and no collinear pairs, we have $\dim V_j \geqslant 2$, for all $j \in \{1, \ldots, k\}$. Therefore, there exists a non-singular 6×6 matrix C such that all the matrices in $C\mu' C^\top$ are block-diagonal, and the block structures of all these matrices fall simultaneously into one of the following cases.

Case 1: There are three 2×2 blocks.

Case 2: The first block is 2×2, the second block is 4×4.
Case 3: There are two 3×3 blocks.

We complete the 6×6 matrix C to the matrix

$$\mathcal{C} = \left(\begin{array}{c|c} C & O_{6\times 3} \\ \hline O_{3\times 6} & I_{3\times 3} \end{array} \right)$$

of the same format as the matrices in (2.2). Using this notation, we write

$$\mathcal{C}\mathcal{L}_\circ\mathcal{C}^\top \subseteq \mathcal{C}\Lambda'\mathcal{C}^\top \tag{10.1}$$

for the negation of the statement $\mathcal{L}_\circ \not\subseteq \Lambda'$ needed to complete the proof, and hence we need to get a contradiction from (10.1). We remark that the upper right 6×3 block of the left-hand side of (10.1) is $C\mathcal{L}_*$, where $\mathcal{L}_*$ is the space as in Definition 6.1, and this allows us to use the results of Section 6. In particular, a restriction of $C\mu'C^\top$ to the first 2×2 block as in the case 1 should have dimension three by Lemma 8.6, and there should be at least three more elements in $C\mu'C^\top$ with non-zeros at this block, by Lemma 6.3, because the matrices in $\mathcal{C}\Lambda'\mathcal{C}^\top$ corresponding to the other blocks leave the first two rows of $C\mathcal{L}_*$ untouched. Therefore, we should have at least $3+3=6$ matrices in μ' associated with the 2×2 block, and a similar consideration with the 4×4 block shows that it is non-zero in at least $7+5=12$ such matrices. Therefore, each of the cases 1 and 2 requires at least 18 matrices to be present in μ', but this is a contradiction because μ' contains $c \leqslant 17$ elements.

Now we focus on the case 3. A reasoning similar to the previous paragraph shows that the case $\dim A\mathcal{L}_* = 3$ of Lemma 6.4 should apply, whenever A is the matrix formed by either the first three rows or the last three rows of C. Then every of the 3×3 diagonal blocks is non-zero on $5+3=8$ matrices in μ', and Lemma 6.4 gives

$$C = \left(\begin{array}{c|c} p_1 G & p_2 G \\ \hline p_3 H & p_4 H \end{array} \right)$$

with G and H being 3×3 matrices and with scalar p_1, p_2, p_3, p_4. An application of Lemma 4.5 with

$$b_{13} = b_{23} = b_{31} = b_{32} = 0,\ b_{33} = 1,\ b_{11} = p_4,\ b_{12} = p_2,\ b_{21} = -p_3,\ b_{22} = -p_1$$

reduces the situation to the case when C is block-diagonal with two 3×3 blocks, and then the matrices in μ' are also block-diagonal with two 3×3 blocks. Therefore, there exists at most $17-8-8=1$ matrix in α which has non-zeros in the upper middle block with respect to the partition in (2.2), and this matrix does not suffice to span all the upper middle blocks in $\mathcal{L}$ because their space is three-dimensional. □

We are ready to prove one of the main results of this section.

Theorem 10.8. *The option (1) in Theorem 5.6 does not realize.*

Proof. We are going to reach a contradiction from the condition (1).

Special case 1: The family μ has no non-zero collinear pairs.

According to Lemma 10.7, the family μ' cannot split, and, using Lemma 8.6, we conclude that μ should span a subspace of dimension at least 11. Up to the relabeling of the indexes, we assume that the upper left 6×6 blocks of

$$\alpha_5 \otimes \alpha_5, \ldots, \alpha_{15} \otimes \alpha_{15}$$

are linearly independent. Also, we have $m_{16} = m_{17} = 0$ directly from the condition (1) in Theorem 5.6. Therefore, the linear span of the matrices $\alpha_j \otimes \alpha_j$ with $5 \leqslant j \leqslant 17$ has a zero intersection with $\mathcal{L}_\circ$, which contradicts to Lemma 10.6.

Special case 2: The family μ has a non-zero collinear pair.

We can assume $m_1 = m_2$ without loss of generality. We are going to reach a contradiction in a way similar to the special case 1 by constructing a linear subspace Q of the space Λ as in (3.1) such that:

- the dimension of Q is at least 12,
- the intersection $\mathcal{L}_\circ \cap Q$ is zero.

We define Q as the span of the set B which includes the following 12 matrices:

$$\alpha_{16} \otimes \alpha_{16},\ \alpha_{17} \otimes \alpha_{17}, \tag{10.2}$$

$$\alpha_1 \otimes \alpha_1 - \alpha_2 \otimes \alpha_2 \tag{10.3}$$

and also a total of nine of those $\alpha_j \otimes \alpha_j$ for which $m_j \otimes m_j$ are linearly independent, which is possible by Theorem 7.1. According to Lemma 10.6, there should exist a non-zero element $\psi \in \mathcal{L}_\circ$ that represents as a linear combination of the matrices in B. However, every matrix in B except the nine mentioned last have all zeros in the upper left 6×6 block, and hence a linear combination of B can belong to $\mathcal{L}_\circ$ only if all these nine matrices are given zero coefficients. Also, the matrices (10.2) do not contribute to the upper right 6×3 block of $\mathcal{L}_\circ$, and we get a contradiction because the upper right 6×3 block of the matrix (10.3) has rank at most one while every such block for non-zero matrices in $\mathcal{L}_\circ$ has rank at least two.

Cases 1 and 2 cover all possibilities, so the proof is complete. □

Now we can focus on the option (2) in Theorem 5.6.

Lemma 10.9. *If the option (2) in Theorem 5.6 is true, then* $\dim \operatorname{span} \mu = 11$ *and* $\mu' = \mu$*, and the family* μ *is almost split as witnessed by a pair* (U, V) *with either*
(2A) $\dim U = 2$, $\dim V = 5$,
(2B) $\dim U = 3$, $\dim V = 4$.

Proof. Lemma 10.7 shows that μ' does not split, and then Lemma 8.6 implies $\dim \operatorname{span} \mu' \geqslant 11$. So we see that

$$11 \geqslant \dim \operatorname{span} \mu \geqslant \dim \operatorname{span} \mu' \geqslant 11,$$

in which the first inequality follows from Lemma 10.6. We get

$$\dim \operatorname{span} \mu = \dim \operatorname{span} \mu' = 11$$

and hence $\mu' = \mu$. Now we apply Lemma 9.3 to see that μ is almost split, and it remains to note that the cases (2A) and (2B) cover all the possibilities for an almost splitting of a family of 6×6 matrices. □

Now we turn down the assumptions (2A) and (2B) in Lemma 10.9.

Theorem 10.10. *The option (2) in Theorem 5.6 does not realize.*

Proof. We need to deduce a contradiction from the assumption (2) in Theorem 5.6. Using Lemma 10.9, we find a non-singular 6×6 matrix C such that either

- all the matrices in $C\mu C^\top$ have the form Φ_{25}, or
- all the matrices in $C\mu C^\top$ have the form Φ_{34},

where

$$\Phi_{25} = \begin{pmatrix} * & * & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \end{pmatrix} \quad \text{and} \quad \Phi_{34} = \begin{pmatrix} * & * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \end{pmatrix}$$

with $*$'s being the placeholders for arbitrary scalars. Since the matrices in $C\mu C^\top$ are all rank-one, any such matrix has all non-zeros concentrated in either the upper left block (of size 2×2 for Φ_{25} and 3×3 for Φ_{34}) or bottom right block (of size 5×5 for Φ_{25} and 4×4 for Φ_{34}). The set of latter matrices is to be called the *upper part* of $C\mu C^\top$, and the former matrices are the corresponding *lower part*.

We proceed in a way similar to Lemma 10.7. We have

$$\mathcal{C}\mathcal{L}_\circ\mathcal{C}^\top \subseteq \mathcal{C}\Lambda\mathcal{C}^\top = \operatorname{span} C\mu C^\top$$

with

$$\mathcal{C} = \left(\begin{array}{c|c} C & O_{6\times 3} \\ \hline O_{3\times 6} & I_{3\times 3} \end{array}\right)$$

and, in order to deduce a contradiction, we pick an arbitrary basis B of $\operatorname{span}\mu$ and consider the two possible cases separately.

Case 1: All the matrices in $C\mu C^\top$ have the form Φ_{25}.

Using Lemma 6.2, we conclude that there should be at least two upper matrices in μ which do not belong to B, and Lemma 6.5 shows that at least 5 lower matrices of μ lie outside B. Therefore, μ contains at least $|B|+2+5$, which is a contradiction because $|B| = 11$ (by Lemma 10.9) and $|\mu| = 17$ (by Definition 10.1).

Case 2: All the matrices in $C\mu C^\top$ have the form Φ_{34}.

An argument as in the case 1 leads to a similar contradiction unless we have

$$\dim A_1\mathcal{L}_* + \dim A_2\mathcal{L}_* \leqslant 6, \tag{10.4}$$

where A_1 is the matrix obtained by taking the first two rows of C, and A_2 is the matrix formed by the last three rows of C. According to Lemmas 6.3 and 6.4, the inequality (10.4) is only possible if

$$C = \left(\begin{array}{c|c} p_1K & p_2K \\ \hline \xi & \gamma \\ \hline p_3H & p_4H \end{array}\right) \tag{10.5}$$

in which K is a rank-two 2×3 matrix, H is a rank-three 3×3 matrix, ξ and γ are unknown 1×3 matrices, and p_1, p_2, p_3, p_4 are scalars. Since the form Φ_{34} cannot invalidate upon any change of the third row of C, as long as the matrix obtained from C in this way remains non-singular, we can assume that

$$C = \left(\begin{array}{c|c} p_1G & p_2G \\ \hline p_3H & p_4H \end{array}\right) \tag{10.6}$$

with G being a non-singular 3×3 matrix. We remark that the matrix (10.6) is indeed non-singular, because the condition $p_1p_4 \neq p_2p_3$ is both necessary for the non-singularity of (10.5) and sufficient for the non-singularity of (10.6). An application of Lemma 4.5 with

$$b_{13} = b_{23} = b_{31} = b_{32} = 0,\ b_{33} = 1,\ b_{11} = p_4,\ b_{12} = p_2,\ b_{21} = -p_3,\ b_{22} = -p_1$$

reduces the situation to the case when $\mathcal{C}$ is block-diagonal with two 3×3 blocks. Then the matrices in μ are products of the form

$$\left(\begin{array}{c|c} K_1 & O_{3\times 3} \\ \hline O_{3\times 3} & K_2 \end{array}\right)\begin{pmatrix} * & * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \end{pmatrix}\left(\begin{array}{c|c} K_3 & O_{3\times 3} \\ \hline O_{3\times 3} & K_4 \end{array}\right)$$

with fixed matrices K_1, K_2, K_3, K_4 of size 3×3. We note that the columns of the upper right block of all the matrices in μ span lie in a subspace of dimension at most one (namely, this subspace is K_1 times the third coordinate vectors in $\mathbb{C}^3$). This corresponds to the fact that the upper middle blocks of the matrices in Λ have their columns in a fixed subspace of dimension at most one, and, in view of Assumption 3.2, the matrices in (2.2) satisfy the same property. This is a contradiction because the upper middle blocks of (2.2) can be of rank two.

As explained in the first paragraph, the cases 1 and 2 cover all possibilities. □

Theorems 10.8 and 10.10 invalidate both the options (1) and (2) in Theorem 5.6, so we deduce the correctness of Theorem 2.2 from Theorem 5.6.

11. Concluding remarks

We showed that the symmetric and partially symmetric ranks of the tensor corresponding to the 3×3 determinant are equal to 18. However, we did not manage to generalize our approach to compute the rank of this tensor.

Question 11.1. What is the rank of the tensor of the 3×3 determinant?

The results in the above mentioned papers [10, 11] imply that this rank is either 17 or 18. If the correct value is 17, then the 3×3 determinant gives a small counterexample to a recently disproved conjecture of Comon [34, 37] and disproves the partially symmetric version of this conjecture, which remains open [5, 17, 34].

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