Comparative statics for oligopoly: Flawless mathematics applied to a faulty result

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Abstract

The assumption that each player conditions on the *endogenous* actions of his rivals when the players are unable to cooperate is inconsistent with the assumption of rational, optimising behavior.

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1. Introduction

Contrary to conventional wisdom, the Nash equilibrium cannot be obtained as a solution of a set of interdependent maximum problems, or *game*. The misconception is so deeply entrenched in the accepted body of knowledge that few, if any, seem to notice it. The literature on the comparative statics for oligopoly (initiated by Dixit, 1986; a recent contribution is Jinji, 2014) confirms this observation.

2. The fault

Consider an oligopoly model with I firms. Let a_i be the decision variable (usually called the *action* or *strategy* in game theory) of firm i, $i = 1, \dots, I$, and let $\omega_i = w_i(\mathbf{a}|\boldsymbol{\theta})$ be the payoff for firm i, where $\mathbf{a}$ is the vector of actions and $\boldsymbol{\theta}$ is a vector of parameters. The functions $w_i(\cdot)$ are assumed to be (twice) continuously differentiable. According to Nash (1951), each player conditions on the actions of his rivals when the players are unable to cooperate. Therefore the "first-order conditions" for a Nash equilibrium are

$$w_{ii} = 0, \qquad i = 1, \dots, I, \tag{1}$$

where $w_{ii} := \partial w_i / \partial a_i$, the *partial* derivative of $w_i(\cdot)$ with respect to a_i.

The first step of the comparative static analysis is to *totally* differentiate (1) with respect to $\mathbf{a}$ and $\boldsymbol{\theta}$. The subsequent mathematics in the literature is flawless. Regrettably the starting point of the analysis is faulty. Deriving the first-order conditions of a game is itself a kind of comparative static analysis, which duly starts by *totally* differentiating the payoff functions with respect to the actions, at given value of $\boldsymbol{\theta}$, to obtain first-order approximations. The result is

$$\mathbf{W}\,\mathrm{d}\mathbf{a} = \mathrm{d}\boldsymbol{\omega}, \tag{2}$$

where $\mathbf{W} := [\partial w_i / \partial a_j]$, the $I \times I$ matrix of partial first-order derivatives of the payoff functions. If the matrix $\mathbf{W}$ is diagonal, the I maximum problems are independent of each other and there is no need to study them jointly. A game corresponds to the case that the off-diagonal elements of $\mathbf{W}$ are generally non-zero. Now there is no (correct) alternative to studying the I problems jointly, for they are interdependent: they constitute a vector maximisation

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problem. One part of the first-order conditions is that the matrix $\mathbf{W}$ have deficient row rank; for if not, a solution of (2) with (strictly) positive vector $\mathrm{d}\boldsymbol{\omega}$ certainly exists. At a point where $\mathbf{W}$ is singular, a vector $\mathrm{d}\mathbf{a}$ exists such that

$$\mathbf{W}\,\mathrm{d}\mathbf{a} = 0. \tag{3}$$

Non-cooperative game theory still treats the maximum problems as independent when deriving (1), ignoring the terms of (3) contributed by the off-diagonal elements of $\mathbf{W}$. There would be no fault if, when evaluated at the Nash equilibrium, the off-diagonal elements would turn out to be zero, but that is generally not true. We thus see that (1) is inconsistent with the first-order approximations of the payoff functions.

3. The remedy

The source of the fault is clearly Nash's claim that each player conditions on the *endogenous* actions of his rivals when the players are unable to cooperate. That restrictions on the possibilities for cooperation may affect the outcomes of real-life interactive decision problems seems plausible. But why would we model such cases by introducing a "solution concept" that corrupts the treatment of vector maximisation problems? Compare the proposal of a new "solution concept" for the problem of budget-constrained utility maximisation, one that applies when a consumer is pressed for time. Whatever the proposal, it would be absurd and would make Gary Becker turn around in his grave. We all know how we may deal with the issue: specify the time intensities of the actions, add a time constraint to the model, and maximise the utility function subject to two inequality restrictions, one stating that total expenditure may not exceed the given budget and the other that total time use may not exceed the time available. Similarly, if there are imminent restrictions on the possibilities for cooperation, the specification of the model must reflect this fact: it must include variables that represent the actions involved in communication and coordination, define their domains, and indicate how they affect the payoffs of the players. The new model will still be a vector maximisation problem, possibly with (in)equality restrictions on the possibilities for cooperation (and maybe also on other actions). Kuhn and Tucker (1950, Section 6) teach us how to deal with such problems.

Further reading

The issue is more fully discussed in: Nieuwenhuis, A., 2017. Reconsidering Nash: The Nash equilibrium is inconsistent. doi: `https://dx.doi.org/10.13140/RG.2.2.29069.03043`. Paper presented at the 21st Annual ESHET Conference, 18–20 May, 2017, Antwerp.

References

Dixit, A.K., 1986. Comparative statics for oligopoly. Int. Econ. Rev. 27, 107–22.
Jinji, N., 2014. Comparative statics for oligopoly: A generalized result. Econ. Lett. 124, 79–82.
Kuhn, H.W., Tucker, A.W., 1950. Nonlinear programming, in: Neyman, J. (Ed.), Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability (held in 1951), University of California Press, Berkeley. pp. 481–92.
Nash, J., 1951. Non-cooperative games. Ann. of Math. 54, 286–95.