ON THE ERDŐS-TURÁN ADDITIVE BASE CONJECTURE

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ABSTRACT. In this paper we formulate and prove several variants of the Erdős-Turán additive bases conjecture.

1. Introduction and Problem Statement

Let $\mathbb{S}$ be a subset of the natural numbers $\mathbb{N}$ and $k \in \mathbb{N}$ be fixed. Then $\mathbb{S}$ is said to be an **additive base** of order k if every natural number can be expressed as a sum of k elements of $\mathbb{S}$. The weak Goldbach conjecture suggests the set of prime numbers is an additive base of order **three** [1]. The Erdős-Turán additive bases conjecture is the assertion that all additive bases qualifies very sufficiently to be an additive additive base in the very large. In particular we have the following conjecture of Erdős and Turán (see [2])

Conjecture 1.1 (Erdős-Turán). Let $\mathbb{B} \subset \mathbb{N}$ and consider

$$r_{\mathbb{B}}(n) := \# \left\{ (a,b) \in \mathbb{B}^2 \mid a + b = n \right\}.$$

If $r_{\mathbb{B}}(n) > 0$ for all sufficiently large values of n, then

$$\limsup_{n \longrightarrow \infty} r_{\mathbb{B}}(n) = \infty.$$

This conjecture has garnered the attention of many authors but remains unresolved [3]. By introducing the language of **Circles of Partition** and associated statistics we reformulate the conjecture in the following manner

Conjecture 1.2 (Erdős-Turán). Let $\mathbb{B} \subset \mathbb{N}$ and consider

$$\mathcal{G}_{\mathbb{B}}(n) = \nu(n, \mathbb{B}) = \# \left\{ \mathbb{L}_{[x],[y]} \; \hat{\in} \; \mathcal{C}(n, \mathbb{B}) \right\}.$$

If $\mathcal{G}_{\mathbb{B}}(n) > 0$ for all sufficiently large values of n, then

$$\limsup_{n \longrightarrow \infty} \mathcal{G}_{\mathbb{B}}(n) = \infty.$$

We exploiting the notion of the density of circles of partition, the notion of ascending, descending and stationary circles of partition and the l th fold energy of circle of partitions to study the Erdős-Turán additive bases conjecture.

2. The Circle of Partition

In this section we introduce the notion of the circle of partition. We study this notion in-depth and explore some potential applications in the following sequel.

Definition 2.1. Let $n \in \mathbb{N}$ and $\mathbb{M} \subset \mathbb{N}$. We denote with

$$\mathcal{C}(n, \mathbb{M}) = \{[x] \mid x, y \in \mathbb{M}, n = x + y\}$$

the Circle of Partition generated by n with respect to the subset $\mathbb{M}$. We will abbreviate this in the further text as CoP. We call members of $\mathcal{C}(n, \mathbb{M})$ as points and denote them by $[x]$. For the special case $\mathbb{M} = \mathbb{N}$ we denote the CoP shortly as $\mathcal{C}(n)$.

Definition 2.2. We denote the line $\mathbb{L}_{[x],[y]}$ joining the point $[x]$ and $[y]$ as an axis of the CoP $\mathcal{C}(n, \mathbb{M})$ if and only if $x + y = n$. We say the axis point $[y]$ is an axis partner of the axis point $[x]$ and vice versa. We do not distinguish between $\mathbb{L}_{[x],[y]}$ and $\mathbb{L}_{[y],[x]}$, since it is essentially the the same axis. The point $[x] \in \mathcal{C}(n, \mathbb{M})$ such that $2x = n$ is the **center** of the CoP. If it exists then it is their only point which is not an axis point. The line joining any two arbitrary point which are not axes partners on the CoP will be referred to as a **chord** of the CoP. The length of the chord joining the points $[x], [y] \in \mathcal{C}(n, \mathbb{M})$, denoted as $\mathcal{D}([x], [y])$ is given by

$$\mathcal{D}([x], [y]) = |x - y|.$$

It is important to point out that the **median** of the weights of each co-axis point coincides with the center of the underlying CoP if it exists. That is to say, given all the axes of the CoP $\mathcal{C}(n, \mathbb{M})$ as

$$\mathbb{L}_{[u_1],[v_1]}, \mathbb{L}_{[u_2],[v_2]}, \ldots, \mathbb{L}_{[u_k],[v_k]}$$

then the following relations hold

$$\frac{u_1 + v_1}{2} = \frac{u_2 + v_2}{2} = \cdots = \frac{u_k + v_k}{2} = \frac{n}{2}$$

which is equivalent to the conditions for any of the pair of axes $\mathbb{L}_{[u_i],[v_i]}, \mathbb{L}_{[u_j],[v_j]}$ for $1 \leq i, j \leq k$

$$\mathcal{D}([u_i], [u_j]) = \mathcal{D}([v_i], [v_j])$$

and

$$\mathcal{D}([v_j], [u_i]) = \mathcal{D}([u_j], [v_i]).$$

The above language in many ways could be seen as a criterion determining the plausibility of carrying out a partition in a specified set. Indeed this feasibility is trivial if we take the set $\mathbb{M}$ to be the set of natural numbers $\mathbb{N}$. The situation becomes harder if we take the set $\mathbb{M}$ to be a special subset of natural numbers $\mathbb{N}$, as the corresponding CoP $\mathcal{C}(n, \mathbb{M})$ may not always be non-empty for all $n \in \mathbb{N}$. One archetype of problems of this flavour is the binary Goldbach conjecture, when we take the base set $\mathbb{M}$ to be the set of all prime numbers $\mathbb{P}$. One could imagine the same sort of difficulty if we extend our base set to other special subsets of the natural numbers.

Remark 2.3. It is important to notice that a typical CoP need not have a center. In the case of an absence of a center then we say the circle has a deleted center. It is easy to see that the CoP $\mathcal{C}(n)$ contains all points whose weights are positive integers from 1 to $n-1$ inclusive:

$$\mathcal{C}(n) = \{[x] \mid x \in \mathbb{N}, x < n\}.$$

Therefore the CoP $\mathcal{C}(n)$ has $\left\lfloor \frac{n-1}{2} \right\rfloor$ different axes.

In the sequel we will denote the assignment of an axis $\mathbb{L}_{[x],[y]}$ to a CoP $\mathcal{C}(n,\mathbb{M})$ as

$$\mathbb{L}_{[x],[y]} \,\hat{\in}\, \mathcal{C}(n,\mathbb{M}) \text{ which means}$$
$$[x],[y] \in \mathcal{C}(n,\mathbb{M}) \quad \textbf{and} \quad x+y = n$$

and the number of axes of a CoP as

$$\nu(n,\mathbb{M}) := \#\{\mathbb{L}_{[x],[y]} \,\hat{\in}\, \mathcal{C}(n,\mathbb{M})\}.$$

Additionally we let

$$\mathbb{N}_n = \{m \in \mathbb{N} \mid m \leq n\}$$

be the **sequence** of the first n natural numbers. Further we will denote

$$\|[x]\| := x$$

as the **weight** of the point $[x]$ and correspondingly the weight set of points in the CoP $\mathcal{C}(n,\mathbb{M})$ as $\|\mathcal{C}(n,\mathbb{M})\|$.

Proposition 2.4. *Each axis is uniquely determined by points $[x] \in \mathcal{C}(n,\mathbb{M})$.*

Proof. Let $\mathbb{L}_{[x],[y]}$ be an axis of the CoP $\mathcal{C}(n,\mathbb{M})$. Suppose as well that $\mathbb{L}_{[x],[z]}$ is also an axis with $z \neq y$. Then it follows by Definition 2.2 that we must have $n = x + y = x + z$ and therefore $y = z$. This cannot be and the claim follows immediately. $\square$

Corollary 2.5. *Each point of a CoP $\mathcal{C}(n,\mathbb{M})$ has exactly one axis partner.*

Proof. Let $[x] \in \mathcal{C}(n,\mathbb{M})$ be a point without an axis partner. Then holds for every point $[y] \neq [x]$

$$\|[x]\| + \|[y]\| \neq n.$$

This contradiction to the Definition 2.1. Due to Proposition 2.4 the case of more than one axis partners is impossible. This completes the proof. $\square$

3. **The Density of Points on the Circle of Partition**

In this section we introduce the notion of density of points on CoP $\mathcal{C}(n,\mathbb{M})$ for $\mathbb{M} \subseteq \mathbb{N}$. We launch the following language in that regard. We exploit this notion in a careful manner to study the Erdős-Turán additive bases conjecture.

Definition 3.1. Let $\mathbb{H} \subset \mathbb{N}$. Then the quantity

$$\mathcal{D}(\mathbb{H}) = \lim_{n \to \infty} \frac{|\mathbb{H} \cap \mathbb{N}_n|}{n}$$

denotes the density of $\mathbb{H}$.

Definition 3.2. Let $\mathcal{C}(n, \mathbb{M})$ be CoP with $\mathbb{M} \subset \mathbb{N}$ and $n \in \mathbb{N}$. Suppose $\mathbb{H} \subset \mathbb{M}$ then by the density of points $[x] \in \mathcal{C}(n, \mathbb{M})$ such that $x \in \mathbb{H}$, denoted $\mathcal{D}(\mathbb{H}_{\mathcal{C}(\infty, \mathbb{M})})$, we mean the quantity

$$\mathcal{D}\left(\mathbb{H}_{\mathcal{C}(\infty, \mathbb{M})}\right) = \lim_{n \longrightarrow \infty} \frac{\#\{\mathbb{L}_{[x],[y]} \,\hat{\in}\, \mathcal{C}(n, \mathbb{M})|\ \{x, y\} \cap \mathbb{H} \neq \emptyset\}}{\nu(n, \mathbb{M})}.$$

Proposition 3.3. *Let $\mathbb{H} \subset \mathbb{M}$ with $\mathbb{M} \subseteq \mathbb{N}$ and suppose $\mathcal{D}(\mathbb{H}_{\mathcal{C}(\infty, \mathbb{M})})$ exists. Then the following properties hold:*

(i) $\mathcal{D}(\mathbb{M}_{\mathcal{C}(\infty, \mathbb{M})}) = 1$ *and* $\mathcal{D}(\mathbb{H}_{\mathcal{C}(\infty, \mathbb{M})}) \leq 1$.

(ii) $1 - \lim\limits_{n \longrightarrow \infty} \dfrac{\nu(n, \mathbb{M} \setminus \mathbb{H})}{\nu(n, \mathbb{M})} = \mathcal{D}(\mathbb{H}_{\mathcal{C}(\infty, \mathbb{M})})$.

(iii) *If* $|\mathbb{H}| < \infty$ *then* $\mathcal{D}(\mathbb{H}_{\mathcal{C}(\infty, \mathbb{M})}) = 0$.

Proof. It is easy to see that **Property** (i) and **Property** (iii) are both easy consequences of the definition of density of points on the CoP $\mathcal{C}(n, \mathbb{M})$. We establish **Property** (ii), which is the less obvious case. We observe by the uniqueness of the axes of CoPs that we can write

$$1 = \lim_{n \longrightarrow \infty} \frac{\nu(n, \mathbb{M})}{\nu(n, \mathbb{M})}$$

$$= \lim_{n \longrightarrow \infty} \frac{\#\{\mathbb{L}_{[x],[y]} \,\hat{\in}\, \mathcal{C}(n, \mathbb{M})|\ x \in \mathbb{H}\ , y \in \mathbb{M} \setminus \mathbb{H}\}}{\nu(n, \mathbb{M})}$$

$$+ \lim_{n \longrightarrow \infty} \frac{\nu(n, \mathbb{H})}{\nu(n, \mathbb{M})} + \lim_{n \longrightarrow \infty} \frac{\nu(n, \mathbb{M} \setminus \mathbb{H})}{\nu(n, \mathbb{M})}$$

$$= \mathcal{D}(\mathbb{H}_{\mathcal{C}(\infty, \mathbb{M})}) + \lim_{n \longrightarrow \infty} \frac{\nu(n, \mathbb{M} \setminus \mathbb{H})}{\nu(n, \mathbb{M})}$$

and (ii) follows immediately. $\qquad\qquad\square$

Proposition 3.4. *Let $\mathcal{C}(n)$ with $n \in \mathbb{N}$ be a CoP and $\mathbb{H} \subset \mathbb{N}$. Then the following inequality holds*

$$\lim_{n \longrightarrow \infty} \frac{\left\lfloor \frac{|\mathbb{H} \cap \mathbb{N}_n|}{2} \right\rfloor}{\left\lfloor \frac{n-1}{2} \right\rfloor} \leq \mathcal{D}(\mathbb{H}_{\mathcal{C}(\infty)}) \leq \lim_{n \longrightarrow \infty} \frac{|\mathbb{H} \cap \mathbb{N}_n|}{\left\lfloor \frac{n-1}{2} \right\rfloor}.$$

Proof. The upper is obtained from a configuration where no two points $[x], [y] \in \mathcal{C}(n)$ such that $x, y \in \mathbb{H}$ lie on the same axis of the CoP. The lower bound however follows from a configuration where any two points $[x], [y] \in \mathcal{C}(n)$ with $x, y \in \mathbb{H}$ are joined by an axis of the CoP. $\qquad\qquad\square$

Remark 3.5. Though we are nowhere near the proof of this conjecture, we prove a weaker version by imposing some suitable conditions. The result is encapsulated in the following theorem.

Theorem 3.6. *Let $\mathbb{B} \subset \mathbb{N}$ with*

$$\lim_{n \longrightarrow \infty} \frac{|\mathbb{B} \cap \mathbb{N}_n|}{n} > 0$$

such that

$$\#\left\{\mathbb{L}_{[x],[y]} \,\hat{\in}\, \mathcal{C}(n)|\ x \in \mathbb{N} \setminus \mathbb{B},\ y \in \mathbb{B}\right\} \leq \nu(n, \mathbb{B}).$$

If $\mathcal{G}_{\mathbb{B}}(n) = \nu(n, \mathbb{B}) > 0$ for all sufficiently large values of n, then

$$\limsup_{n \longrightarrow \infty} \mathcal{G}_{\mathbb{B}}(n) = \infty.$$

Proof. Suppose $\mathbb{B} \subset \mathbb{N}$ and let $\mathcal{G}_{\mathbb{B}}(n) > 0$ for all sufficiently large values of n. Suppose to the contrary that

$$\limsup_{n \longrightarrow \infty} \mathcal{G}_{\mathbb{B}}(n) < \infty.$$

Consider the CoP $\mathcal{C}(n, \mathbb{B})$, then we note that by the uniqueness of axes of CoPs we can compute the density of points $[x] \in \mathcal{C}(n)$ with $||[x]|| \in \mathbb{B}$ in the following way

$$
\begin{aligned}
\mathcal{D}(\mathbb{B}_{\mathcal{C}(\infty)}) &= \lim_{n \longrightarrow \infty} \frac{\#\{\mathbb{L}_{[x],[y]} \,\hat{\in}\, \mathcal{C}(n) | \{x, y\} \cap \mathbb{B} \neq \emptyset\}}{\left\lfloor \frac{n-1}{2} \right\rfloor} \\
&= \lim_{n \longrightarrow \infty} \frac{\#\{\mathbb{L}_{[x],[y]} \,\hat{\in}\, \mathcal{C}(n) | \; x \in \mathbb{N} \setminus \mathbb{B}, \; y \in \mathbb{B}\}}{\left\lfloor \frac{n-1}{2} \right\rfloor} + \lim_{n \longrightarrow \infty} \frac{\nu(n, \mathbb{B})}{\left\lfloor \frac{n-1}{2} \right\rfloor} \\
&\leq 2 \lim_{n \longrightarrow \infty} \frac{\nu(n, \mathbb{B})}{\left\lfloor \frac{n-1}{2} \right\rfloor} \\
&= 0
\end{aligned}
$$

by virtue of the earlier assumption. By applying Proposition 3.4, it follows that

$$\lim_{n \longrightarrow \infty} \frac{\left\lfloor \frac{|\mathbb{B} \cap \mathbb{N}_n|}{2} \right\rfloor}{\left\lfloor \frac{n-1}{2} \right\rfloor} = 0.$$

It follows that $\mathcal{D}(\mathbb{B}) = 0$, thereby contradicting the requirement of the statement. $\square$

4. Ascending, Descending and Stationary Circles of Partition

In this section we introduce the notion of **ascending**, **descending** and **stationary** CoPs between generators. We formalize this notion in the following language. We exploit this notion to improve on the result concerning the Erdős-Turán additive bases conjecture in section 2.

Definition 4.1. Let $\mathbb{M} \subset \mathbb{N}$ with $\mathcal{C}(n, \mathbb{M})$ be a CoP. Then we say the CoP $\mathcal{C}(n, \mathbb{M})$ is **ascending** from n to the **spot** m if for $n < m$ holds

$$\nu(n, \mathbb{M}) < \nu(m, \mathbb{M}).$$

Similarly, we say it is **descending** from n to the **spot** m if for $n < m$ then

$$\nu(n, \mathbb{M}) > \nu(m, \mathbb{M}).$$

We say it is **globally** ascending (resp. descending) if at $\forall m \in \mathbb{N}$ it is ascending (resp. descending). We say the CoP $\mathcal{C}(n, \mathbb{M})$ is **stationary** from n to the **spot** m if for $n < m$ then

$$\nu(n, \mathbb{M}) = \nu(m, \mathbb{M}).$$

Similarly, we say it is **globally stationary** if it is stationary at all spots $m \in \mathbb{N}$. If the CoP $\mathcal{C}(n, \mathbb{M})$ is neither globally ascending, descending nor stationary, then we say it is globally **oscillatory**.

Theorem 4.2. *Let* $\mathbb{H} \subset \mathbb{N}$ *and* $\mathcal{C}(n, \mathbb{H})$ *be a CoP. If*

$$\lim_{n \longrightarrow \infty} \frac{|\mathbb{H} \cap \mathbb{N}_n|}{n} > 0$$

with

$$\lim_{n \longrightarrow \infty} \frac{|(\mathbb{N} \setminus \mathbb{H}) \cap \mathbb{N}_n|}{n} < \frac{1}{2} \lim_{n \longrightarrow \infty} \frac{|\mathbb{H} \cap \mathbb{N}_n|}{n}$$

then $\mathcal{C}(n, \mathbb{H})$ *is ascending at infinitely many spots.*

Proof. Let $\mathcal{C}(n, \mathbb{H})$ be a CoP and assume to the contrary that there are finitely many spots at which it is ascending. Let us name and arrange the spots as follows $m_1 < m_2 < \cdots < m_k$. It follows that

$$\nu(n, \mathbb{H}) \geq \nu(m_{k+1}, \mathbb{H}) \geq \cdots \nu(m_{k+i}, \mathbb{H}) \geq \cdots$$

for all $i \geq 1$. The upshot is that

$$\lim_{n \longrightarrow \infty} \frac{\nu(n, \mathbb{H})}{\left\lfloor \frac{n-1}{2} \right\rfloor} = 0.$$

Next, by virtue of uniqueness of axes of CoPs, we can compute the density of points with weight in $\mathbb{H}$ on the CoP $\mathcal{C}(n)$ as follows

$$
\begin{aligned}
\mathcal{D}(\mathbb{H}_{\mathcal{C}(\infty)}) &= \lim_{n \longrightarrow \infty} \frac{\# \left\{ \mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n) | \ \{x, y\} \cap \mathbb{H} \neq \emptyset \right\}}{\left\lfloor \frac{n-1}{2} \right\rfloor} \\
&= \lim_{n \longrightarrow \infty} \frac{\# \left\{ \mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n) | \ x \in \mathbb{H}, \ y \in \mathbb{N} \setminus \mathbb{H} \right\}}{\left\lfloor \frac{n-1}{2} \right\rfloor} + \lim_{n \longrightarrow \infty} \frac{\nu(n, \mathbb{H})}{\left\lfloor \frac{n-1}{2} \right\rfloor} \\
&= \lim_{n \longrightarrow \infty} \frac{\# \left\{ \mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n) | \ x \in \mathbb{H}, \ y \in \mathbb{N} \setminus \mathbb{H} \right\}}{\left\lfloor \frac{n-1}{2} \right\rfloor} \\
&\leq \lim_{n \longrightarrow \infty} \frac{|(\mathbb{N} \setminus \mathbb{H}) \cap \mathbb{N}_n|}{\left\lfloor \frac{n-1}{2} \right\rfloor} \\
&\leq 2 \lim_{n \longrightarrow \infty} \frac{|(\mathbb{N} \setminus \mathbb{H}) \cap \mathbb{N}_n|}{n}.
\end{aligned}
$$

Invoking Proposition 3.4, we have the inequality

$$\lim_{n \longrightarrow \infty} \frac{|\mathbb{H} \cap \mathbb{N}_n|}{n} \leq 2 \lim_{n \longrightarrow \infty} \frac{|(\mathbb{N} \setminus \mathbb{H}) \cap \mathbb{N}_n|}{n}.$$

This, however, violates the requirement of the statement, thereby ending the proof.
$\square$

Remark 4.3. Next we obtain from this result another weak variant of the Erdős-Turán conjecture. Roughly speaking, it purports very dense sequences sufficiently qualifies to be an additive base.

Corollary 4.4. Let $\mathbb{H} \subset \mathbb{N}$ with $\mathcal{D}(\mathbb{H}) > 0$ such that $\mathcal{D}(\mathbb{N} \setminus \mathbb{H}) < \frac{1}{2}\mathcal{D}(\mathbb{H})$. If

$$r_{\mathbb{H}}(n) := \# \left\{ (a, b) \in \mathbb{H}^2 | \ a + b = n \right\}$$

then $\lim_{n \longrightarrow \infty} r_{\mathbb{H}}(n) = \infty$.

5. The lth Fold Energy of Circles of Partition

In this section we introduce and study the notion of the lth fold energy of CoPs and exploit some applications in this context. This notion tends to more effective and extends very much to sequences not necessarily having a positive density.

Definition 5.1. Let $\mathbb{M} \subset \mathbb{N}$ and $\mathcal{C}(n, \mathbb{M})$ be a CoP. Then by the lth-fold energy of the CoP $\mathcal{C}(n, \mathbb{M})$, we mean the quantity

$$\mathcal{E}(l, \mathbb{M}) := \sum_{n=3}^{\infty} \frac{\nu(n^l, \mathbb{M})}{\left\lfloor \frac{n^l - 1}{2} \right\rfloor}$$

for a fixed $l \in \mathbb{N}$.

It is important to remark that the lth energy of a typical CoP $\mathcal{C}(n, \mathbb{M})$ could either be infinite or finite. In that latter case it certainly should have a finite value. To that effect we state the following proposition.

Proposition 5.2. Let $\mathbb{J}^l \subset \mathbb{N}$ be the set of all l^{th} powers. Then $\mathcal{E}(l, \mathbb{J}^l) < \infty$ for all $l \geq 3$ and $\mathcal{E}(2, \mathbb{J}^2) = \infty$.

Proof. Let $l \geq 3$ be fixed and consider the CoP $\mathcal{C}(n^l, \mathbb{J}^l)$, where $\mathbb{J}^l \subset \mathbb{N}$ is the set of all l^{th} powers. Then it follows from the configuration of CoPs the following inequality

$$\mathcal{E}(l, \mathbb{J}^l) = \sum_{n=3}^{\infty} \frac{\nu(n^l, \mathbb{J}^l)}{\left\lfloor \frac{n^l - 1}{2} \right\rfloor}$$
$$\leq \sum_{n=3}^{\infty} \frac{\frac{n}{2}}{\left\lfloor \frac{n^l - 1}{2} \right\rfloor}$$
$$\ll \sum_{n=3}^{\infty} \frac{1}{n^{l-1}} < \infty$$

for all $l \geq 3$. $\qquad \square$

Proposition 5.3. Let $\mathbb{M} \subset \mathbb{N}$ and $\mathcal{C}(n, \mathbb{M})$ be a CoP. If $\mathcal{E}(l, \mathbb{M}) = \infty$ for $l \geq 2$, then $\mathcal{C}(n^l, \mathbb{M})$ is ascending at infinitely many spots.

Proof. Let $\mathcal{E}(l, \mathbb{M}) = \infty$ and assume to the contrary that the CoP $\mathcal{C}(n^l, \mathbb{M})$ is ascending at finitely many spots. Then it follows that

$$\lim_{n \longrightarrow \infty} \nu(n^l, \mathbb{M}) < \infty.$$

This implies $\mathcal{E}(l, \mathbb{M}) < \infty$, thereby contradicting the requirement of the statement. $\qquad \square$

Theorem 5.4. Let $\mathbb{B} \subset \mathbb{N}$ with $\# \{n \leq x \mid n \in \mathbb{B}\} \sim x^{1-\epsilon}$ for any $0 < \epsilon \leq \frac{1}{2}$ and consider

$$\mathcal{G}_{\mathbb{B}}(n) = \nu(n, \mathbb{B})$$

If $\mathcal{G}_{\mathbb{B}}(n) > 0$ for all sufficiently large values of n, then

$$\limsup_{n \longrightarrow \infty} \mathcal{G}_{\mathbb{B}}(n) = \infty.$$

Proof. First we compute the two fold energy $\mathcal{E}(2, \mathbb{B})$ of the CoP $\mathcal{C}(n, \mathbb{B})$. Since $\mathcal{G}_{\mathbb{B}}(n) > 0$ for all sufficiently large values of n, it follows that $\mathcal{G}_{\mathbb{B}}(n^2) > 0$ for all sufficiently large values of n so that for all k large enough there exist some constant $\mathcal{L} = \mathcal{L}(k) > 0$ such that we can write

$$\sum_{n=3}^{k} \frac{\mathcal{G}_{\mathbb{B}}(n^2)}{\left\lfloor \frac{n^2-1}{2} \right\rfloor} = \mathcal{L}(k)(1 + o(1)) \sum_{n=3}^{k} \frac{\left\lfloor \frac{n^{2-2\epsilon}-1}{2} \right\rfloor}{\left\lfloor \frac{n^2-1}{2} \right\rfloor}$$

$$\gg_k \sum_{n=3}^{k} \frac{1}{n^{2\epsilon}}.$$

By taking limits on both sides as $k \longrightarrow \infty$ and noting that $0 < \epsilon \leq \frac{1}{2}$, we deduce $\mathcal{E}(2, \mathbb{B}) = \infty$. Appealing to Proposition 5.3, it follows that

$$\limsup_{n \longrightarrow \infty} \mathcal{G}_{\mathbb{B}}(n^2) = \infty.$$

Since $\{n^2 \in \mathbb{N}\} \subset \{n \in \mathbb{N}\}$, it follows that $\limsup_{n \longrightarrow \infty} \mathcal{G}_{\mathbb{B}}(n) = \infty$. $\qquad\square$

Let $\mathbb{B}$ be an additive base of order 2, then it is well-known that

$$\# \{n \leq x \mid n \in \mathbb{B}\} \geq \sqrt{x}.$$

In line with this tied with Theorem 5.4 the solution to the Erdős-Turán additive bases conjecture is an easy consequence.
[1].

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