On the zeros of the Riemann zeta function

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Abstract. Zeros and the pole of the Riemann zeta function $\zeta(s)$ correspond to simple poles of the logarithmic derivative $f(s) = \frac{d}{ds} \ln \zeta(s)$. In $Re\{s\} > 1$ the function $f(s)$ has an absolutely convergent sum expression $f(s) = \sum_{j=1}^{\infty} h_j(s)$ where $h_j(s) = h_1(js)$ and $h_1(s) = -\sum_{m=1}^{\infty} \ln(p_m) p_m^{-s}$, a sum over all primes $p_m > 1$. When the Taylor series of $f(s)$ is evaluated at a point $(l, 0)$, $l >> 1$, the absolute values of the coefficients of the Taylor series decrease in a negatively exponential manner when l increases. The function $f(s)$ has simple poles in the area $Re\{s\} < 1$. The pole gives the function $r/(s - s_k)$, which can be evaluated into a Taylor series at $(l, 0)$. The coefficients of the Taylor series of the pole decrease as l^{-1} as a function of l. This implies that in the sum of all poles of $f(s)$ poles must cancel other poles so that the negatively exponential behavior of the coefficients of the Taylor series dominates. The function of $x = l^{-1}$ arising from the pole $-1/(s-1)$ at $s = 1$ is $-x/(1-x)$. The poles of $f(s)$ at even negative integers give the function $-xC$. These two negative functions cannot cancel poles s_k that are on the x-axis and $0 < s_k < 1$. Thus, such poles do not exist. Pole pairs s_k, s_k^* give the function $x + x/(1-x)$ that cancels the sum $-xC - x/(1-x)$ when $C = 1$ if only if every pole s_k has $Re\{s\} = \frac{1}{2}$. The convergence of the coefficient of every power $i > 0$ of x larger to zero at least as $O(x)$ is shown possible for this solution.

Key words: Riemann zeta function, Riemann Hypothesis, Number Theore.

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1 Definitions

The Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \tag{1}$$

where s is a complex number. The zeta function can be continued analytically to the whole complex plane except for $s = 1$ where the function has a simple pole. The zeta function has trivial zeros at even negative integers. It does not have zeros in $Re\{s\} \geq 1$. The nontrivial zeros lie in the strip $0 < x < 1$. Let

$$P = \{p_1, p_2, \ldots | p_j \text{ is a prime}, p_{j+1} > p_j > 1, j \geq 1\}$$

be the set of all primes (larger than one). Let $s = x + iy$, $x, y \in \mathbb{R}$ and $x > \frac{1}{2}$. The Riemann zeta function can be expressed as

$$\zeta(s) = \prod_{j=1}^{\infty} (1 - p_j^{-s})^{-1}, \tag{2}$$

This infinite product converges absolutely if $Re\{s\} > 1$. See e.g. [1] for the basic facts of $\zeta(s)$.

2 An introductory lemma and the theorem

Lemma 1. *The functions*

$$h_j(s) = -\sum_{j=1}^{\infty} \ln(p_j) p_j^{-js} \quad , \quad j > 0 \tag{3}$$

are related by $h_j(s) = h_1(js)$. *The functions* $h_j(s)$ *have analytic continuations to* $Re\{s\} > 0$ *with the exception of isolated first-order poles. The poles of* $h_j(s)$ *that are not on the x-axis appear in pole pairs: close to* s_k, *where* $Im\{s_k\} > 0$, $h_j(s)$

is of the type

$$h_j(s) = \frac{r}{s - s_k} + f_1(s) \tag{4}$$

and close to s_k^**, where* s_k^* *is a complex conjugate of* s_k*,* $h_J(s)$ *is of the type*

$$h_j(s) = \frac{r}{s - s_k^*} + f_2(s)$$

The functions $f_1(s)$ *and* $f_2(s)$ *are analytic close to* s_k *and* s_k^* *respectively. If the pole is at the x-axis, there is only one pole of the type (4) with* $Im\{s_k\} = 0$.

Proof. The claim

$$h_j(s) = h_1(js) \tag{5}$$

follows directly from (3).

The function $h_1(s)$ converges absolutely if $Re\{s\} > 1$ because

$$\sum_{j=1}^{\infty} p_j^{-s}$$

converges absolutely for $Re\{s\} > 1$ and $|\ln p_j| < |p_j^{\alpha}|$ for any fixed $\alpha > 0$ if j is sufficiently large. Therefore

$$|\ln(p_j)p_j^{-s}| < 2|p_j^{-s+\alpha}|$$

for any fixed $\alpha > 0$ if j is sufficiently large. Therefore, by (5), $h_j(s)$ converges absolutely if $Re\{s\} > \frac{1}{j}$.

From (2) follows

$$\zeta'(s)\zeta(s)^{-1} = \frac{d}{ds}\ln\zeta(s) = \sum_{j=1}^{\infty} h_j(s).$$

The derivative $\zeta'(s)$ is analytic in all points except for $s = 1$. The function $h_1(s)$ is continued analytically to $Re\{s\} > \frac{1}{2}$ by

$$h_1(s) = \zeta(s)^{-1}\zeta'(s) - g(s) \tag{6}$$

where

$$g(s) = \sum_{j=2}^{\infty} h_j(s).$$

The function $\zeta(s)^{-1}$ is analytic except for at points where $\zeta(s)$ has a zero or a pole. The function $g(s)$ is analytic for $Re\{s\} > \frac{1}{2}$ because each $h_j(s)$, $j > 1$, is analytic in $Re\{s\} > \frac{1}{j}$. Thus, the right side of (6) is defined and analytic for $\frac{1}{2} < Re\{s\}$ except for at points where $\zeta(s)$ has a zero or a pole. At those isolated points $h_1(s)$ has a pole.

At a pole s_k of $\zeta(s)$ the zeta function has the expansion

$$\zeta(s) = \frac{C}{(s - s_k)^k} + \text{higher order terms.}$$

If $Re\{s\} > \frac{1}{2}$ the function $h_1(s)$ is of the form

$$h_1(s) = \zeta'(s)\zeta(s)^{-1} - g(s) = \frac{r}{s - s_k} + f_1(s)$$

where $f_1(s)$ is analytic close to s_k and $r = -k < 0$ is an integer. The function $\zeta(s)$ has only one pole, at $s_k = 1 = (1, 0)$, and it is a simple pole, thus $r = -1$.

At a zero s_k of $\zeta(s)$ the zeta function has the expansion

$$\zeta(s) = C(s - s_k)^k + \text{higher order terms.}$$

If $Re\{s\} > \frac{1}{2}$ the function

$$h_1(s) = \zeta'(s)\zeta(s)^{-1} - g(s) = \frac{r}{s - s_k} + f_1(s)$$

where $f_1(s)$ is analytic close to s_k and $r = k > 0$ is an integer. It is known that $\zeta(s)$ has many zeros with $Re\{s_k\} = 1/2$.

Thus, $h_1(s)$ has only first-order poles for $Re\{s\} > \frac{1}{2}$ and therefore $h_j(s)$ has only first-order poles for $Re\{s\} > \frac{1}{2j}$. At every pole of $h_1(s)$ in $Re\{s\} > \frac{1}{2}$ the value of r is an integer.

As $h_1(s)$ is continued to $Re\{s\} > \frac{1}{2}$ by (6), the equation (5) continues $h_j(s)$ to $Re\{s\} > \frac{1}{2j}$. Then (6) continues $h_1(s)$ to $Re\{s\} > \frac{1}{4}$. The function $h_1(s)$ has isolated poles at $Re\{s\} > \frac{1}{4}$. Each pole is a first-order pole, but the value of r at a pole does not need to be an integer.

We can repeat the procedure inductively: If $h_1(s)$ is continued to $Re\{s\} > \frac{1}{2^i}$ by (6), the equation (5) continues $h_j(s)$ to $Re\{s\} > \frac{1}{2^i j}$. Then (6) continues $h_1(s)$ to $Re\{s\} > \frac{1}{2^{i+1}}$. By induction, all $h_j(s)$ are analytically continued to $Re\{s\} > 0$.

In this inductive process $h_1(s)$ gets isolated first-order poles. In these poles s_k the values $r = r_k$ can be positive or negative, and they do not need to be integers. If $h_1(s)$ has a pole

$$h_1(s) = \frac{r}{s - s_k} + f_1(s)$$

(here $f_1(s)$ is analytic close to s_k), then $h_j(s) = h_1(js)$ has a pole at $j^{-1}s_k$ and the r value is $j^{-1}r$ since

$$h_j(s) = h_1(js) = \frac{j^{-1}r}{s - j^{-1}s_k} + f_1(js).$$

The function $h_1(s)$ is symmetric with respect to the real axis. By (4) $h_j(s)$, $j > 1$, is also symmetric with respect to the real axis. Therefore poles of each $h_j(s)$, $j > 0$, appear as pairs s_k and s_k^*. In the special case where s_k is real there is only one pole, not a pair. □

Theorem 1. *All poles of* $\sum_{j=1}^{\infty} h_j(s)$ *in* $0 < Re\{s\} < 1$ *have the real part* $\frac{1}{2}$.

Proof. Let us consider a function $f(s)$ that has a first-order pole at s_0 and write $z_1 = s - s_0$. The function $f(s)$ does not have a Taylor series at s_0, but the function

$z_1 f(z1+s_0)$ has a Taylor series at $z_1 = 0$ and $f(s)$ can be expressed as

$$f(s) = \frac{c_{-1}}{z_1} + \sum_{k=0}^{\infty} c_k z_1^k. \tag{7}$$

Let us evaluate $f(s)$ at another point at $s_0 + l$, $l > 0$, by first writing $z_1 = l - z_2$ where $|z_1| << 1$, inserting $z_1 = l - z_2$ to the series expression of $f(s)$, and then considering the result when $|z_2| << 1$. The function

$$f_1(z_1) = f(z_1 + s_0) - \frac{c_{-1}}{z_1} \tag{8}$$

has the Taylor series at $z_1 = l - z_2$ where $|z_1| << 1$ as

$$f_1(l - z_2) = \sum_{m=0}^{\infty} c_m (l - z_2)^m$$

$$= \sum_{m=0}^{\infty} \sum_{i=0}^{m} \frac{m!}{i!(m-i)!} l^i (-z_2)^{m-i} c_m$$

$$= \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \frac{(k+i)!}{i!k!} l^i (-1)^k c_{k+i} z_2^k = \sum_{k=0}^{\infty} b_k z_2^k.$$

Thus

$$b_k = \sum_{i=0}^{\infty} \frac{(k+i)!}{i!k!} l^i (-1)^k c_{k+i}.$$

As

$$c_k = \frac{1}{k!} \frac{d^k}{dz_1^k} f_1(s)|_{z_1=0}$$

we can express

$$b_k = \left(\sum_{i=0}^{\infty} \frac{1}{i!} l^i \frac{d^i}{dz_1^i} \right) \frac{1}{k!} (-1)^k \frac{d^k}{dz_1^k} f_1(s)|_{z_1=0}. \tag{9}$$

If there is no pole of $f(s)$ at $s_0 + l$, the function

$$f_1(l - z_2) = \sum_{k=0}^{\infty} b_k z_2^k$$

is analytic and defined by its Taylor series as powers of z_2 where the series converges.

The pole of $f(s)$ at c_{-1} can be evaluated as a Taylor series of z_2 at $s_0 + l$ as

$$\frac{c_{-1}}{l - z_2} = \frac{c_{-1}}{l} \frac{1}{1 - z_2 l^{-1}} = \frac{c_{-1}}{l} \sum_{k=0}^{\infty} \left(\frac{z_2}{l}\right)^k .$$

We can subtract a set of first-order poles of $f(s)$ in points $s_j \in A$ and define

$$f_1(z_1) = f(s) - \sum_{j \in A} \frac{r_j}{s - s_j} \tag{10}$$

where $r_j = c_{-1,j}$ and express

$$s - s_k = (s - s_0) - (s_j - s_0) = z_1 - s_j + s_0 = l - z_2 - (s_j - s_0).$$

At the point $s_0 + l$ the set of poles is

$$\sum_{j \in A} \frac{r_j}{s - s_j} = \sum_{j \in A} \frac{r_j}{l - z_2 - (s - s_0)} = x \sum_{j \in A} \frac{r_j}{1 - a_j x} \tag{11}$$

where $a_j = s_j - s_0$ and $x = (l - z_2)^{-1}$.

Let us consider

$$f(s) = \sum_{k=1}^{k_1} \frac{r_k}{s - s_k} + f_1(s) \tag{12}$$

$$f_1(s) = - \sum_{j=1}^{j_{\max}} \ln(p_j) p_j^{-s}.$$

Let $l >> 1$. The Taylor series of the set of poles points s_k at s_0 in powers of z_1 is

$$- \sum_{i=0}^{\infty} \left(\sum_{k=1}^{k_1} r_k (s_k - s_0)^{-i-1} \right) z_1^i$$

and the Taylor series at $s_0 + l$ in powers of $z_2 = l - z_1$ is

$$\sum_{i=0}^{\infty} \left(\sum_{k=1}^{k_1} r_k (s_0 + l - s_k)^{-i-1} \right) z_2^i.$$

For each k the coefficient of the ith power of z_1 at s_0 is $c_i = r_k(s_k - s_0)^{-i-1}$ while the coefficient of z_2 at $s_0 + l$ is

$$b_i = r_k(s_0 + l - s_k)^{-i-1} = r_k l^{-i-1} + r_k(i+1)(s_k - s_0) l^{-i-2} + \cdots$$

The absolute value of the coefficient b_i of the Taylor series in powers of z_2 at $s_0 + l$ decreases as

$$|\sum_{k=1}^{k_1} r_k| l^{-i-1}$$

as a function of $l >> 1$.

The part $f_1(s)$ of $f(s)$ satisfies

$$|f_1(s+l)| = \left| - \sum_{j=1}^{j_{\max}} \ln(p_j) p_j^{-s-l} \right| = \left| - \sum_{j=1}^{j_{\max}} \ln(p_j) p_j^{-s} e^{-l \ln p_j} \right|$$

$$\leq \left| e^{-l \ln 2} \right| \left| - \sum_{j=1}^{j_{\max}} \ln(p_j) p_j^{-s} \right| = e^{-l \ln 2} |f_1(s)|. \tag{13}$$

The absolute value of the coefficient b_i of the Taylor series in powers of z_2 at $s_0 + l$ decreases as

$$|b_i| \leq e^{-l \ln 2} |c_i|.$$

This is negative exponential decrease and much faster than the hyperbolic decrease for the set of poles.

When $l \to \infty$, the hyperbolic contribution from the poles must vanish: every nonzero coefficient of the Taylor series of $f(s)$ at $(l, 0)$ when $l \to \infty$ must decrease as a negative exponential $e^{-l \ln(2)}$. This negative exponential of l decreases faster than any negative power of l. For each power i of x, the coefficient in the power series of x coming from the sum of the poles must go to zero at least as $O(x)$ leaving the negatively exponentially decreasing coefficient from $f_1(s)$ in (12) to dominate.

The sum of the poles decreases as $O(x)$, $x = l^{-1}$, and goes to zero when $x \to 0$ assuming that the x-coordinate of every pole of $f(s)$ is smaller or equal to one, but the required convergence that each coefficient of the power series of x must go separately to zero at least as $O(x)$ is a stronger condition. The requirement that the coefficient of a power i of x the sum of poles decreases at least as $O(x)$ means that that the poles of $f(s)$ partially cancel each others when l grows. Poles cannot completely cancel: a pole at s_k with $r = r_k$ can be completely cancelled only by a pole at s_k with $r = -r_k$. The sum of poles has the poles of its terms, but at $l >> 1$ there can be partial cancellation so that the Taylor series coefficients decrease sufficiently fast as a function of l.

Let $j_{\max} \to \infty$ in (13). Then $f(s) = h_1(s)$. If $Re\{s\} = l >> 1$, the sum (13) taken to infinity converges absolutely. The inequality (13) holds when $j_{\max} \to \infty$ and the absolute values of the coefficients of the Taylor series at $s_0 + l$ for the function $h_1(s)$ must decrease in negative exponential manner as a function of l when $l \to \infty$. It follows that every $h_j(s) = h_1(js)$ also has the same negatively exponential dependence of the coefficients of the Taylor series at $(l, 0)$ on l when $l >> 1$. Consequently the sum of the poles of the functions $h_j(s) = h_1(js)$ has same negatively exponential dependence for coefficients at $(l, 0)$ on l when $l >> 1$. Therefore the sum of the poles of the function

$$f(s) = \frac{d}{ds} \ln \zeta(s) = \sum_{j=1}^{\infty} h_j(s)$$

must satisfy the requirement that the coefficient of each power of x decreases at least as $O(x)$ when l grows to infinity.

We did not continue $h_j(s)$ to the area $Re\{s\} \le 0$ in Lemma 1, but the function $f(s)$ is analytically continued to $Re\{s\} \le 0$ by

$$f(s) = \frac{d}{ds} \ln \zeta(s)$$

to all points where $\zeta(s) \ne 0$ and we can find all poles of $f(s)$.

The function $f(s)$ has the following poles in $Re\{s\} > 0$:

(i) There is a pole with $r = -1$ at $s = 1$.

(ii) There is a set A of pole pairs $h_1(s)$ at s_k and s_k^* where s_k has a nonzero imaginary part, and the r-value r_k is positive. All we know of s_k is that $0 < Re\{s_k\} < 1$, and that that there exist poles s_k with the real part $\frac{1}{2}$.

(iii) There may be a set A_1 of poles $s_{k,1}$ of $h_1(s)$ with $r_{k,1}$ a positive integer and the pole s_k is real, $0 < s_k < 1$. No such pole is known.

Inserting $s = s_0 + l$, $s_0 = 0$, $x = l^{-1}$ to the expression of a pole (4) on the x-axis gives (ignoring the analytic function part in (4))

$$\frac{r_k}{s - s_k} = \frac{x r_k}{1 - a_k x}.$$

Here a_k is a real number. A pole pair in the positive and negative y-axis can be written as

$$\frac{r_k}{s - s_k} = \frac{x r_k}{1 - (1 + i\alpha_k) a_k x}$$
$$\frac{r}{s - s_k^*} = \frac{x r_k}{1 - (1 - i\alpha_k) a_k x}.$$

Here $x = (l - z_2)^{-1} > 0$ is a real number and small if l is large, $a_k = Re\{s_k\}$ and α_k is chosen positive. We will always take s_0 as 0. The number l is the distance from $s_0 = 0$ to the observation point on the x-axis, $(l, 0)$, where the Taylor series with z_2 is evaluated and $|z_2| << 1$. As z_2 is the variable of the Taylor series at $(l, 0)$, the expressions are valid for any small z_2 and we select $z_2 = 0$ for easier notations. Thus, $x = l^{-1}$. The pole (i) at $s = 1$ gives the power series of x where

$a_k = 1$ and $r = -1$

$$\frac{xr}{1-(a_k x)} = \frac{-x}{1-x} = -x\sum_{m=0}^{\infty} x^m.$$

The zeros of $\zeta(s)$ in the area $Re\{s\} \leq 0$ are the so called trivial zeros at even negative integers. They come from the formula

$$\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1}$$

where $B_m = 0$ if $m > 1$ is odd. Zeta does not have a zero at $s = 0$. From the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin(2^{-1}\pi s)\Gamma(1-s)\zeta(1-s) \tag{14}$$

we can deduce that the trivial zeros are zeros of $\sin(2^{-1}\pi s)$ and therefore first-order zeros. Thus, at a point $s_k = -2k$, $k > 0$ integer, the function $f(s)$ has a first-order pole with the r-value 1.

A pole at $s_k = -2k$, $k > 0$, is

$$\frac{r_k}{s - s_k} = \frac{1}{s+2k}.$$

We can evaluate the Taylor series of z_1 at s_0 and the Taylor series of z_2 at $s_0 + l$ for any such pole and for a finite sum of such poles:

$$\frac{1}{s_0 + z_1 + 2k} = \frac{1}{s_0 + 2k}\sum_{i=0}^{\infty}(-1)^i (s_0 + 2k))^{-1} z_1^i$$

$$\frac{1}{s_0 + l - z_2 + 2k} = \frac{1}{s_0 + l + 2k}\sum_{i=0}^{\infty}(s_0 + l + 2k))^{-1} z_2^i$$

but if sum the index k goes to infinity, the series diverges at every finite point $s_0 + l$. We will evaluate the sum of these poles at $s_0 = 0$, conclude that the contribution is negative, and present a way to move a finite but growing sum of these poles to $(l, 0)$.

First we find out the sign of the infinity of the sum of the poles $s_k = -2k$ at $s_0 = 0$ and $z_1 = 0$. Notice that for a point $s_j = -k$ the pole at that point, with the r-value r, when evaluated to a Taylor series at $s_0 = 0$ and $z_1 = 0$ is

$$\frac{r}{s - s_j} = \frac{r}{k}.$$

This is the inverse of a pole with the same r but with $s_j = k$ when evaluated to a Taylor series at $s_0 = 0$ and $z_1 = 0$. As an example, $s_j = 1$ is the pole at $s = 1$ with $r = -1$. When evaluated at $s_0 = z_1 = 0$ it is the inverse of a pole with $r = -1$ but $s_k = -1$. Thus, the pole at $s_k = -2k$ with $r = 1 > 0$ is the same at $s_0 = 0$ as a a pole at $s_k = 2k$ with $r = -1 < 0$. We see that any sum of the poles $s_k = -2k$ gives a negative infinity when evaluated at $s_0 = 0$.

The type of infinity of the sum of all poles $s = -2k$ at $s_0 = 0$ can be calculated. Using the facts that $\zeta(s)$ has a simple pole at $s = 1$

$$\zeta(s) = \frac{a}{s-1} + g(s)$$

where $g(s)$ is analytic at $s = 1$ and that $\lim_{s\to 1}(s-1)\zeta(s) = 1$, so $a = 1$, we can write

$$\zeta(1) = \lim_{s\to 1} \frac{1 + (s-1)f(1)}{s-1} = \lim_{s\to 1} \frac{1}{s-1} = \lim_{s\to 0} \frac{1}{s}$$

This result gives

$$\sum_{k=1}^{\infty} \frac{1}{2k} = \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k^1} = \frac{1}{2}\zeta(1) = \lim_{s\to 0} \frac{1}{2}\frac{1}{s}.$$

Thus, the sum of the poles at $s_k = -2k$ appears as a simple pole when evaluated at $s_0 = 0$. The pole has a negative r-value with $r = -1$ at $s_0 = 0$. However, it is not a simple pole. A simple pole with $r = -1/2$ is

$$\lim_{s\to s_0} (-1/2)1/(s - s_0).$$

It is moved to $s_0 + l$ by writing

$$\lim_{s\to s_0}(-1/2)1/(s-l-s_0) = (-1/2)/l = -x/2$$

where $x = l^{-1}$. This pole is finite for every $l > 0$, but the sum of the poles $s_k = -2k$ is infinite at every finite l. This is so because the infinity $\lim_{s\to s_0}(-1/2)/(s-s_0)$ is not caused by the pole being physically at s_0, the infinity comes from the sum of the numbers $1/(l+2k)$.

If we subtract all poles $s_k = -2k$ from $f(s)$, then $f_1(s)$ is infinite at every point. Because of this reason all poles $s_k = -2k$ cannot be moved to $(l, 0)$ at the same time. We can only move at a given $l > 0$ such a subset of poles (like a finite set) that the sum gives a finite number when moved to $(l, 0)$. All poles have to be moved at some point as the sum of all poles of $f(s)$ should be zero at $l \to \infty$. Thus, we must move more poles when l grows until all poles are moved when $l \to \infty$. The choice of which subsums of poles are moved for each l cannot influence the result. We will make a convenient choice for these sums: let us choose a suitable growing function $N(l)$ and move the subsum of poles $s_k = -2k$ satisfying $k \leq N(l)$. A finite sum up to $N(l)$ can be moved to $s_0 + l$, and when $N(l)$ increases with l, all poles $-2k$ are included in the finite sum when $k \leq N(l)$. The tail of the infinite sum that is outside the finite sum up to $N(l)$ goes to zero when $l \to \infty$.

Thus, we take a finite sum

$$\sum_{k=1}^{[N(l)]} \frac{1}{2k}.$$

As it is a finite sum, it can be moved to $(l, 0)$ without creating an infinity. If $N(l)$ is sufficiently large and fixed, and $l = 0$, the moved sum is $-x/2 - \epsilon(l)$. The number $\epsilon(l)$ depends only on $N(l)$ and we can select a function $N(l)$ such that $\epsilon < \min\{\epsilon_0, e^{-l}\}$ where $\epsilon_0 > 0$ is small. Then $\epsilon(l)$ decreases with l faster than any power of $x = l^{-l}$. The number $N(l)$ increases when l grows, and therefore the absolute value of the sum grows with l. It gives a function $-xC(l) - \epsilon(l)$. This function cannot have any higher powers of x, only the first power, because every power of x can be continued to s_0 and there would be the power of x also at s_0, but at s_0 the function is $-x/2$ when $N(0) \to \infty$. In the limit $x \to 0$ the function $xC(l)$ must be of the order $O(x)$ because all other poles give contributions of $O(x)$ and the sum of all poles must vanish when $l \to \infty$. Since $C(l) \geq 1/2$ is a growing function and limited from above, the function $-xC(l) - \epsilon$ must converge to $-xC$, where $C > 1/2$ is a finite real number. The number ϵ goes to zero, as it decreases faster than any power of x. The number C will be determined later in this proof.

The poles (iii) of A_1 sum to a series of the type $x\sum_{m=0}^{\infty} c_m x^m$ where every c_m is nonnegative. Since all of these poles are in the area $0 < s_k < 1$ and they are isolated and therefore do not have a concentration point at $s = 1$, the power series of x coming from these poles cannot be of the type $b(x + x^2 + x^3 \cdots)$, which is the type of the power series of the pole at $s = 1$. It follows that the poles of A_1 cannot be cancelled the pole at $s = 1$ giving the contribution $-x/(1-x)$. Adding the contribution $-xC$ from the the sum of poles $-2k$ does not help to cancel any poles of A_1. The poles of (ii) yield a power series of x where the coefficient of every x^i is nonnegative. They cannot cancel poles of A_1. Thus, the poles of A_1 cannot be cancelled by any set of other poles in the limit $l \to \infty$. Therefore the set A_1 must be empty.

The pole pairs in A can be cancelled by the poles in $s = 1$ and in $-2k$, as will be seen later. The coefficient of the power one of x can be cancelled by sum of the corresponding coefficient -1 of the pole at $s = 1$ and the coefficient $-C$ coming from the poles in $Re\{s_k\} \leq 0$. Higher than power one coefficients of x coming from a sum of pole pairs in A can be cancelled only by the pole at $s = 1$ since $-xC$ does not have higher powers of x.

The two poles (ii) of a pole pair have a real sum:

$$\frac{xr_k}{1 - a_k(1 + i\alpha_k)x} + \frac{xr_k}{1 - a_k(1 - i\alpha_k)x} = xr_k \frac{2(1 - a_k x)}{1 - 2a_k x + (1 + \alpha_k^2)(a_k x)^2}.$$

We expand the sum S of the poles of a pole pair omitting the multiplier xr_k for simplicity in this calculation up to (16):

$$S = \frac{2(1-a_kx)}{1-2a_kx+\alpha_k^2(a_kx)^2} = \frac{2-2a_kx}{1+\alpha_k^2(a_kx)^2}\frac{1}{1-2a_kx\gamma_k^{-1}}$$

where $\gamma_k = 1+\alpha_k^2(a_kx)^2$.

$$= \frac{2-2a_kx}{\gamma_k}\sum_{i=0}^{\infty}(2a_kx\gamma_k^{-1})^i.$$

Writing $\beta_{k,i} = (2a_k)^i\gamma_k^{-i-1}$ we get

$$S = 2\sum_{i=0}^{\infty}\beta_{k,i}x^i - 2a_k\sum_{i=0}^{\infty}\beta_{k,i}x^{i+1} = \sum_{i=0}^{\infty}2\beta_{k,i}x^i - 2a_k\sum_{i=1}^{\infty}\beta_{k,i-1}x^i$$

$$= 2\beta_{k,0} + \sum_{i=1}^{\infty}(2\beta_{k,i} - 2a_k\beta_{k,i-1})x^i.$$

For $i > 0$

$$2\beta_i - 2a_k\beta_{k,i-1} = 2\frac{(2a_k)^{i-1}}{\gamma_k^i}(2a_k\gamma_k^{-1} - a_k)$$

$$= \frac{(2a_k)^i}{\gamma_k^{i+1}}(2-\gamma_k) = \beta_{k,i}(2-\gamma_k).$$

This gives an equation for every $i > 0$

$$2\beta_i - 2a_k\beta_{k,i-1} = 2\beta_{k,i} - \gamma_k\beta_{k,i}.$$

Inserting $\gamma_k = 1 + (\alpha_k a_k x)$ yields for $i > 0$

$$2a_k\beta_{k,i-1} = \gamma_k\beta_{k,i} = \beta_{k,i} + x^2(\alpha_k a_k)^2\beta_{k,i}.$$

For every k when $l >> 1$ and thus for $0 < x = l^{-1} << 1$ and $i > 0$ holds

$$2a_k\beta_{k,i-1} = \gamma_k\beta_{k,i} = \beta_{k,i} + O(x^2).$$

The coefficient of the the power x^i, $i > 0$, is

$$2\beta_{k,i} - 2a_k\beta_{k,i-1} = \beta_{k,i} + O(x^2). \tag{16}$$

The coefficient of the power of x^{i+1} in the power series $-x/(1-x)$ of the pole in $s = 1$ is -1 for every $i > 0$. The coefficient of x^{i+1} in the power series of the sum of poles (ii) is

$$\sum_{k\in A} r_k(2\beta_{k,i} - 2a_k\beta_{k,i-1})$$

where we have included the multiplier xr_k that was so far omitted. Summing the powers of i from $i = 2$ to $i = i_1 + 1$ and inserting (16) gives the equation where the coefficients of the pole pairs (ii) must cancel the coefficients of the pole (i) to the degree of $O(x^2)$:

$$i_1 = -\sum_{i=2}^{i_1+1}(-1) = \sum_{i=2}^{i_1+1}\sum_{k\in A} r_k\beta_{k,i} + O(x^2). \tag{17}$$

For each k, when $x \to 0$ and $i > 0$, holds

$$\beta_{k,i} = 2a_k\beta_{k,i-1}. \tag{18}$$

If every $a_k = \frac{1}{2}$ the recursion equation (18) gives $\beta_{k,i+1} = \beta_{k,i}$ for every k. For every k the power series of x for $i > 1$ is of the form $x\beta_{k,1}(x + x^2 + x^3 + \cdots)$. This is the same form as the power series $-x(x + x^2 + x^3 + \cdots)$ for the pole $s = 1$ for $i > 1$. The power series for the poles k for $i > 1$ add to one power series of the type $xb(x + x^2 + x^3 + \cdots)$. We see that if every $a_k = \frac{1}{2}$, the sum of poles (ii) cancels all powers $i > 1$ in the pole in $s = 1$ when $x \to 0$ and the coefficient of each power $i > 1$ of x converges to the negative of the coefficient of the power i of x in the power series for the pole in $s = 1$ as $O(x^2)$.

Assume that one a_k is not $\frac{1}{2}$. The functional equation (14) shows that if there exists a zero $s_0 = x_0 + iy_0$ of $\zeta(s)$ with $0 < x_0 < \frac{1}{2}$ then there exists a zero of $\zeta(s)$

at a symmetric point in $\frac{1}{2} < x < 1$. This implies that we can find $s_{k'}$ such that $2a_{k'} > 1$. The form of (18) for a nonzero x, $i > 0$, is

$$\beta_{k,i} = 2a_k\beta_{k,i-1} + O(x^2). \tag{19}$$

For $a_{k'}$ the recursion (19) gives $\beta_{k',i} = \beta_{k',1}(2a_{k'})^i + O(x^2)$ From (17) we get (20):

$$i_1 = \sum_{i=2}^{i_1+1}\sum_{k\in A} r_k\beta_{k,i} + O(x^2) \geq \beta_{k',1}(2a_{k'})^{i_1} + O(x^2). \tag{20}$$

The right side in (20) grows as $\beta_{k',1}(2a_{k'})^{i_1}$ as a function of i_1 while the left side is linear in i_1. This is a contradiction. Thus, every a_k must be $\frac{1}{2}$.

By (20) each $a_k = \frac{1}{2}$. Inserting $a_k = 2^{-1}$ to (21) gives

$$\beta_i = \sum_{k\in A} r_k(1 + 2^{-2}(\alpha_k x)^2)^{-i-1}. \tag{22}$$

The recursion equation for $\beta_{k,j}$ is $\beta_{k,i} = (2a_k/\gamma_k)\beta_{k,i-1}$. As $2a_k = 1$ and since $\gamma_k \geq 1$, this implies that $\beta_{k,i-1} \geq \beta_{k,i}$ for all $i > 0$. Recursion (18) for $a_k = \frac{1}{2}$ shows that for every $i > 0$ the value $\beta_{k,i}$ is the same when $x \to 0$. Since $\gamma_k \to 1$ when $x \to 0$, β_i is the same for every $i \geq 0$. Equation (20) implies that $\beta_i = 1$ for every $i > 0$. In the limit $x \to 0$ holds $\beta_{k,0} = \beta_{k,1}$. Therefore also $\beta_i = 1$ when $x \to 0$.

The claim of Theorem 1, i.e., that each $a_k = \frac{1}{2}$ for $s_k \in A$ and A_1 is empty, is already proven. The reason for this result is that since the poles of $f(s)$ at $Re\{s\} \leq 0$ give $-xC$, all powers $i > 1$ of x in the series $-x/(1-x)$ for the pole at $s = 1$ have to be cancelled by the poles of A and A_1. This series to be cancelled by the poles of A and A_1 is $-x^2 - x^3 - x^4 - \cdots$. No sum of poles in A_1 can give this series because each pole in A_1 is smaller than one and larger than zero. A pole pair s_k, s_k^* in A gives this series if and only if $Re\{s_k\} = \frac{1}{2}$. If even one s_k has the real part not at $\frac{1}{2}$, (20) gives a contradiction. Thus, all poles of A have the real part as one half.

Let us still check if the solution is possible. We check if all

$$\beta_i = \sum_{k\in A} r_k \beta_{k,i} \tag{21}$$

can have the value 1 as the solution gives, and if all coefficients of the power series of x can go to zero at least as $O(x)$ when $l \to \infty$.

Because $x \to 0$, the values of α_k in (22) must grow to infinity with k. The set A is necessarily infinite. We renumber the poles of (ii) so that (α_k) is a growing sequence and the sum $k \in A$ is the sum $k = 1$ to infinity. Since $a_k = \frac{1}{2}$ by (20) we can evaluate

$$2\beta_{k,i} - 2a_k\beta_{k,i-1} = \beta_{k,i}(2-\gamma_k)$$

and get

$$\beta_{k,i} = \beta_{k,i-1}\left(1 - \frac{(0.5\alpha_k x)^2}{1+(0.5\alpha_k x)^2}\right).$$

Let $l >> 1$ be fixed. If $\alpha_k >> l = x^{-1}$, then

$$\frac{(0.5\alpha_k x)^2}{1+(0.5\alpha_k x)^2}$$

is close to one and $\beta_{k,i}$ is close to zero. This means that large values of α_k contribute very little to the Taylor series at $s_0 + l$. The sum in (22) can be finite and there is no reason why it could not be one as the solution gives.

The contribution from the poles at $s_k = -2k$ is $-xC$, from the pole at $s = 1$ it is $-x/(1-x)$, from the poles of A_1 it is 0, and the contribution of the pole pairs of A is approaching the series $x(2 + x + x^2 + x^3 + \cdots)$ when $x \to 0$ as $O(x^2)$ separately for the coefficient of each power i of x^i. The sum of these contributions when $l \to \infty$ is

$$-Cx - \frac{x}{1-x} + 0 + x(2 + x + x^2 + x^3 + \cdots) = (-3/2 + 2\beta_0)x = (1-C)x. \tag{24}$$

The sum (24) must be zero. This is possible when $C = 1$.

The convergence of the coefficients of the powers of x to zero in (24) when x grows is $O(x^2)$ for the coefficient of each power $i > 1$ of x^i separately, which fulfills the convergence criterion.

For the power one of x we only get the result that the convergence as at least $O(x)$ is possible. The term β_0 converges to β_1 as $O(x^2)$ since every $\beta_{k,i} = \beta_{k,i-1} + O(x^2)$. Each β_i converges to 1 as $O(x^2)$. The contribution from the poles at $-2k$ is $-xC(l) - \epsilon(l)$ where $0 < \epsilon(l) < e^{-1}$ goes to zero very fast. The sum $-xC(l) - \epsilon(l) - x + x\beta_0$ can go to zero at least as fast as $O(x)$ as the solution requires. The solution is possible. □

All known facts of the Riemann zeta function that are used in this proof can be found in [1]. The history and background of the Riemann Hypothesis are well described in the book [2]. As the problem is still open, recently published results do not add so much to the topic. As they are not needed in this proof, they are not referred to.

References

1. E. T. Whittaker and G. N. Watson, *A Course in Modern Analysis*, Cambridge, University Press, 1952.
2. K. Sabbagh, *The Riemann Hypothesis, the greatest unsolved problem in mathematics*, Farrar, Strauss and Giroux, New York, 2002.