The Kepler two body problem in the language of geometric algebra

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This text is for the young people, as an extension to the book [3].

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geometric algebra, *Kepler's laws*, *Laplace-Runge-Lenz vector*, *eccentricity*, *energy conservation*, *angular momentum*

The Kepler problem can be nicely treated in the language of geometric algebra, without coordinates. From *Newton's laws*, we have (see **Fig. 1**)

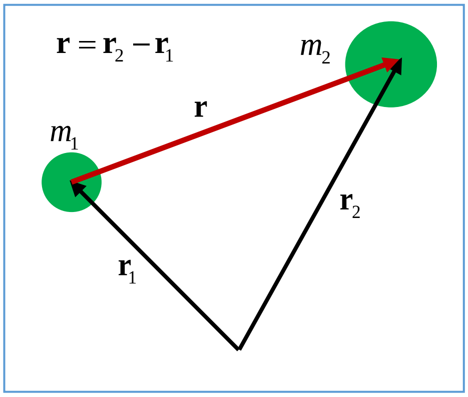

Fig. 1 The Kepler problem

$$\mathbf{F}_{21} = -\mathbf{F}_{12} = \frac{Gm_1m_2}{r^2}\hat{\mathbf{r}},$$

$$\ddot{\mathbf{r}}_1 = \frac{Gm_2}{r^2}\hat{\mathbf{r}}, \quad \ddot{\mathbf{r}}_2 = -\frac{Gm_1}{r^2}\hat{\mathbf{r}},$$

($\hat{\mathbf{r}}$ is a unit vector)

$$\ddot{\mathbf{r}} = \ddot{\mathbf{r}}_2 - \ddot{\mathbf{r}}_1 = -\frac{G(m_1+m_2)}{r^2}\hat{\mathbf{r}} = -\frac{\mu}{r^2}\hat{\mathbf{r}}, \quad \mu \equiv G(m_1+m_2).$$

Denoting $\mathbf{v} = \dot{\mathbf{r}}$, $\mathbf{p} = m\dot{\mathbf{r}}$, $m = \frac{m_1m_2}{m_1+m_2}$ (the *reduced mass*), we define the *angular momentum* of the system

$$\mathbf{L} = \mathbf{r} \wedge \mathbf{p}, \quad \mathbf{l} = \mathbf{r} \wedge \dot{\mathbf{r}},$$

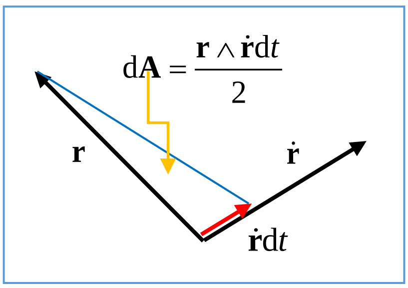

Fig. 2 The area swept

whence, due to $\ddot{\mathbf{r}} \parallel \hat{\mathbf{r}}$, we have

$$\dot{\mathbf{l}} = \dot{\mathbf{r}} \wedge \dot{\mathbf{r}} + \mathbf{r} \wedge \ddot{\mathbf{r}} = 0 \Rightarrow \dot{\mathbf{L}} = 0 \Rightarrow \mathbf{L} = const.$$

From **Fig. 2**, we see that the area swept by $\mathbf{r}$ in a small time is

$$\dot{\mathbf{A}} = \mathbf{l}/2 = const$$

(*Kepler's second law*).

Energy conservation

From

$$\hat{\mathbf{r}} \cdot \dot{\hat{\mathbf{r}}} = \frac{1}{2} \frac{\mathrm{d}\left(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}\right)}{\mathrm{d}t} = 0 ,$$

($\hat{\mathbf{r}} \perp \dot{\hat{\mathbf{r}}} \Rightarrow \hat{\mathbf{r}} \wedge \dot{\hat{\mathbf{r}}} = \hat{\mathbf{r}}\dot{\hat{\mathbf{r}}}$) we have

$$\frac{1}{2} \frac{\mathrm{d}\left(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}\right)}{\mathrm{d}t} = \ddot{\mathbf{r}} \cdot \dot{\mathbf{r}} = -\frac{\mu}{r^2} \hat{\mathbf{r}} \cdot \dot{\mathbf{r}} = -\frac{\mu}{r^2} \hat{\mathbf{r}} \cdot \left(\dot{r}\hat{\mathbf{r}} + r\dot{\hat{\mathbf{r}}}\right) = -\frac{\mu}{r^2} \dot{r} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mu}{r}\right),$$

that is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{v^2}{2} - \frac{\mu}{r}\right) \equiv \frac{\mathrm{d}E}{\mathrm{d}t} = 0 \Rightarrow E = const ,$$

where E is the *total energy*, T is the *kinetic energy*, while U is the *potential energy*

$$E = \frac{v^2}{2} - \frac{\mu}{r} ,\ T = \frac{v^2}{2} ,\ U = -\frac{\mu}{r} .$$

Laplace-Runge-Lenz vector

From $\hat{\mathbf{r}}\dot{\hat{\mathbf{r}}} = -\dot{\hat{\mathbf{r}}}\hat{\mathbf{r}}$ and $\dot{\mathbf{l}} = 0$, we have

$$\mathbf{l} = \mathbf{r} \wedge \dot{\mathbf{r}} = \left(r\hat{\mathbf{r}}\right) \wedge \left(\dot{r}\hat{\mathbf{r}} + r\dot{\hat{\mathbf{r}}}\right) = r^2 \hat{\mathbf{r}}\dot{\hat{\mathbf{r}}} ,$$

$$\mathbf{l}\ddot{\mathbf{r}} = r^2 \hat{\mathbf{r}}\dot{\hat{\mathbf{r}}}\ddot{\mathbf{r}} = -\mu \hat{\mathbf{r}}\dot{\hat{\mathbf{r}}}\hat{\mathbf{r}} = \mu \dot{\hat{\mathbf{r}}} ,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{l}\dot{\mathbf{r}} - \mu\hat{\mathbf{r}}\right) = 0 ,$$

and we can define the constant vector

$$\mathbf{c} = \mathbf{l}\dot{\mathbf{r}} - \mu\hat{\mathbf{r}} ,$$

also known as the *Laplace-Runge-Lenz vector*. Note how easily we came to an important result. Now we can write

$$\mathbf{l}\dot{\mathbf{r}}\mathbf{r} = \mathbf{c}\mathbf{r} + \mu\hat{\mathbf{r}}\mathbf{r} \Rightarrow$$

$$\mathbf{l}\left(\dot{\mathbf{r}} \cdot \mathbf{r} + \dot{\mathbf{r}} \wedge \mathbf{r}\right) = \mathbf{c} \cdot \mathbf{r} + \mathbf{c} \wedge \mathbf{r} + \mu r \Rightarrow$$

$$\mathbf{l}\left(\dot{\mathbf{r}} \cdot \mathbf{r} - \mathbf{l}\right) = \mathbf{c} \cdot \mathbf{r} + \mathbf{c} \wedge \mathbf{r} + \mu r ,$$

whence, comparing grades, we have

$$\left(\dot{\mathbf{r}} \cdot \mathbf{r}\right)\mathbf{l} = \mathbf{c} \wedge \mathbf{r} \qquad (*)$$

$$-\mathbf{l}^2 = \mathbf{c}\cdot\mathbf{r} + \mu r \qquad (**)$$

From (*), we see that for $\dot{\mathbf{r}} \perp \mathbf{r}$ (a *major axis* for planetary motions) it follows $\mathbf{c} \parallel \mathbf{r}$, which means that the LRL vector is always parallel to the major axis.

Trajectories

As $\mathbf{l}^2 < 0$, we define $h^2 = -\mathbf{l}^2$. Defining also $\mathbf{c}\cdot\mathbf{r} = cr\cos\theta$, $p = h^2/\mu$, and $e = c/\mu$, the relation (**) gives

$$r = \frac{p}{1 + e\cos\theta}. \qquad (***)$$

This is an equation of a *conic*, where e is the *eccentricity* and p is the *semi-latus rectum*.

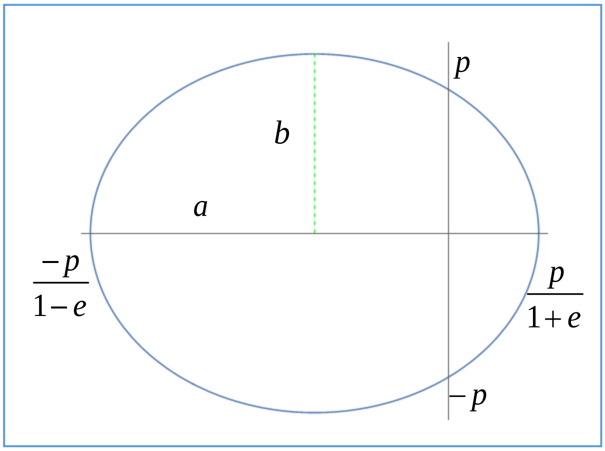

Fig. 3 An ellipse

Note that for circular motion we have $e = 0$, for $0 < e < 1$ we have an *ellipse*, for $e = 1$ we have a *parabola*, while for $e > 1$ we have a *hyperbola*. For $0 < e < 1$ (*ellipse*, *Kepler's first law*), we have (for $\theta = 0$ and $\theta = \pi$)

$$a = \frac{1}{2}\left(\frac{p}{1+e} + \frac{p}{1-e}\right) = \frac{p}{1-e^2},$$

which is the *semi-major axis*.

For the *semi-minor axis*, we have $b = \sqrt{ap}$. From the expression for the surface of an ellipse, for the period T we have

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{h}{2} \Rightarrow$$

$$ab\pi = \frac{hT}{2} \Rightarrow T^2 = \frac{4\pi^2}{\mu}a^3$$

(*Kepler's third law*).

Vis-Viva Equation

Consider the expression for the energy once again. Defining a new vector $\mathbf{e} = \mathbf{c}/\mu$, we can write

$$\mathbf{l}\dot{\mathbf{r}} = \mu(\mathbf{e} + \hat{\mathbf{r}}).$$

From the property $\mathbf{l}\dot{\mathbf{r}} = (\mathbf{r} \wedge \dot{\mathbf{r}})\dot{\mathbf{r}} = -\dot{\mathbf{r}}(\mathbf{r} \wedge \dot{\mathbf{r}})$ (see [3]), we get

$$(\mathbf{l}\dot{\mathbf{r}})^2 = -(\mathbf{r} \wedge \dot{\mathbf{r}})\dot{\mathbf{r}}\dot{\mathbf{r}}(\mathbf{r} \wedge \dot{\mathbf{r}}) = -v^2\mathbf{l}^2 = v^2h^2,$$

$$v^2h^2 = \mu^2(\mathbf{e} + \hat{\mathbf{r}})^2 = \mu^2(e^2 + 2\mathbf{e} \cdot \hat{\mathbf{r}} + 1).$$

From (***), we have

$$\mathbf{e} \cdot \hat{\mathbf{r}} = \frac{h^2}{\mu r} - 1,$$

whence follows that

$$E = \frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu^2}{2h^2}(1 - e^2),$$

and we see that the sing of the total energy is related to the eccentricity e. From the previous definitions, we can also write

$$E = -\frac{\mu}{2a} \Rightarrow v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$$

(*Vis-Viva Equation*).

Finally, we can add that there is a much better approach to this problem, using *eigenspinors* (see [3], Sect. 2.8).

References

[1] Arnold, V.I.: *Mathematical Methods of Classical Mechanics*, Springer, 1989

[2] Hestenes, D.: *New Foundations for Classical Mechanics*, Kluwer Academic, Dordrecht, 1999

[3] Josipović, M.: *Geometric Multiplication of Vectors - An Introduction to Geometric Algebra in Physics*, Birkhäuser, 2019