Fibonacci series from power series

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Abstract

We show how every power series gives rise to a Fibonacci series and a companion series involving Lucas numbers. For illustrative purposes, Fibonacci series arising from trigonometric functions, inverse trigonometric functions, the gamma function and the digamma function are derived. Infinite series involving Fibonacci and Bernoulli numbers and Fibonacci and Euler numbers are also obtained.

1 Introduction

The Fibonacci numbers, F_n, and the Lucas numbers, L_n, are defined, for all integers n by the Binet formulas:

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad L_n = \alpha^n + \beta^n\,, \tag{1.1}$$

where α and β are the zeros of the characteristic polynomial, $x^2 - x - 1$, of the Fibonacci sequence. Thus $\alpha + \beta = 1$ and $\alpha\beta = -1$; so that $\alpha = (1+\sqrt{5})/2$ (the golden ratio) and $\beta = -1/\alpha = (1-\sqrt{5})/2$. Koshy [10] and Vajda [13] have written excellent books dealing with Fibonacci and Lucas numbers.

Our task in this paper is to show how every power series gives rise to a Fibonacci series and a companion series involving Lucas numbers. For example, in §4, we shall demonstrate that the inverse tangent series,

$$\sum_{j=1}^\infty \frac{(-1)^j z^{2j-1}}{2j-1} \equiv \tan^{-1} z, \quad |z| \le 1\,,$$

gives rise to the following Fibonacci series:

$$\sum_{j=1}^\infty \frac{(-1)^{j-1}}{2j-1} F_{2rj+s} z^{2j-1} = \frac{F_{s+r}}{2}\tan^{-1}\left(\frac{zL_r}{1-(-1)^r z^2}\right) + \frac{L_{s+r}}{2\sqrt{5}}\tan^{-1}\left(\frac{F_r z\sqrt{5}}{1+(-1)^r z^2}\right),$$

valid for any integers r and s and any real or complex variable z such that $|z| \le 1$; with a companion result for the Lucas numbers.

We have the particularly beautiful identity

$$\sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{2j-1} F_{2j-1} = \frac{\pi}{2\sqrt{5}}\,.$$

Take another example. The power series

$$\sum_{j=2}^{\infty} \zeta(j) \frac{z^j}{j} \equiv \log \Gamma(1-z) - \gamma z, \quad |z| < 1\,,$$

where γ is the Euler-Mascheroni constant, $\zeta(n)$ is the Riemann zeta function and $\Gamma(x)$ is the gamma function, leads to the following series involving Lucas numbers and the zeta function (Theorem 13, §7):

$$\sum_{j=2}^{\infty} \frac{\zeta(j)}{j} L_{rj} z^j = \log \Gamma(1-\alpha^r z)\Gamma(1-\beta^r z) - \gamma z L_r\,;$$

particular instances of which are

$$\sum_{j=2}^{\infty} (-1)^j \frac{\zeta(j)}{j} L_j = \log(-\pi \csc(\pi\beta)) + \gamma$$

and

$$\sum_{j=2}^{\infty} (-1)^j \frac{\zeta(j)}{2^j j} L_{3j} = \log\left(-\frac{\pi\sqrt{5}}{8}\sec(\pi\beta)\right) + 2\gamma\,.$$

In §2, we prove the theorem regarding how to obtain Fibonacci series from power series. Illustrative Examples are then presented in §3 – §7.

2 Fibonacci series from power series

Theorem. *For real or complex z, let a given well-behaved function $h(z)$ have, in its domain, the representation $h(z) = \sum_{j=c_1}^{c_2} g(j) z^{f(j)}$ where $f(j)$ and $g(j)$ are given real sequences and $c_1, c_2 \in [-\infty, \infty]$. Let r and s be integers. Then,*

$$\sum_{j=c_1}^{c_2} F_{rf(j)+s}\, g(j) z^{f(j)} = \frac{F_s}{2}\left(h(\alpha^r z) + h(\beta^r z)\right) + \frac{L_s}{2\sqrt{5}}\left(h(\alpha^r z) - h(\beta^r z)\right)\,, \qquad \text{(F)}$$

$$\sum_{j=c_1}^{c_2} L_{rf(j)+s}\, g(j) z^{f(j)} = \frac{L_s}{2}\left(h(\alpha^r z) + h(\beta^r z)\right) + \frac{F_s\sqrt{5}}{2}\left(h(\alpha^r z) - h(\beta^r z)\right)\,, \qquad \text{(L)}$$

whenever the series on the left hand side of each of (F) *and* (L) *converges.*

Proof. We have

$$\sum_{j=c_1}^{c_2} g(j)z^{f(j)} = h(z)\,. \tag{2.1}$$

Writing $\alpha^r z$ for z in (2.1) and multiplying both sides by α^s, we obtain

$$\sum_{j=c_1}^{c_2} g(j)\alpha^{rf(j)+s}z^{f(j)} = \alpha^s h(\alpha^r z)\,. \tag{2.2}$$

Similarly, writing $\beta^r z$ for z in (2.1) and multiplying both sides by β^s, we obtain

$$\sum_{j=c_1}^{c_2} g(j)\beta^{rf(j)+s}z^{f(j)} = \beta^s h(\beta^r z)\,. \tag{2.3}$$

From (2.2) and (2.3), we have

$$\frac{1}{2}\sum_{j=c_1}^{c_2} g(j)L_{rf(j)+s}z^{f(j)} + \frac{\sqrt{5}}{2}\sum_{j=c_1}^{c_2} g(j)F_{rf(j)+s}z^{f(j)} = \alpha^s h(\alpha^r z)\,, \tag{2.4}$$

and

$$\frac{1}{2}\sum_{j=c_1}^{c_2} g(j)L_{rf(j)+s}z^{f(j)} - \frac{\sqrt{5}}{2}\sum_{j=c_1}^{c_2} g(j)F_{rf(j)+s}z^{f(j)} = \beta^s h(\beta^r z)\,, \tag{2.5}$$

where we have used the fact that, for any integer m,

$$\alpha^m = \frac{L_m + F_m\sqrt{5}}{2} \text{ and } \beta^m = \frac{L_m - F_m\sqrt{5}}{2}\,. \tag{2.6}$$

Subtraction of (2.5) from (2.4) while making use of (2.6) again to resolve α^s and β^s produces identity (F). Addition of (2.4) and (2.5) gives identity (L). □

Setting $s = 0$ in (F) and (L), we have the particular cases,

$$\sum_{j=c_1}^{c_2} F_{rf(j)}g(j)z^{f(j)} = \frac{1}{\sqrt{5}}\left(h(\alpha^r z) - h(\beta^r z)\right)\,, \tag{F1}$$

$$\sum_{j=c_1}^{c_2} L_{rf(j)}g(j)z^{f(j)} = h(\alpha^r z) + h(\beta^r z)\,. \tag{L1}$$

In §3 – §7 we will apply identities (F) and (L) to derive Fibonacci series from certain power series.

3 Fibonacci series from trigonometric functions

Theorem 1. *If r and s are integers and z is a real or complex variable, then,*

$$\begin{aligned}\sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!}F_{2rj+s}z^{2j+1} &= F_{s-r}\sin\left(\frac{zL_r}{2}\right)\cos\left(\frac{F_r z\sqrt{5}}{2}\right)\\ &\quad + \frac{L_{s-r}}{\sqrt{5}}\sin\left(\frac{F_r z\sqrt{5}}{2}\right)\cos\left(\frac{zL_r}{2}\right)\,,\end{aligned} \tag{3.1}$$

$$\begin{aligned}\sum_{j=0}^{\infty}\frac{(-1)^j}{(2j+1)!}L_{2rj+s}z^{2j+1}&=L_{s-r}\sin\left(\frac{zL_r}{2}\right)\cos\left(\frac{F_rz\sqrt5}{2}\right)\\&\qquad+F_{s-r}\sqrt5\sin\left(\frac{F_rz\sqrt5}{2}\right)\cos\left(\frac{zL_r}{2}\right)\,.\end{aligned}\tag{3.2}$$

Proof. Consider the Maclaurin series expansion of $\sin z$:

$$\sum_{j=0}^{\infty}(-1)^j\frac{z^{2j+1}}{(2j+1)!}=\sin z\,.$$

Use $g(j)=(-1)^j/((2j+1)!)$, $f(j)=2j+1$, $c_1=0$, $c_2=\infty$ and $h(z)=\sin z$ in identities (F) and (L); noting the identities

$$\sin x\pm\sin y=2\sin\left(\frac{x\pm y}{2}\right)\cos\left(\frac{x\mp y}{2}\right)\,.$$

□

In particular,

$$\sum_{j=0}^{\infty}\frac{(-1)^j}{(2j+1)!}F_{r(2j+1)}z^{2j+1}=\frac{2}{\sqrt5}\sin\left(\frac{F_rz\sqrt5}{2}\right)\cos\left(\frac{zL_r}{2}\right)\,,\tag{3.3}$$

$$\sum_{j=0}^{\infty}\frac{(-1)^j}{(2j+1)!}L_{r(2j+1)}z^{2j+1}=2\sin\left(\frac{zL_r}{2}\right)\cos\left(\frac{F_rz\sqrt5}{2}\right)\,.\tag{3.4}$$

Example 1. *If r is an integer, then,*

$$\sum_{j=1}^{\infty}\frac{\zeta(2j)}{(2j+1)B_{2j}}\frac{F_{r(2j+1)}}{F_r^{2j+1}5^j}=\frac12\,,\tag{3.5}$$

$$\sum_{j=1}^{\infty}\frac{\zeta(2j)}{2^{2j+1}(2j+1)B_{2j}}\frac{F_{r(2j+1)}}{L_r^{2j+1}}=\frac{F_r}{4L_r}\,,\tag{3.6}$$

$$\sum_{j=1}^{\infty}\frac{\zeta(2j)}{(2j+1)B_{2j}}\frac{L_{r(2j+1)}}{L_r^{2j+1}}=\frac12\,,\tag{3.7}$$

$$\sum_{j=1}^{\infty}\frac{\zeta(2j)}{2^{2j+1}(2j+1)B_{2j}}\frac{L_{r(2j+1)}}{F_r^{2j+1}5^j}=\frac{L_r}{4F_r}\,.\tag{3.8}$$

Proof. To prove (3.5) and (3.6), set $z=2\pi/(F_r\sqrt5)$, $z=\pi/L_r$, in (3.3), in turn. To prove (3.7) and (3.8), set $z=2\pi/L_r$, $z=\pi/(F_r\sqrt5)$, in (3.4), in turn. Note the use of (7.1). □

Theorem 2. *If r and s are integers and z is a real or complex variable, then,*

$$\begin{aligned}\sum_{j=0}^{\infty}\frac{(-1)^j}{(2j)!}F_{2rj+s}z^{2j}&=F_s\cos\left(\frac{zL_r}{2}\right)\cos\left(\frac{F_rz\sqrt5}{2}\right)\\&\quad-\frac{L_s}{\sqrt5}\sin\left(\frac{zL_r}{2}\right)\sin\left(\frac{F_rz\sqrt5}{2}\right),\end{aligned}\tag{3.9}$$

$$\begin{aligned}\sum_{j=0}^{\infty}\frac{(-1)^j}{(2j)!}L_{2rj+s}z^{2j}&=L_s\cos\left(\frac{zL_r}{2}\right)\cos\left(\frac{F_rz\sqrt5}{2}\right)\\&\quad-F_s\sqrt5\sin\left(\frac{zL_r}{2}\right)\sin\left(\frac{F_rz\sqrt5}{2}\right).\end{aligned}\tag{3.10}$$

Proof. Consider the Maclaurin series for $\cos z$:

$$\sum_{j=0}^{\infty}(-1)^j\frac{z^{2j}}{2j!}=\cos z\,.$$

Use $f(j)=2j$, $g(j)=(-1)^j/(2j!)$, $c_1=0$, $c_2=\infty$ and $h(z)=\cos z$ in (F) and (L), noting the identities

$$\cos x+\cos y=2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right),$$

$$\cos x-\cos y=-2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right).$$

□

In particular,

$$\sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{(2j)!}F_{2rj}z^{2j}=\frac{2}{\sqrt5}\sin\left(\frac{zL_r}{2}\right)\sin\left(\frac{F_rz\sqrt5}{2}\right),\tag{3.11}$$

$$\sum_{j=0}^{\infty}\frac{(-1)^j}{(2j)!}L_{2rj}z^{2j}=2\cos\left(\frac{zL_r}{2}\right)\cos\left(\frac{F_rz\sqrt5}{2}\right).\tag{3.12}$$

Example 2. *If r is an integer, then,*

$$\sum_{j=1}^{\infty}\frac{\zeta(2j)}{B_{2j}}\frac{F_{2rj}}{L_r^{2j}}=0\,,\tag{3.13}$$

$$\sum_{j=1}^{\infty}\frac{\zeta(2j)}{B_{2j}}\frac{F_{2rj}}{F_r^{2j}5^j}=0\,,\tag{3.14}$$

$$\sum_{j=1}^{\infty}\frac{\zeta(2j)}{2^{2j}B_{2j}}\frac{L_{2rj}}{L_r^{2j}}=1\,,\tag{3.15}$$

$$\sum_{j=1}^{\infty}\frac{\zeta(2j)}{2^{2j}B_{2j}}\frac{L_{2rj}}{F_r^{2j}5^j}=1\,.\tag{3.16}$$

Proof. Set $z=2\pi/L_r$, $z=2\pi/(F_r\sqrt5)$ in (3.11) to prove (3.13) and (3.14). Set $z=\pi/L_r$, $z=\pi/(F_r\sqrt5)$ in (3.12) to prove (3.15) and (3.16). Note the use of (7.1) □

4 Fibonacci series from the inverse tangent function

Theorem 3. *Let r and s be integers. Let z be a real or complex variable such that $|z| \le 1$. Then,*

$$\begin{aligned}\sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{2j-1} F_{2rj+s} z^{2j-1} &= \frac{F_{s+r}}{2} \tan^{-1}\left(\frac{zL_r}{1-(-1)^r z^2}\right)\\ &\quad + \frac{L_{s+r}}{2\sqrt{5}} \tan^{-1}\left(\frac{F_r z\sqrt{5}}{1+(-1)^r z^2}\right),\end{aligned} \tag{4.1}$$

$$\begin{aligned}\sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{2j-1} L_{2rj+s} z^{2j-1} &= \frac{L_{s+r}}{2} \tan^{-1}\left(\frac{zL_r}{1-(-1)^r z^2}\right)\\ &\quad + \frac{F_{s+r}\sqrt{5}}{2} \tan^{-1}\left(\frac{F_r z\sqrt{5}}{1+(-1)^r z^2}\right).\end{aligned} \tag{4.2}$$

Proof. Consider the inverse tangent series

$$\sum_{j=1}^{\infty} \frac{(-1)^j z^{2j-1}}{2j-1} = \tan^{-1} z\,.$$

Use $f(j) = 2j-1$, $g(j) = (-1)^j/(2j-1)$, $c_1 = 1$, $c_2 = \infty$ and $h(z) = \tan^{-1} z$ in identities (F) and (L). Note the arctangent identities

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy}\,, \quad \text{if } xy < 1\,, \tag{4.3}$$

$$\tan^{-1} x - \tan^{-1} y = \tan^{-1} \frac{x-y}{1+xy}\,, \quad \text{if } xy > -1\,. \tag{4.4}$$

□

In particular, we have

$$\sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{2j-1} F_{r(2j-1)} z^{2j-1} = \frac{1}{\sqrt{5}} \tan^{-1}\left(\frac{F_r z\sqrt{5}}{1+(-1)^r z^2}\right), \tag{4.5}$$

$$\sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{2j-1} L_{r(2j-1)} z^{2j-1} = \tan^{-1}\left(\frac{zL_r}{1-(-1)^r z^2}\right). \tag{4.6}$$

Example 3. *We have*

$$\sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{2j-1} F_{2j-1} = \frac{\pi}{2\sqrt{5}}\,, \tag{4.7}$$

$$\sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{2j-1} L_{2(2j-1)} = \frac{\pi}{2}\,. \tag{4.8}$$

Example 4. *If r is a non-zero integer, then,*

$$\sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{2j-1} \frac{F_{r(2j-1)}}{F_r^{2j-1}} = \frac{1}{\sqrt{5}} \tan^{-1}\left(\frac{F_r^2\sqrt{5}}{F_{r-1}F_{r+1}}\right), \tag{4.9}$$

$$\sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{2j-1} \frac{L_{r(2j-1)}}{F_r^{2j-1}5^j} = \frac{1}{\sqrt{5}} \tan^{-1}\left(\frac{F_{2r}\sqrt{5}}{L_{r-1}L_{r+1}}\right). \tag{4.10}$$

Proof. Set $z = 1/F_r$ in (4.5) and $z = 1/(F_r\sqrt{5})$ in (4.6).

Note the use of Cassini's formulas:

$$F_{n-1}F_{n+1} = F_n^2 + (-1)^n, \tag{4.11}$$

$$L_{n-1}L_{n+1} = 5F_n^2 + (-1)^{n-1}. \tag{4.12}$$

□

5 Infinite series involving Fibonacci numbers and Bernoulli numbers

The Bernoulli numbers, B_j, are defined by the generating function

$$\frac{z}{e^z - 1} = \sum_{j=0}^{\infty} B_j \frac{z^j}{j!}, \quad z < 2\pi. \tag{5.1}$$

The first few Bernoulli numbers are

$$B_0 = 1,\ B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6},\ B_3 = 0,\ B_4 = -\frac{1}{30},\ B_5 = 0,\ B_6 = \frac{1}{42},\ B_7 = 0,\ \ldots. \tag{5.2}$$

Basic properties of the Bernoulli polynomials are highlighted in recent articles by Frontczak [3] and by Frontczak and Goy [7] where new identities involving Fibonacci and Bernoulli numbers, and Lucas and Euler numbers are presented. Additional information on Bernoulli polynomials can be found in Erdélyi et al [2, §1.13].

Theorem 4. *Let r and s be integers and z any real or complex variable such that* $z < 2\pi\alpha^{-r}$. *Then,*

$$\begin{aligned}\sum_{j=1}^{\infty} (-1)^j \frac{B_{2j}}{(2j)!} F_{2rj+s} z^{2j} &= \frac{zF_{r+s}}{4}\left(\cot\left(\frac{\alpha^r z}{2}\right) + \cot\left(\frac{\beta^r z}{2}\right)\right) - F_s \\ &\quad + \frac{zL_{r+s}}{4\sqrt{5}}\left(\cot\left(\frac{\alpha^r z}{2}\right) - \cot\left(\frac{\beta^r z}{2}\right)\right),\end{aligned} \tag{5.3}$$

$$\begin{aligned}\sum_{j=1}^{\infty} (-1)^j \frac{B_{2j}}{(2j)!} L_{2rj+s} z^{2j} &= \frac{zL_{r+s}}{4}\left(\cot\left(\frac{\alpha^r z}{2}\right) + \cot\left(\frac{\beta^r z}{2}\right)\right) - L_s \\ &\quad + \frac{zF_{r+s}\sqrt{5}}{4}\left(\cot\left(\frac{\alpha^r z}{2}\right) - \cot\left(\frac{\beta^r z}{2}\right)\right).\end{aligned} \tag{5.4}$$

Proof. Setting $x = iz$, z real, in the following identity [9, Formula **1.213**]:

$$\sum_{j=1}^{\infty} \frac{B_{2j}}{(2j)!} x^{2j} = \frac{x}{e^x - 1} + \frac{x}{2} - 1, \quad |x| < 2\pi\,,$$

and taking the real part, we have

$$\sum_{j=1}^{\infty} (-1)^j \frac{B_{2j}}{(2j)!} z^{2j} = \frac{z}{2} \cot\left(\frac{z}{2}\right) - 1\,.$$

Now use $f(j) = 2j$, $g(j) = B_{2j}/(2j)!$, $c_1 = 1$, $c_2 = \infty$ and $h(z) = z/2\cot(z/2) - 1$ in (F) and (L). The identities of Theorem 4 follow after some algebra, including also the use of identities (2.6). Note that in the final simplification, we used

$$F_r L_s + F_s L_r = 2F_{r+s}, \quad \text{Vajda [13, (16a)]}\,,$$

and

$$L_r L_s + 5F_r F_s = 2L_{r+s}, \quad \text{Vajda [13, (17a)+(17b)]}\,.$$

□

In particular, for integer r and $z < 2\pi\alpha^{-r}$, we have

$$\sum_{j=1}^{\infty} (-1)^j \frac{B_{2j}}{(2j)!} F_{r(2j-1)} z^{2j} = \frac{z}{2\sqrt{5}} \left(\cot\left(\frac{\alpha^r z}{2}\right) - \cot\left(\frac{\beta^r z}{2}\right)\right) + (-1)^r F_r\,, \tag{5.5}$$

$$\sum_{j=1}^{\infty} (-1)^j \frac{B_{2j}}{(2j)!} L_{r(2j-1)} z^{2j} = \frac{z}{2} \left(\cot\left(\frac{\alpha^r z}{2}\right) + \cot\left(\frac{\beta^r z}{2}\right)\right) - (-1)^r L_r\,. \tag{5.6}$$

Example 5. *Let r be an integer. Then,*

$$\sum_{j=1}^{\infty} (-1)^j \frac{B_{2j}}{(2j)!} \frac{F_{r(2j-1)}}{F_r^{2j} 5^j} (2\pi)^{2j} = (-1)^r F_r, \quad r \neq 0\,, \tag{5.7}$$

$$\sum_{j=1}^{\infty} (-1)^j \frac{B_{2j}}{(2j)!} \frac{L_{r(2j-1)}}{L_r^{2j}} (2\pi)^{2j} = (-1)^{r-1} L_r\,. \tag{5.8}$$

Proof. Set $z = 2\pi/(F_r\sqrt{5})$ in (5.5) and $z = 2\pi/L_r$ in (5.6). □

Note that, in view of identity (7.1), identities (5.7) and (5.8) can also be written as

$$\sum_{j=1}^{\infty} \frac{\zeta(2j)}{F_r^{2j-1} 5^j} F_{r(2j-1)} = \frac{(-1)^{r-1}}{2} F_r^2\,, \tag{5.9}$$

$$\sum_{j=1}^{\infty} \frac{\zeta(2j)}{L_r^{2j-1}} L_{r(2j-1)} = \frac{(-1)^r}{2} L_r^2\,, \tag{5.10}$$

which are the same identities (7.59) and (7.62) of Example 7.

6 Infinite series involving Fibonacci numbers and Euler numbers

The Euler numbers E_j are defined by the exponential generating function:

$$\frac{2}{e^z + e^{-z}} = \sum_{j=0}^{\infty} \frac{E_j}{j!} z^j .$$

The first few Euler numbers are

$$E_0 = 1,\ E_2 = -1,\ E_4 = 5,\ E_6 = -61,\ \ldots,\ \text{with } E_{2j+1} = 0 \text{ for } j \geq 0\,.$$

Theorem 5. *If r and s are integers and z is a real or complex variable, then,*

$$\begin{aligned}\sum_{j=0}^{\infty} \frac{|E_{2j}|}{(2j)!} F_{2rj+s} z^{2j} &= \frac{F_s}{2}\left(\sec(\alpha^r z) + \sec(\beta^r z)\right)\\ &\quad + \frac{L_s}{2\sqrt{5}}\left(\sec(\alpha^r z) - \sec(\beta^r z)\right)\,,\end{aligned} \tag{6.1}$$

$$\begin{aligned}\sum_{j=0}^{\infty} \frac{|E_{2j}|}{(2j)!} L_{2rj+s} z^{2j} &= \frac{L_s}{2}\left(\sec(\alpha^r z) + \sec(\beta^r z)\right)\\ &\quad + \frac{F_s\sqrt{5}}{2}\left(\sec(\alpha^r z) - \sec(\beta^r z)\right)\,.\end{aligned} \tag{6.2}$$

Proof. Consider the identity [9, Formula **1.411** 9.]

$$\sum_{j=0}^{\infty} \frac{|E_{2j}|}{(2j)!} z^{2j} = \sec z, \quad z^2 < \pi^2/4\,,$$

Use $f(j) = 2j$, $g(j) = |E_{2j}|/(2j)!$, $c_1 = 0$, $c_2 = \infty$ and $h(z) = \sec z$ in (F) and (L). □

In particular,

$$\sum_{j=0}^{\infty} \frac{|E_{2j}|}{(2j)!} F_{2rj} z^{2j} = \frac{1}{\sqrt{5}}\left(\sec(\alpha^r z) - \sec(\beta^r z)\right)\,, \tag{6.3}$$

$$\sum_{j=0}^{\infty} \frac{|E_{2j}|}{(2j)!} L_{2rj} z^{2j} = \sec(\alpha^r z) + \sec(\beta^r z)\,. \tag{6.4}$$

7 Infinite series involving Fibonacci numbers and the Riemann zeta function

As noted by Frontczak and Goy [8], studies in infinite series involving Fibonacci numbers and Riemann zeta numbers have not been previously documented. The narrative has changed, however, following research results by the aforementioned authors, as contained in their recent papers: Frontczak [4, 5, 6] and Frontczak and Goy [8]. In this section, we explore more infinite series involving the Fibonacci numbers and the Riemann zeta numbers.

The Riemann zeta function, $\zeta(n)$, $n \in \mathbb{C}$, defined by

$$\zeta(n) = \sum_{j=1}^{\infty} \frac{1}{j^n}, \quad \Re(n) > 1\,,$$

is analytically continued to all $n \in \mathbb{C}$ with $\Re(n) > 0$, $n \neq 1$ through

$$\zeta(n) = \frac{1}{1 - 2^{1-n}} \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j^n}\,.$$

For positive even arguments, the numbers $\zeta(2n)$ are directly related to the Bernoulli numbers, B_{2n}:

$$\zeta(2n) = (-1)^{n+1} \frac{(2\pi)^{2n}}{2(2n)!} B_{2n}\,. \tag{7.1}$$

No such simple formula is known for the zeta function at odd integer arguments.

More information on the Riemann zeta function can be found in the books by Edwards [1] and Srivastava and Choi [12].

The zeta number generating functions, found in Srivastava and Choi [12, p. 270–271, p. 280–281], also Erdélyi et al [2, §1.7.1], which we require to establish the infinite series here, are expressed in terms of the Gamma function and the digamma fumction.

The Gamma function is defined for $\Re(z) > 0$ by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt = \int_0^\infty (\log(1/t))^{z-1}\, dt\,,$$

and is extended to the rest of the complex plane, excluding the non-positive integers, by analytic continuation. The Gamma function has a simple pole at each of the points $z = \cdots, -3, -2, -1, 0$. The Gamma function extends the classical factorial function to the complex plane: $\Gamma(z) = (z-1)!$.

The digamma function, $\psi(z)$, is the logarithmic derivative of the Gamma function:

$$\psi(z) = \frac{d}{dz} \log \Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}\,.$$

7.1 Functional equations for the Gamma and the digamma fumction

Here is a list of basic functional equations for the gamma function (see Erdélyi et al [2, §1.2]):

$$\Gamma(z+1) = z\Gamma(z)\,, \tag{7.2}$$

$$\Gamma(z)\Gamma(-z) = -\pi \csc(\pi z)/z\,, \tag{7.3}$$

$$\Gamma(z)\Gamma(1-z) = \pi \csc(\pi z)\,, \tag{7.4}$$

$$\Gamma(1/2+z)\Gamma(1/2-z) = \pi \sec(\pi z)\,, \tag{7.5}$$

$$\Gamma(1+z)\Gamma(1-z) = \pi z \csc(\pi z)\,. \tag{7.6}$$

Writing $-x$ for z and $-y$ for z, in turn, in (7.2), we find

$$\Gamma(1-x)\Gamma(1-y) = xy\Gamma(-x)\Gamma(-y)\,. \tag{7.7}$$

More functional equations that are required for our discussion will now be derived.

A consequence of (7.3) is

$$\Gamma(-x)\Gamma(-y) = \frac{2\pi^2}{xy} \frac{1}{\Gamma(x)\Gamma(y)} \frac{1}{\cos(x-y)-\cos(x+y)}\,, \tag{7.8}$$

so that

$$\begin{aligned} \log\left(\Gamma(-x)\Gamma(-y)\right) = \log(2\pi^2/(xy)) - \log\left(\Gamma(x)\Gamma(y)\right) \\ + \log\left(\cos(x+y)-\cos(x-y)\right)\,. \end{aligned} \tag{7.9}$$

From (7.6), it follows that if $x+y=1$, then,

$$\Gamma\left(\frac{1}{2}+x\right)\Gamma\left(\frac{1}{2}+y\right) = \left(\frac{1}{2}-x\right)\pi \sec(\pi x)\,. \tag{7.10}$$

We require the following basic properties of the digamma function:

$$\psi(z+1) = \psi(z) + \frac{1}{z}\,, \tag{7.11}$$

$$\psi(z) - \psi(-z) = -\pi \cot \pi z - \frac{1}{z}\,, \tag{7.12}$$

$$\psi(1+z) - \psi(1-z) = \frac{1}{z} - \pi \cot \pi z\,, \tag{7.13}$$

$$\psi(z) - \psi(1-z) = -\pi \cot \pi z\,, \tag{7.14}$$

$$\psi\left(\frac{1}{2}+z\right) - \psi\left(\frac{1}{2}-z\right) = \pi \tan \pi z\,, \tag{7.15}$$

$$\psi(z+n) = \psi(z) + \sum_{j=1}^{n} \frac{1}{z+j-1}\,, \tag{7.16}$$

$$\psi(mz) = \log m + \frac{1}{m}\sum_{j=0}^{m-1} \psi\left(z+\frac{j}{m}\right)\,. \tag{7.17}$$

As a consequence of identity (7.12) we have the following useful identity:

$$\psi(-x) - \psi(-y) = \psi(x) - \psi(y) + \pi(\cot(\pi x) - \cot(\pi y)) - \frac{x-y}{xy}\,. \tag{7.18}$$

We observe from relation (7.11) that if $x - y = 1$, then,

$$\psi(x) - \psi(y) = \frac{1}{y}, \quad y \neq 0\,, \tag{7.19}$$

while relation (7.14) is equivalent to saying that if $x + y = 1$, then,

$$.\psi(x) - \psi(y) = -\pi\cot(\pi x) \tag{7.20}$$

From relation (7.13), we have

$$\begin{aligned}\psi(1+x) - \psi(1+y) &= \psi(1-x) - \psi(1-y)\\ &\quad + \frac{1}{x} - \frac{1}{y} - \pi(\cot(\pi x) - \cot(\pi y))\,.\end{aligned} \tag{7.21}$$

Thus, if $x + y = 1$, $x \notin \mathbb{Z}$, then,

$$\psi(1+x) - \psi(1+y) = -\pi\cot(\pi x) + \frac{1}{x} - \frac{1}{1-x}, \quad x \notin \{0,1\}\,, \tag{7.22}$$

while if $x + y = 2$, $x \notin \mathbb{Z}$, we have

$$\psi(1+x) - \psi(1+y) = -\frac{1}{1-x} + \frac{1}{x} - \frac{1}{2-x} - \pi\cot(\pi x), \quad x \notin \{0,1,2\}\,. \tag{7.23}$$

If $x + y = 1$, identity (7.15) also implies

$$\psi(1/2+x) - \psi(1/2+y) = -\pi\tan(\pi x) + \frac{2}{2x-1}, \quad x \neq 1/2\,, \tag{7.24}$$

while if $x - y = 1$, we have

$$\psi(1/2+x) - \psi(1/2+y) = \frac{2}{2x-1}, \quad x \neq 1/2\,. \tag{7.25}$$

Writing $z - 1$ for z in (7.13) gives

$$\psi(2-z) = \psi(z) + \pi\cot(\pi(z-1)) - \frac{1}{z-1}\,, \tag{7.26}$$

and hence, also

$$\psi(2+z) = \psi(-z) - \pi\cot\pi(z+1) + \frac{1}{z+1}\,. \tag{7.27}$$

From (7.26) we get

$$\begin{aligned}\psi(2-x) - \psi(2-y) &= \psi(x) - \psi(y) - \frac{2\pi\sin(\pi(x-y))}{\cos(\pi(x-y)) - \cos(\pi(x+y))}\\ &\quad + \frac{x-y}{1-x-y+xy}\,,\end{aligned} \tag{7.28}$$

which, writing x for $-x$ and y for $-y$ and makng use of (7.18), also gives

$$\psi(2+x) - \psi(2+y) = \psi(x) - \psi(y) - \frac{x-y}{xy} - \frac{x-y}{1+x+y+xy}\,. \tag{7.29}$$

Using $m=2$ in (7.17) gives

$$\psi(2z) = \log 2 + \frac{1}{2}\psi(z) + \frac{1}{2}\psi\left(z+\frac{1}{2}\right), \tag{7.30}$$

which also means that,

$$\begin{aligned}\psi(2x) - \psi(2y) &= \frac{1}{2}\left(\psi(x) - \psi(y)\right)\\ &\quad + \frac{1}{2}\left(\psi\left(x+\frac{1}{2}\right) - \psi\left(y+\frac{1}{2}\right)\right).\end{aligned} \tag{7.31}$$

If $x+y=1$, then relation (7.31), in view of (7.14) and (7.15), gives

$$\psi(2x) - \psi(2y) = -\pi\cot(2\pi x) + \frac{1}{2x-1}; \tag{7.32}$$

while if $x-y=1$, it produces

$$\psi(2x) - \psi(2y) = \frac{1}{2(x-1)} - \frac{1}{2x-1}. \tag{7.33}$$

7.2 Evaluations at the relevant arguments

Lemma 1. *We have*

$$\Gamma(\alpha)\Gamma(\beta) = \pi\csc(\pi\alpha) = \pi\csc(\pi\beta), \tag{7.34}$$

$$\Gamma(\alpha^2)\Gamma(\beta^2) = -\pi\csc(\pi\beta), \tag{7.35}$$

$$\Gamma(-\alpha)\Gamma(-\beta) = -\pi\csc(\pi\beta), \tag{7.36}$$

$$\Gamma(-\alpha^2)\Gamma(-\beta^2) = -\pi\csc(\pi\beta), \tag{7.37}$$

$$\Gamma(\alpha^3/2)\Gamma(\beta^3/2) = \frac{\pi\sqrt{5}}{2}\sec(\pi\beta), \tag{7.38}$$

$$\Gamma(\alpha^r/L_r)\Gamma(\beta^r/L_r) = \pi\csc(\pi\alpha^r/L_r), \tag{7.39}$$

$$\Gamma(-\alpha^r/L_r)\Gamma(-\beta^r/L_r) = (-1)^r\pi L_r^2\csc(\pi\alpha^r/L_r). \tag{7.40}$$

Proof. Setting $z=\alpha$ in (7.4) gives (7.34). Use of $x=-\alpha$, $y=-\beta$ in (7.7) gives (7.35). In view of (7.7), we have (7.36) and (7.37). Identity (7.38) is proved by setting $x=\beta$ in identity (7.10). To prove (7.39), set $z=\alpha^r/L_r$ in (7.4). □

Lemma 2. *We have*

$$\psi(\alpha) - \psi(\beta) = -\pi\cot\pi\alpha, \tag{7.41}$$

$$\psi(\alpha^2) - \psi(\beta^2) = -\pi\cot\pi\alpha + \sqrt{5}, \tag{7.42}$$

$$\psi(2\alpha) - \psi(2\beta) = \frac{1}{\sqrt{5}} - \pi \cot \pi\sqrt{5}\,, \tag{7.43}$$

$$\psi(\alpha^3) - \psi(\beta^3) = \frac{7\sqrt{5}}{10} - \pi \cot \pi\sqrt{5}\,, \tag{7.44}$$

$$\psi\left(\frac{\alpha^3}{2}\right) - \psi\left(\frac{\beta^3}{2}\right) = \frac{2}{\sqrt{5}} - \pi \cot \frac{\pi\sqrt{5}}{2}\,, \tag{7.45}$$

$$\psi(\alpha\sqrt{5}) - \psi(\beta\sqrt{5}) = -\frac{\alpha}{\sqrt{5}}\,, \tag{7.46}$$

$$\psi\left(\frac{\alpha^r}{L_r}\right) - \psi\left(\frac{\beta^r}{L_r}\right) = -\pi \cot\left(\frac{\pi\alpha^r}{L_r}\right) = \tan\left(\frac{\pi}{2}\frac{F_r\sqrt{5}}{L_r}\right)\,, \tag{7.47}$$

$$\psi\left(\frac{\alpha^r}{F_r\sqrt{5}}\right) - \psi\left(\frac{\beta^r}{F_r\sqrt{5}}\right) = \frac{F_r\sqrt{5}}{\beta^r}\,, \tag{7.48}$$

$$\psi\left(\frac{-\alpha^r}{L_r}\right) - \psi\left(\frac{-\beta^r}{L_r}\right) = -\tan\left(\frac{\pi}{2}\frac{F_r\sqrt{5}}{L_r}\right) - (-1)^r F_{2r}\sqrt{5}\,, \tag{7.49}$$

$$\psi\left(-\frac{\alpha^r}{F_r\sqrt{5}}\right) - \psi\left(-\frac{\beta^r}{F_r\sqrt{5}}\right) = \frac{F_r\sqrt{5}}{\beta^r} - (-1)^r 5F_r^2\,. \tag{7.50}$$

Proof. Setting $z = \alpha$ in (7.14) proves (7.41). Identity (7.43) comes from choosing $x = \alpha$, $y = \beta$ in (7.32). Setting $x = \alpha$, $y = \beta$ in (7.22) proves (7.42). Use of $x = 2\alpha$, $y = 2\beta$ in (7.23) produces (7.44). Setting $(x, y) = (\alpha, \beta)$ in (7.24) gives (7.45). Identities (7.47) and (7.48) are proved be setting $(x, y) = (\alpha^r/L_r, \beta^r/L_r)$ in (7.19) and $(x, y) = (\alpha^r/(F_r\sqrt{5}), \beta^r/(F_r\sqrt{5}))$ in (7.20). Identities (7.49) and (7.50) are obtained from (7.47) and (7.48) wth the aid of (7.18). □

7.3 Fibonacci-Zeta infinite series

Theorem 6. *Let r be an integer. Let z be a real or complex variable such that $|z| < \alpha^{-r}$. Then,*

$$\begin{aligned}\sum_{j=1}^{\infty} \zeta(j+1) F_{rj} z^j = &-\frac{1}{\sqrt{5}}\left(\psi(\alpha^r z) - \psi(\beta^r z)\right) \\ &- \frac{\pi}{\sqrt{5}}\left(\cot(\pi z\alpha^r) - \cot(\pi z\beta^r)\right)\,.\end{aligned}$$

Proof. Consider [12, p. 270, identity (13)]:

$$\sum_{j=1}^{\infty} \zeta(j+1) z^j = -\psi(1-z) - \gamma, \quad |z| < 1\,.$$

Use $f(j) = j$, $g(j) = \zeta(j+1)$, $c_1 = 1$, $c_2 = \infty$ and $h(z) = -\psi(1-z) - \gamma$ in identity (F1). □

Example 6. *If r is an integer with $|r|>1$, then*

$$\sum_{j=1}^{\infty}\frac{\zeta(j+1)}{L_r^j}F_{rj}=\frac{\pi}{\sqrt{5}}\tan\left(\frac{\pi}{2}\frac{F_r\sqrt{5}}{L_r}\right)\,, \tag{7.51}$$

$$\sum_{j=1}^{\infty}(-1)^{j-1}\frac{\zeta(j+1)}{L_r^j}F_{rj}=\frac{\pi}{\sqrt{5}}\tan\left(\frac{\pi}{2}\frac{F_r\sqrt{5}}{L_r}\right)-(-1)^rF_{2r}\,. \tag{7.52}$$

Proof. Set $z=\pm 1/L_r$ in the identity of Theorem 6 and use (7.47) and (7.49). □

Theorem 7. *Let r and s be integers. Let z be a real or complex variable such that $|z|<\alpha^{-r}$. Then,*

$$\begin{split}\sum_{j=1}^{\infty}\frac{\zeta(2j)}{2j}F_{2rj+s}z^{2j}&=\frac{F_s}{4}\log\left(\frac{(-1)^r2\pi^2z^2}{\cos(\pi zF_r\sqrt{5})-\cos(\pi zL_r)}\right)\\&\quad+\frac{L_s}{4\sqrt{5}}\log\left((-1)^r\alpha^{2r}\frac{\sin\pi\beta^rz}{\sin\pi\alpha^rz}\right)\,,\end{split} \tag{7.53}$$

$$\begin{split}\sum_{j=1}^{\infty}\frac{\zeta(2j)}{2j}L_{2rj+s}z^{2j}&=\frac{L_s}{4}\log\left(\frac{(-1)^r2\pi^2z^2}{\cos(\pi zF_r\sqrt{5})-\cos(\pi zL_r)}\right)\\&\quad+\frac{F_s\sqrt{5}}{4}\log\left((-1)^r\alpha^{2r}\frac{\sin\pi\beta^rz}{\sin\pi\alpha^rz}\right)\,.\end{split} \tag{7.54}$$

Proof. Consider [12, p. 271, identity (17)]:

$$\sum_{j=1}^{\infty}\zeta(2j)\frac{z^{2j}}{j}=\log\left(\frac{\pi z}{\sin\pi z}\right),\quad |z|<1\,.$$

Use $f(j)=2j$, $g(j)=\zeta(2j)/j$, $c_1=1$, $c_2=\infty$ and $h(z)=\log(\pi z/\sin(\pi z))$ in identities (F) and (L) . □

Theorem 8. *Let r and s be integers. Let z be a real or complex variable such that $|z|<\alpha^{-r}$. Then,*

$$\begin{split}\sum_{j=1}^{\infty}\zeta(2j)F_{2rj+s}z^{2j-1}&=\frac{F_{s+r}}{2}\left(-\frac{\pi\sin(\pi zL_r)}{\cos(\pi zF_r\sqrt{5})-\cos(\pi zL_r)}+\frac{(-1)^rL_r}{2z}\right)\\&\quad+\frac{L_{s+r}}{2\sqrt{5}}\left(\frac{\pi\sin(\pi zF_r\sqrt{5})}{\cos(\pi zF_r\sqrt{5})-\cos(\pi zL_r)}-\frac{(-1)^rF_r\sqrt{5}}{2z}\right)\,,\end{split} \tag{7.55}$$

$$\begin{split}\sum_{j=1}^{\infty}\zeta(2j)L_{2rj+s}z^{2j-1}&=\frac{L_{s+r}}{2}\left(-\frac{\pi\sin(\pi zL_r)}{\cos(\pi zF_r\sqrt{5})-\cos(\pi zL_r)}+\frac{(-1)^rL_r}{2z}\right)\\&\quad+\frac{F_{s+r}\sqrt{5}}{2}\left(\frac{\pi\sin(\pi zF_r\sqrt{5})}{\cos(\pi zF_r\sqrt{5})-\cos(\pi zL_r)}-\frac{(-1)^rF_r\sqrt{5}}{2z}\right)\,.\end{split} \tag{7.56}$$

Proof. Consider [12, p. 271, identity (18)]:

$$\sum_{j=1}^{\infty}\zeta(2j)z^{2j-1}=-\frac{\pi}{2}\cot\pi z+\frac{1}{2z},\quad |z|<1\,.$$

Use $f(j)=2j-1$, $g(j)=\zeta(2j)$, $c_1=1$, $c_2=\infty$ and $h(z)=-(\pi/2)\cot(\pi z)+1/(2z)$ in identities (F) and (L). □

In particular, setting $s=-r$ in the identities of Theorem 8 yields

$$\sum_{j=1}^{\infty}\zeta(2j)F_{r(2j-1)}z^{2j-1}=\frac{1}{\sqrt5}\left(\frac{\pi\sin(\pi zF_r\sqrt5)}{\cos(\pi zF_r\sqrt5)-\cos(\pi zL_r)}-\frac{(-1)^rF_r\sqrt5}{2z}\right),\tag{7.57}$$

$$\sum_{j=1}^{\infty}\zeta(2j)L_{r(2j-1)}z^{2j-1}=-\frac{\pi\sin(\pi zL_r)}{\cos(\pi zF_r\sqrt5)-\cos(\pi zL_r)}+\frac{(-1)^rL_r}{2z}\,.\tag{7.58}$$

Example 7. *If r is an integer, then,*

$$\sum_{j=1}^{\infty}\frac{\zeta(2j)}{F_r^{2j-1}5^j}F_{r(2j-1)}=\frac{(-1)^{r-1}}{2}F_r^2\,,\tag{7.59}$$

$$\sum_{j=1}^{\infty}\frac{\zeta(2j)}{F_r^{2j-1}5^j}L_{r(2j-1)}=\frac{\pi}{\sqrt5}\tan\left(\frac{\pi}{2}\frac{L_r}{F_r\sqrt5}\right)+\frac{(-1)^r}{2}F_{2r}\,,\tag{7.60}$$

$$\sum_{j=1}^{\infty}\frac{\zeta(2j)}{L_r^{2j-1}}F_{r(2j-1)}=\frac{\pi}{\sqrt5}\tan\left(\frac{\pi}{2}\frac{F_r\sqrt5}{L_r}\right)-\frac{(-1)^r}{2}F_{2r},\quad |r|>1\,,\tag{7.61}$$

$$\sum_{j=1}^{\infty}\frac{\zeta(2j)}{L_r^{2j-1}}L_{r(2j-1)}=\frac{(-1)^r}{2}L_r^2\,.\tag{7.62}$$

Proof. Set $z=1/(F_r\sqrt5)$ in (7.57) and in (7.58) to obtain (7.59) and (7.60). Set $z=1/L_r$ in (7.57) and in (7.58) to obtain (7.61) and (7.62). □

Frontczak [4, Theorem 2.1] also obtained the special case $r=1$ of identities (7.59) and (7.60).

Theorem 9. *Let r be an integer and z any real or complex variable such that $|z|<2\alpha^{-r}$. Then,*

$$\sum_{j=1}^{\infty}(\zeta(j+1)-1)F_{rj}z^j=-\frac{1}{\sqrt5}\left(\psi(\alpha^rz)-\psi(\beta^rz)\right)-\frac{\pi}{\sqrt5}\left(\cot(\pi z\alpha^r)-\cot(\pi z\beta^r)\right)$$
$$-\frac{F_rz}{(-1)^rz^2-L_rz+1}\,.$$

Proof. Consider [12, p. 280, identity (146)]:

$$\sum_{j=1}^{\infty}(\zeta(j+1)-1)z^j=-\psi(2-z)+1-\gamma,\quad |z|<2\,.$$

Use $f(j)=j$, $g(j)=\zeta(j+1)-1$, $c_1=1$, $c_2=\infty$ and $h(z)=-\psi(2-z)+1-\gamma$ in identity (F1). □

Example 8. *If r is an integer, then*

$$\sum_{j=1}^{\infty} \frac{(\zeta(j+1)-1)}{L_r^j} F_{rj} = \frac{\pi}{\sqrt{5}} \tan\left(\frac{\pi}{2} \frac{F_r\sqrt{5}}{L_r}\right) - (-1)^r F_{2r}\,.$$

Proof. Set $z = 1/L_r$ in the identity of Theorem 9 and use (7.47). □

Frontczak [4, Theorem 2.2, identity (2.3)] obtained the $r = 1$ case of Example 8.

Theorem 10. *Let r be an integer and z any real or complex variable such that $|z| < 2\alpha^{-r}$. Then,*

$$\begin{aligned}
&\sum_{j=1}^{\infty} (\zeta(2j+1)-1) F_{2rj} z^{2j} \\
&\qquad = -\frac{1}{\sqrt{5}} \left(\psi(\alpha^r z) - \psi(\beta^r z)\right) - \frac{\pi}{2\sqrt{5}} \left(\cot(\pi z \alpha^r) - \cot(\pi z \beta^r)\right) \\
&\qquad\qquad - \frac{F_{2r} z^2}{(1+(-1)^r z^2)^2 - L_r^2 z^2} + \frac{1}{2z} (-1)^r F_r\,.
\end{aligned}$$

Proof. Consider [12, p. 280, identity (149)]:

$$\sum_{j=1}^{\infty} (\zeta(2j+1)-1) z^{2j} = -\frac{1}{2} \left(\psi(2+z) + \psi(2-z)\right) + 1 - \gamma, \quad |z| < 2\,.$$

Use $f(j) = 2j$, $g(j) = \zeta(2j+1)-1$, $c_1 = 1$, $c_2 = \infty$ and $h(z) = -\psi(2+z)/2 - \psi(2-z)/2 + 1 - \gamma$ in identity (F1).

Note that we used

$$\begin{aligned}
&\psi(2-x) - \psi(2-y) + \psi(2+x) - \psi(2+y) \\
&\qquad = 2\psi(x) - 2\psi(y) - \frac{2\pi \sin(\pi(x-y))}{\cos(\pi(x-y)) - \cos(\pi(x+y))} \\
&\qquad\qquad + \frac{2(x^2-y^2)}{(1+xy)^2 - (x+y)^2} - \frac{x-y}{xy}\,,
\end{aligned} \tag{7.63}$$

which follows from identities (7.28) and (7.29). □

Example 9. *If r is an integer, then,*

$$\sum_{j=1}^{\infty} \frac{(\zeta(2j+1)-1)}{L_r^{2j}} F_{2rj} = -\frac{L_r^2 F_{2r}}{2(-1)^r L_r^2 + 1} + \frac{(-1)^r}{2} F_{2r}\,, \tag{7.64}$$

$$\sum_{j=1}^{\infty} \frac{(\zeta(2j+1)-1)}{5^j F_r^{2j}} F_{2rj} = -\frac{5F_r^2 F_{2r}}{10(-1)^{r-1} F_r^2 + 1} + \frac{(-1)^{r-1}}{2} F_{2r}\,. \tag{7.65}$$

Proof. Set $z = 1/L_r$, $z = 1/(F_r\sqrt{5})$ in the identity of Theorem 10 and use identities (7.47) and (7.48). We used

$$\cot\left(\frac{\pi\alpha^r}{L_r}\right) + \cot\left(\frac{\pi\beta^r}{L_r}\right) = 0$$

and

$$\cot\left(\frac{\pi\alpha^r}{F_r\sqrt{5}}\right) - \cot\left(\frac{\pi\beta^r}{F_r\sqrt{5}}\right) = 0\,.$$

□

Theorem 11. *Let r be an integer and z any real or complex variable such that $|z| < 2\alpha^{-r}$. Then,*

$$\begin{aligned}\sum_{j=1}^{\infty}(\zeta(2j)-1)\frac{F_{2rj+s}}{j}z^{2j} &= \frac{F_s}{2}\log\left(\frac{(-1)^r 2\pi^2 z^2(1-L_{2r}z^2+z^4)}{\cos(\pi F_r z\sqrt{5})-\cos(\pi L_r z)}\right)\\ &\quad+\frac{L_s}{2\sqrt{5}}\log\left(\frac{\alpha^r-\alpha^{3r}z^2}{\beta^r-\beta^{3r}z^2}\ \frac{\sin(\pi\beta^r z)}{\sin(\pi\alpha^r z)}\right),\end{aligned} \tag{7.66}$$

$$\begin{aligned}\sum_{j=1}^{\infty}(\zeta(2j)-1)\frac{L_{2rj+s}}{j}z^{2j} &= \frac{L_s}{2}\log\left(\frac{(-1)^r 2\pi^2 z^2(1-L_{2r}z^2+z^4)}{\cos(\pi F_r z\sqrt{5})-\cos(\pi L_r z)}\right)\\ &\quad+\frac{F_s\sqrt{5}}{2}\log\left(\frac{\alpha^r-\alpha^{3r}z^2}{\beta^r-\beta^{3r}z^2}\ \frac{\sin(\pi\beta^r z)}{\sin(\pi\alpha^r z)}\right).\end{aligned} \tag{7.67}$$

Proof. Consider [12, p. 281, identity (150)]:

$$\sum_{j=1}^{\infty}(\zeta(2j)-1)\frac{z^{2j}}{j}=\log\left(\frac{\pi z(1-z^2)}{\sin\pi z}\right),\quad |z|<2\,.$$

Use $f(j)=2j$, $g(j)=\zeta(2j)-1$, $c_1=1$, $c_2=\infty$ and $h(z)=\log(\pi z(1-z^2)/\sin(\pi z))$ in identities (F) and (L) . □

Theorem 12. *Let r be an integer and z any real or complex variable such that $|z| < 2\alpha^{-r}$. Then,*

$$\begin{aligned}&\sum_{j=1}^{\infty}(\zeta(2j)-1)F_{2rj+s}z^{2j-1}\\ &\quad=\frac{F_{s+r}}{2}\left(-\frac{\pi\sin(\pi z L_r)}{\cos(\pi F_r z\sqrt{5})-\cos(\pi L_r z)}+\frac{(-1)^r L_r}{2z}\ \frac{3z^4-L_r^2z^2+1}{z^4-L_{2r}z^2+1}\right)\\ &\qquad+\frac{L_{s+r}}{2\sqrt{5}}\left(\frac{\pi\sin(\pi z F_r\sqrt{5})}{\cos(\pi F_r z\sqrt{5})-\cos(\pi L_r z)}-\frac{(-1)^r F_r\sqrt{5}}{2z}\ \frac{3z^4-5F_r^2z^2+1}{z^4-L_{2r}z^2+1}\right),\end{aligned} \tag{7.68}$$

$$\begin{aligned}&\sum_{j=1}^{\infty}(\zeta(2j)-1)L_{2rj+s}z^{2j-1}\\ &\quad=\frac{L_{s+r}}{2}\left(-\frac{\pi\sin(\pi z L_r)}{\cos(\pi F_r z\sqrt{5})-\cos(\pi L_r z)}+\frac{(-1)^r L_r}{2z}\ \frac{3z^4-L_r^2z^2+1}{z^4-L_{2r}z^2+1}\right)\\ &\qquad+\frac{F_{s+r}\sqrt{5}}{2}\left(\frac{\pi\sin(\pi z F_r\sqrt{5})}{\cos(\pi F_r z\sqrt{5})-\cos(\pi L_r z)}-\frac{(-1)^r F_r\sqrt{5}}{2z}\ \frac{3z^4-5F_r^2z^2+1}{z^4-L_{2r}z^2+1}\right).\end{aligned} \tag{7.69}$$

Proof. Consider [12, p. 281, identity (151)]:

$$\sum_{j=1}^{\infty}(\zeta(2j)-1)z^{2j-1}=-\frac{\pi}{2}\cot\pi z+\frac{3z^2-1}{2z(z^2-1)},\quad |z|<2\,.$$

Use $f(j) = 2j-1$, $g(j) = \zeta(2j)-1$, $c_1 = 1$, $c_2 = \infty$ and $h(z) = -\pi\cot(\pi z)/2 + (3z^2-1)/(2z(z^2-1))$ in identities (F) and (L).

Note that we used (Vajda [13, Formula (17c)])

$$L_{2r} + (-1)^r 2 = L_r^2 \tag{7.70}$$

and (Vajda [13, Formula (23)])

$$L_{2r} - (-1)^r 2 = 5F_r^2\,. \tag{7.71}$$

□

Theorem 13. *Let r be an integer and z any real or complex variable such that $|z| < \alpha^{-r}$. Then,*

$$\sum_{j=2}^{\infty} \frac{\zeta(j)}{j} L_{rj} z^j = \log\left(\Gamma(1-\alpha^r z)\Gamma(1-\beta^r z)\right) - \gamma z L_r\,.$$

Proof. Consider [12, p. 270, identity (9)]:

$$\sum_{j=2}^{\infty} \zeta(j)\frac{z^j}{j} = \log\Gamma(1-z) - \gamma z, \quad |z| < 1\,.$$

Use $f(j) = j$, $g(j) = \zeta(j)$, $c_1 = 2$, $c_2 = \infty$ and $h(z) = \log\Gamma(1-z) - \gamma z$ in identity (L1). □

Corollary 14. *If r is an even integer and z is a real variable such that $|z| < \alpha^{-r}$, then,*

$$\sum_{j=2}^{\infty} \frac{\zeta(j)}{j} L_{rj} z^j = 2\log|z| + \log\left(\Gamma(-\alpha^r z)\Gamma(-\beta^r z)\right) - \gamma z L_r\,.$$

Example 10. *We have*

$$\sum_{j=2}^{\infty} (-1)^j \frac{\zeta(j)}{j} L_j = \log(-\pi\csc(\pi\beta)) + \gamma\,.$$

Proof. Set $z = -1$, $r = 1$ in the identity of Theorem 13 and use identity (7.35). □

Example 11. *We have*

$$\sum_{j=2}^{\infty} (-1)^j \frac{\zeta(j)}{2^j j} L_{3j} = \log\left(-\frac{\pi\sqrt{5}}{8}\sec(\pi\beta)\right) + 2\gamma\,.$$

Proof. Set $z = -1/2$, $r = 3$ in the identity of Theorem 13 to obtain

$$\sum_{j=2}^{\infty} (-1)^j \frac{\zeta(j)}{2^j j} L_{3j} = \log\left(\Gamma\left(1+\frac{\alpha^3}{2}\right)\Gamma\left(1+\frac{\beta^3}{2}\right)\right) + 2\gamma\,.$$

Now, by (7.7), we have

$$\Gamma\left(1+\frac{\alpha^3}{2}\right)\Gamma\left(1+\frac{\beta^3}{2}\right) = \frac{\alpha^3}{2}\frac{\beta^3}{2}\Gamma\left(\frac{\alpha^3}{2}\right)\Gamma\left(\frac{\beta^3}{2}\right) = -\frac{\pi\sqrt{5}}{8}\sec(\pi\beta)\,,$$

by (7.38). □

Example 12. *If r is an even integer, then*

$$\sum_{j=2}^{\infty} \frac{\zeta(j)}{L_r^j j} L_{rj} = \log\left(\pi \csc(\pi\alpha^r/L_r)\right) - \gamma\,, \tag{7.72}$$

$$\sum_{j=2}^{\infty} (-1)^j \frac{\zeta(j)}{L_r^j j} L_{rj} = -2\log|L_r| + \log\left(\pi \csc(\pi\alpha^r/L_r)\right) + \gamma\,. \tag{7.73}$$

Proof. Set $z = \pm 1/L_r$ in the identity of Corollary 14 and use identities (7.39) and (7.40). □

Theorem 15. *Let r be an integer and z any real or complex variable such that $|z| < \alpha^{-r}$. Then,*

$$\begin{aligned}\sum_{j=1}^{\infty} \frac{\zeta(2j+1)}{2j+1} L_{r(2j+1)} z^{2j+1} &= \log\left(\Gamma(1-\alpha^r z)\Gamma(1-\beta^r z)\right) - \gamma z L_r \\ &\quad - \frac{1}{2}\log\left(\frac{(-1)^r 2\pi^2 z^2}{\cos(\pi z F_r\sqrt{5}) - \cos(\pi z L_r)}\right).\end{aligned}$$

Proof. Let

$$t(j) = \frac{\zeta(j+1)}{j+1} L_{r(j+1)} z^{j+1}\,.$$

Then,

$$t(2j) = \frac{\zeta(2j+1)}{2j+1} L_{r(2j+1)} z^{2j+1}, \quad t(2j-1) = \frac{\zeta(2j)}{2j} L_{r(2j)} z^{2j}\,.$$

Use these in the summation formula

$$\sum_{j=1}^{\infty} t(2j) = \sum_{j=1}^{\infty} t(j) - \sum_{j=1}^{\infty} t(2j-1)\,,$$

while taking note of Theorem 7 (identity (7.54)) and Theorem 13. □

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